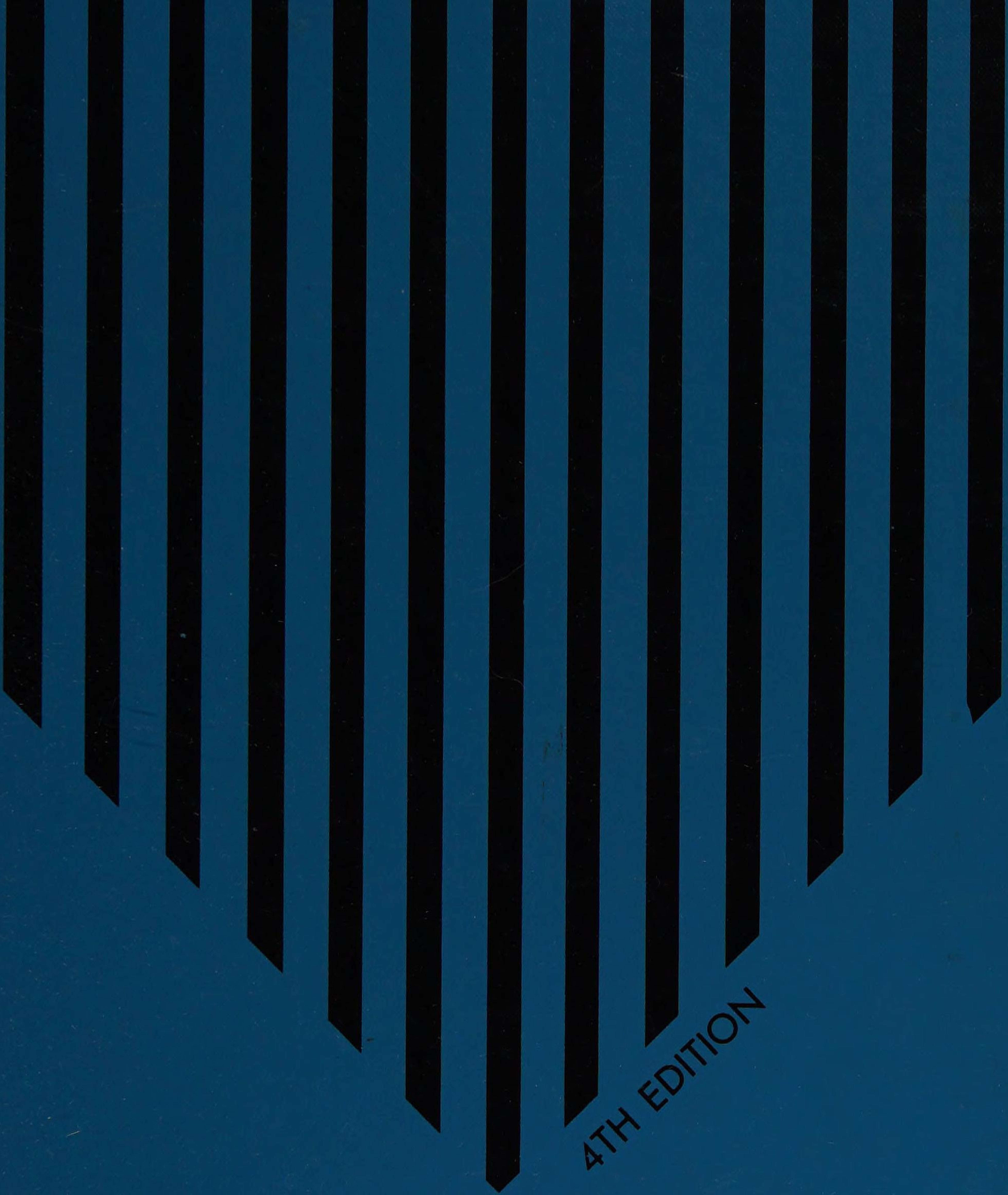

4TH EDITION

introduction to vector analysis

Harry F. Davis / Arthur David Snider

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Introduction to
Vector Analysis

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Vector Analysis

fourth edition

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Contents

Preface ix

CHAPTER 1 VECTOR ALGEBRA **1**

1.1 Definitions 1
1.2 Addition and Subtraction 3
1.3 Multiplication of Vectors by Numbers 6
1.4 Cartesian Coordinates 7
1.5 Space Vectors 9
1.6 Digression 12
1.7 Some Problems in Geometry 14
Summary — Geometric and Coordinate Descriptions 18
1.8 Equations of a Line 20
1.9 Scalar Products 26
1.10 Equations of a Plane 30
1.11 Orientation 33
1.12 Vector Products 36
Optional Reading: The Proof of the Distributive Laws 40
Summary — Multiplying Vectors 42
1.13 Triple Scalar Products 45
1.14 Vector Identities 49
Optional Reading: Tensor Notation 51
Supplementary Problems 55

CHAPTER 2 VECTOR FUNCTIONS OF A SINGLE VARIABLE **58**

2.1 Differentiation 58

2.2 Space Curves, Velocities, and Tangents 62
2.3 Acceleration and Curvature 72
Optional Reading: The Frenet Formulas 79
2.4 Planar Motion in Polar Coordinates 82
2.5 Optional Reading: Tensor Notation 87
Supplementary Problems 87

CHAPTER 3 SCALAR AND VECTOR FIELDS **90**

3.1 Scalar Fields; Isotomic Surfaces; Gradient 90
3.2 Vector Fields and Flow Lines 97
3.3 Divergence 100
3.4 Curl 106
3.5 Del Notation 112
3.6 The Laplacian 115
Optional Reading: Dyadics 117
3.7 Vector Identities 119
Optional Reading: Tensor Notation 123
Supplementary Problems 125

CHAPTER 4 LINE AND SURFACE INTEGRALS **127**

4.1 Line Integrals 127
4.2 Domains: Simply Connected Domains 133
4.3 Conservative Fields 138
4.4 Conservative Fields (Continued) 144
Optional Reading: Proof of Theorem 4.3 for Star-Shaped Domains 148
4.5 Optional Reading: Vector Potentials 152
4.6 Oriented Surfaces 154
4.7 Surface Integrals 165
4.8 Volume Integrals 176
4.9 Introduction to the Divergence Theorem and Stokes' Theorem 183
4.10 The Divergence Theorem 189
4.11 Green's Theorem 195
4.12 Stokes' Theorem 201
4.13 Optional Reading: Transport Theorems 205
Supplementary Problems 213

CHAPTER 5 GENERALIZED ORTHOGONAL COORDINATES **221**

5.1 Cylindrical and Spherical Coordinates 221
5.2 Orthogonal Curvilinear Coordinates 237
5.3 Optional Reading: Matrix Techniques in Vector Analysis 248
5.4 Optional Reading: Linear Orthogonal Transformations 259

Review Problems 272
Appendix A Historical Notes 277

Appendix B Two Theorems of Advanced Calculus 282
Appendix C The Vector Equations of Classical Mechanics 286
Appendix D The Vector Equations of Electromagnetism 298

Answers and Notes 314

Index 337

Preface

Current trends in mathematics education emphasize the relevance and applicability of its various disciplines to other fields. Of course, this is nothing new for vector analysis, which has virtually been the *language* of mechanics and electromagnetism since the beginning of this century. The earlier editions of *Introduction to Vector Analysis* have recognized this aspect of the subject, but the authors of the fourth edition decided to expand the treatment of these applications in the hopes of initiating a broader and more diverse group of readers into the realms of vector techniques. This should also result in increased versatility on the part of those who master the subject.

This revised presentation is the result of streamlining some sections of the text, expanding others, inserting some new ones, and devising a format that, by utilizing many "optional reading" sections, has the flexibility to accommodate a wide range of instructors' tastes. As in the earlier editions our watchword has been expository excellence, so we have preserved our tradition of approaching each new concept on dual levels: first heuristically and geometrically, then analytically and rigorously. For instance, discussions of the physical significance of the cross product, curl, divergence theorem, etc. lead to heuristic derivations of the associated formulas; then the analytic formulations and proofs are presented. This geometric–analytic duality is even extended back to the basic notion of a vector, in this fourth edition. We feel that this approach aids the student in remembering the concepts, encourages him or her to anticipate the associated theorems, and bestows a more profound understanding of the covariance properties.

Some of the revisions in the basic material include: an explicit treatment of the parallel–perpendicular decomposition of a vector (which is utilized in Hilbert space theory as well as in engineering applications), a careful exposition of the *parameter* concept in geometry, reorganized sections on space curves and particle trajectories, a simplified proof of the potential theorem for conservative fields, derivations of Green's formulas, and considerable expansion of the sections introducing line, surface, and volume integrals. Also the material on curvilinear coordinates has been expanded to encompass all of Chapter 5, with one section treating cylindrical and spherical coordinates separately, one section on general orthogonal coordinates, and optional sections on linear orthogonal transformations and the matrix techniques used in handling and applying them.

The old Chapter 5 ("Advanced Topics") has been mostly eliminated; a study revealed that few instructors cover this rather specialized material. However, the "Historical Notes" still appear in the appendices.

Some of the topics covered in the optional readings are: tensor notation and its advantages in proving vector identities, dyadics, more general proofs of certain theorems, the Frenet–Serret formulas and their interpretation, vector potentials, and the transport theorems of fluid dynamics. Appendix B provides coverage of two theorems of advanced calculus that are crucial to vector analysis.

In addition there are two new appendices that may prove valuable to readers. Appendices C and D contain derivations of all of the basic equations of particle mechanics, rigid body dynamics, and (nonrelativistic) electromagnetism. The treatment here is rather more compact than expository, and somewhat more rigorous than one usually finds in the physics textbooks. Some instructors and advanced students may be surprised to see how readily these equations can be related when the full power of vector machinery is invoked.

The sets of exercises in the earlier editions have been enriched by the addition of "Supplementary Problems" at the end of each chapter. These vary greatly in difficulty; some merely provide drill while others involve quite provocative examples from advanced calculus and geometry. The answers to these new exercises are not included so that instructors can draw test material from them.

The authors extend special thanks to Rodrego A. Restrepo, who graciously contributed many of these problems and served as a reviewer. Many thanks also to the other reviewers: Vuryl J. Klassen, Samuel S. McNeary, and Thomas A. Metzger; and to Edwin Clark, Richard Hutchinson, and Samuel Poss, whose contributions have enriched the text.

In closing let us state that it has been our intent to design a textbook that will stand the engineer and scientist in good stead for his/her professional needs, give the aspiring mathematician a firm grasp of the three-dimensional versions of the theorems of higher geometry and their applications, and continue to serve the needs of every reader for some time after the successful (we trust) completion of formal training.

Introduction to
Vector Analysis

CHAPTER 1

Vector Algebra

1.1 DEFINITIONS

The vector concept is closely related to the geometrical idea of a *directed line segment*. Roughly speaking, a vector is a quantity that has direction as well as magnitude. It is represented by an arrow of length equal to its magnitude, pointing in the appropriate direction. Two vectors **A** and **B** are said to be equal, $\mathbf{A} = \mathbf{B}$, if they have the same length and direction.

This description of a vector conveys the intuitive concept, but as a definition it suffers from a lack of precision. Let us go back to basics and see if we can formulate this idea more carefully and unambiguously.

Consider two points P and Q in space. If P and Q are distinct points, there will exist one and only one line passing through them both. That part of the line between P and Q, including both P and Q as endpoints, is called a *line segment*. A line segment is said to be *directed* when the endpoints are given a definite order. The same line segment determines two directed line segments, one denoted PQ and the other QP (or $-PQ$). If P and Q coincide, PQ is said to be *degenerate*, and the line segment is a point.

Now a directed line segment is a quantity with magnitude (the distance between P and Q) and direction (one exception: the degenerate segment, or point!). Historically, vectors were defined to be directed line segments. Experience has taught us, however, that it is convenient to consider two directed line segments as representing the *same* vector if they are parallel translates of one another; that is, if they point in the same direction and have the same length. Thus in Fig. 1.1 we see that PQ, RS, and TU are all equivalent and they represent the same vector. Being careful to consider all the possibilities, we can now formulate these definitions: Two directed line

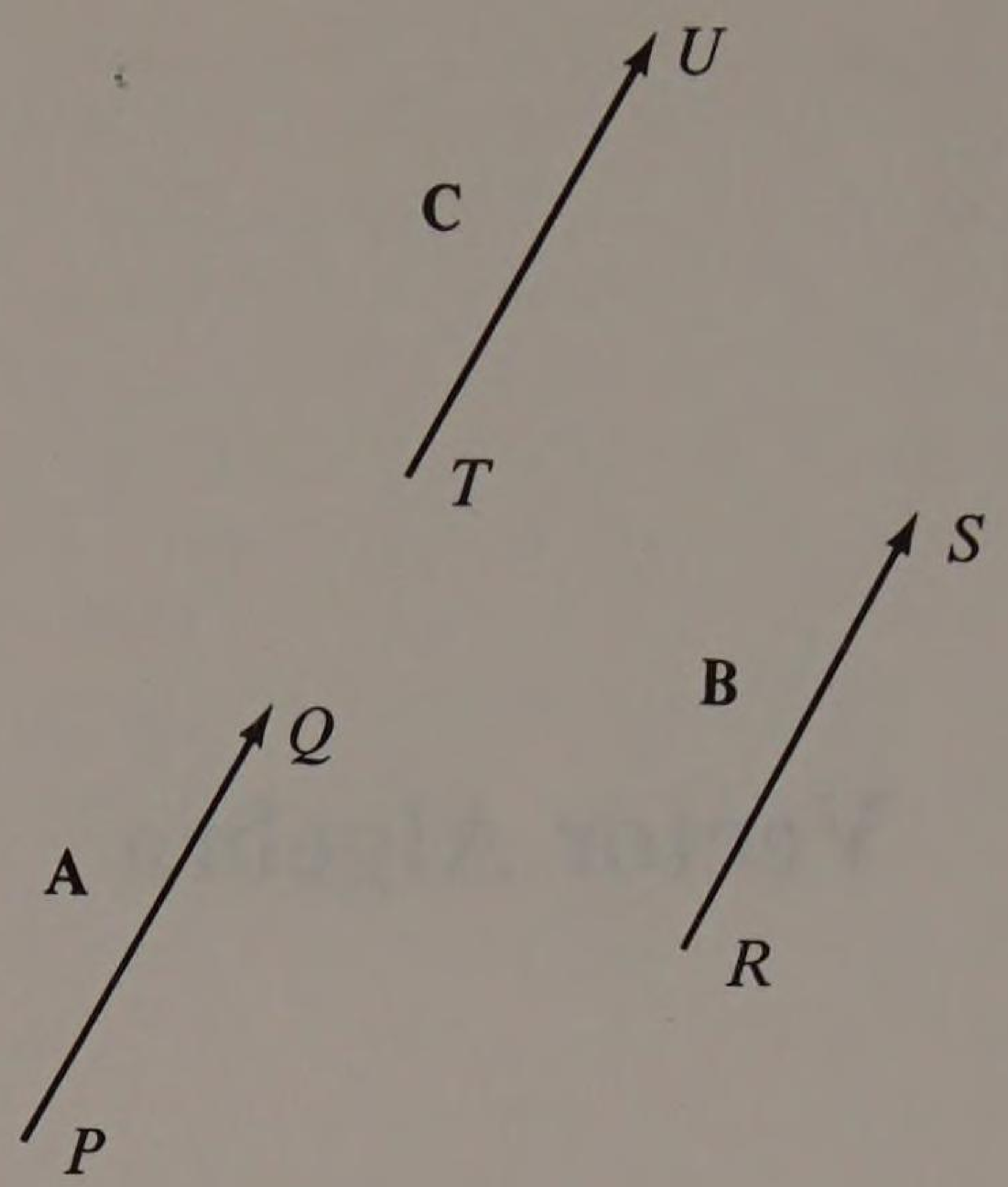

FIGURE 1.1

segments PQ and RS are said to be *equivalent* if PQ and RS have the same length and are parallel (with the proviso that any two *points* are parallel), and also PR and QS have the same length and are parallel (the last condition ensures that RS is not directed opposite to PQ—draw a sketch to see this). *A vector is defined to be a collection of equivalent directed line segments.*

We may represent a vector by any one of the directed line segments in the collection. Thus, we may represent a vector by giving a particular directed line segment PQ, but it is understood that the vector itself is the set of all directed line segments that are equivalent to PQ.

In this book, boldface letters are usually used to denote vectors. In the diagrams, a single directed line segment will often be drawn to represent a vector, and will be labeled by a boldface letter to denote the vector it represents. In Fig. 1.1, PQ is labeled $\mathbf{A}$, RS is labeled $\mathbf{B}$, and TU is labeled $\mathbf{C}$. Since these all represent the same vector, we can write $\mathbf{A} = \mathbf{B} = \mathbf{C}$. Notice, however, that PQ and RS are not the same directed line segment, since they occupy different positions in space, so we would not write $PQ = RS$.

To summarize, $\mathbf{A} = \mathbf{B}$ implies that PQ is parallel to RS, that PQ and RS have the same directed sense, and that the distance between P and Q is the same as the distance between R and S. This common distance is called the *magnitude* of the vector. Any point (degenerate line segment) represents the zero vector $\mathbf{0}$. This vector has zero magnitude and no direction; it is the exception to the intuitive characterization of "vector" given in the first paragraph.

Many of the quantities of physics have magnitude and direction, and thus are conveniently represented by vectors. As examples we mention force, displacement, velocity, acceleration, and magnetic field intensity. Such quantities are represented graphically by arrows of length proportional to the magnitude of the quantity, and pointing in the appropriate direction.

In some books, what we call *directed line segments* are called *bound vectors*, and what we simply call "vectors" are called *free vectors.* The idea is that a "free vector" can be moved freely through space; provided it is always kept parallel to its initial position, and is never allowed to reverse its sense or to vary in magnitude, it does not really "change," whereas a "bound vector" could not be moved about in space. The distinction creates logical difficulties for both the pure mathematician and for the physicist. For the pure mathematician it is difficult to accept such loose terminology as "moving freely through space" in the definition of a quantity that does not fundamentally involve the idea of time or motion at all. For the physicist the difficulty is in determining whether *force* is a bound or a free vector. In many cases the effect produced by a force acting on a body depends not only on its magnitude and direction but also on its point of application. Hence, force might well be regarded as a bound vector; but in deeper theoretical work this becomes extremely awkward. Most physicists regard force as a vector quantity (i.e., a "free" vector), recognizing nevertheless that the *effect* of a force may depend on the point where it is applied.

In this book, the word *scalar* is used as a synonym for *number*. Those quantities of physics that are characterized by numerical magnitude alone (and have nothing to do with direction) are called *scalars* or *scalar quantities.* Examples are mass, time, density, distance, temperature, and speed (as read from a speedometer).

Loosely speaking, you can think of a vector as simply an arrow, but recognize that two arrows are considered equal, from a vector viewpoint, provided they are parallel, have the same directed sense, and the same magnitude.

Suppose you are sitting at a desk with a horizontal surface. How many vectors are there that are perpendicular to this surface, are directed upward, and have magnitude of three inches? *Only one.* There are an infinite number of directed line segments with these properties, but they are *identical* as vectors.

1.2 ADDITION AND SUBTRACTION

The *sum* $\mathbf{A} + \mathbf{B}$ of two vectors may be defined in the following way. Let the vectors be represented so that the terminal point, or tip, of $\mathbf{A}$ coincides with the initial point, or tail, of $\mathbf{B}$. Then $\mathbf{A} + \mathbf{B}$ is represented by the arrow extending from the tail of $\mathbf{A}$ to the tip of $\mathbf{B}$ (Fig. 1.2). It is evident that this definition of addition is compatible with the notion of equivalence; that is, if $\mathbf{A} = \mathbf{A}'$ and $\mathbf{B} = \mathbf{B}'$, then $\mathbf{A} + \mathbf{B} = \mathbf{A}' + \mathbf{B}'$. It is also commutative,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Figure 1.2 demonstrates that this statement is another way of saying that opposite sides of a parallelogram are equal and parallel.

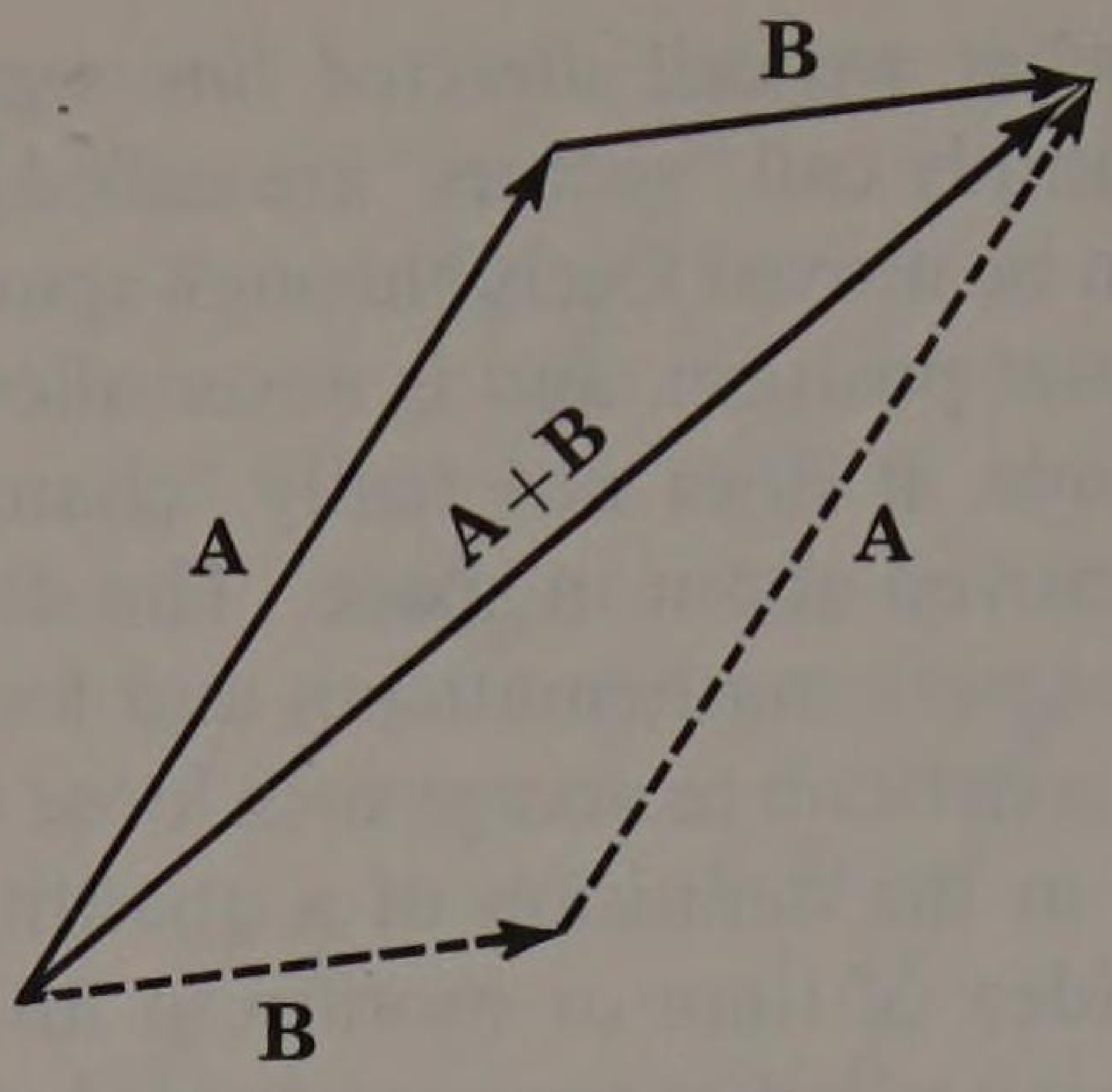

FIGURE 1.2

From Fig. 1.3 we see that vector addition is associative,

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

so that no ambiguity results from writing **A** + **B** + **C** without parentheses.

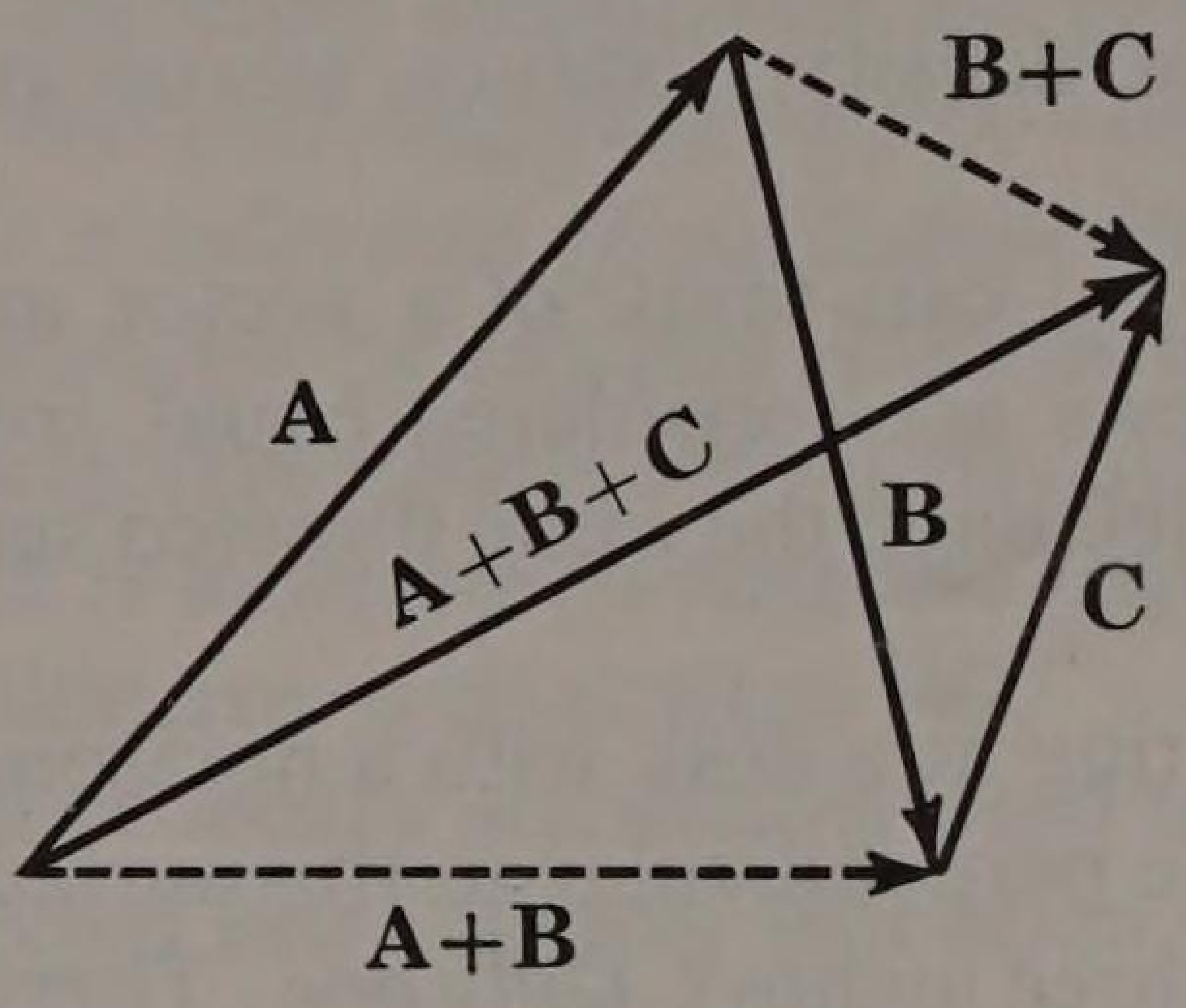

FIGURE 1.3

If **B** is a vector, −**B** is defined to be the vector with the same magnitude as **B** but opposite direction (Fig. 1.4). Subtraction of vectors is defined by adding the negative,

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

The student who ignores this definition and simply memorizes Fig. 1.4 will inevitably confuse **A** − **B** with **B** − **A**, which has the opposite direction. A good way of avoiding confusion is to keep in mind that **A** − **B** is, algebraically, the vector that must be added to **B** to produce **A**; hence it runs from the tip of **B** to the tip of **A**, when **A** and **B** share a common tail.

The above definitions apply to the vector **0** if it is represented by a degenerate line segment. We have $\mathbf{0} = -\mathbf{0}$, $\mathbf{A} - \mathbf{A} = \mathbf{0}$, $\mathbf{A} + \mathbf{0} = \mathbf{A}$, $\mathbf{0} + \mathbf{A} = \mathbf{A}$, for every vector **A**.

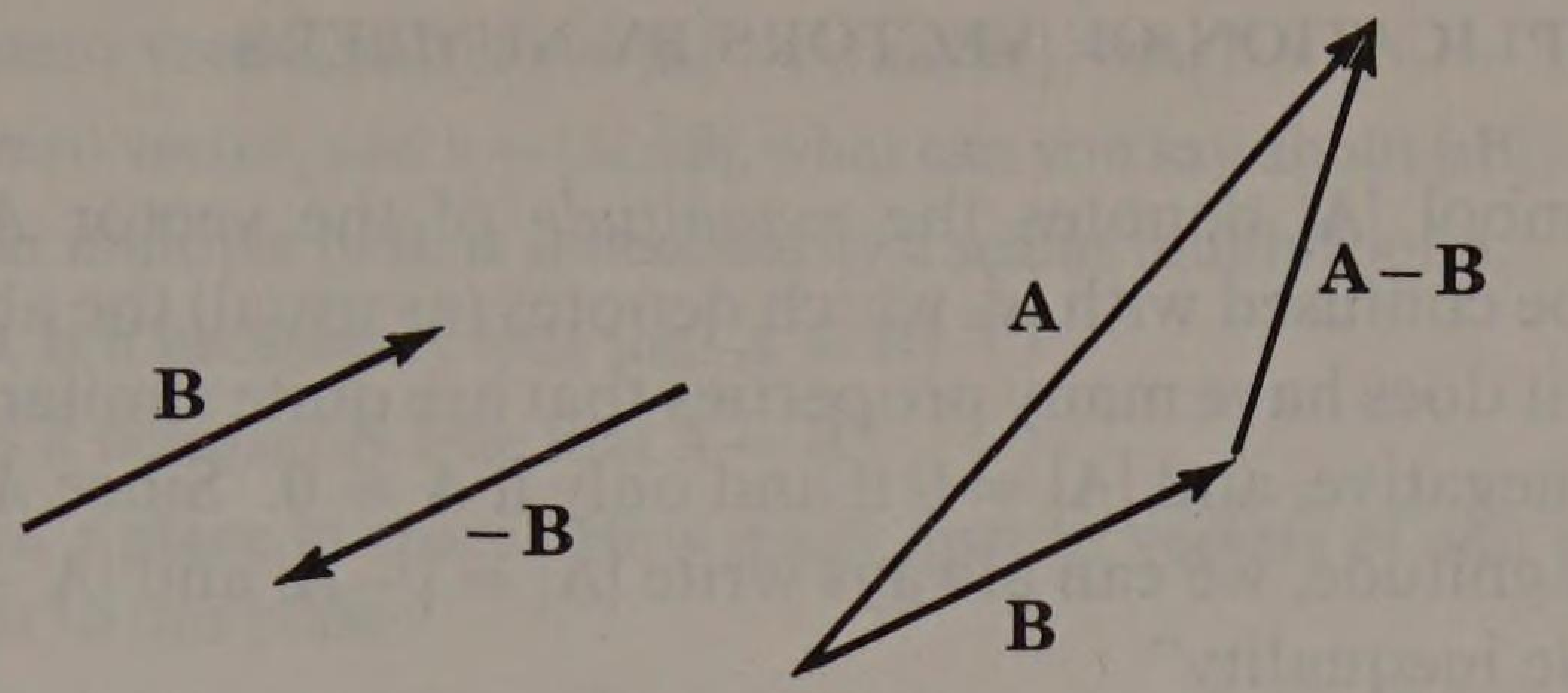

FIGURE 1.4

Exercises

1. If **A** and **B** are represented by arrows whose initial points coincide, what arrow represents **A** + **B**?
2. By drawing a diagram, show that if **A** + **B** = **C**, then **B** = **C** − **A**.
3. Is the following statement correct? If **A**, **B**, **C**, and **D** are nonzero vectors represented by arrows from the origin to the points *A*, *B*, *C*, *D*, and if **B** − **A** = **C** − **D**, then *ABCD* is a parallelogram.
4. Let the sides of a regular hexagon be drawn as arrows, with the terminal point of each arrow at the initial point of the next.
   (a) If **A** and **B** are vectors represented by consecutive sides, find the other four vectors in terms of **A** and **B**.
   (b) What is the vector sum of all six vectors?

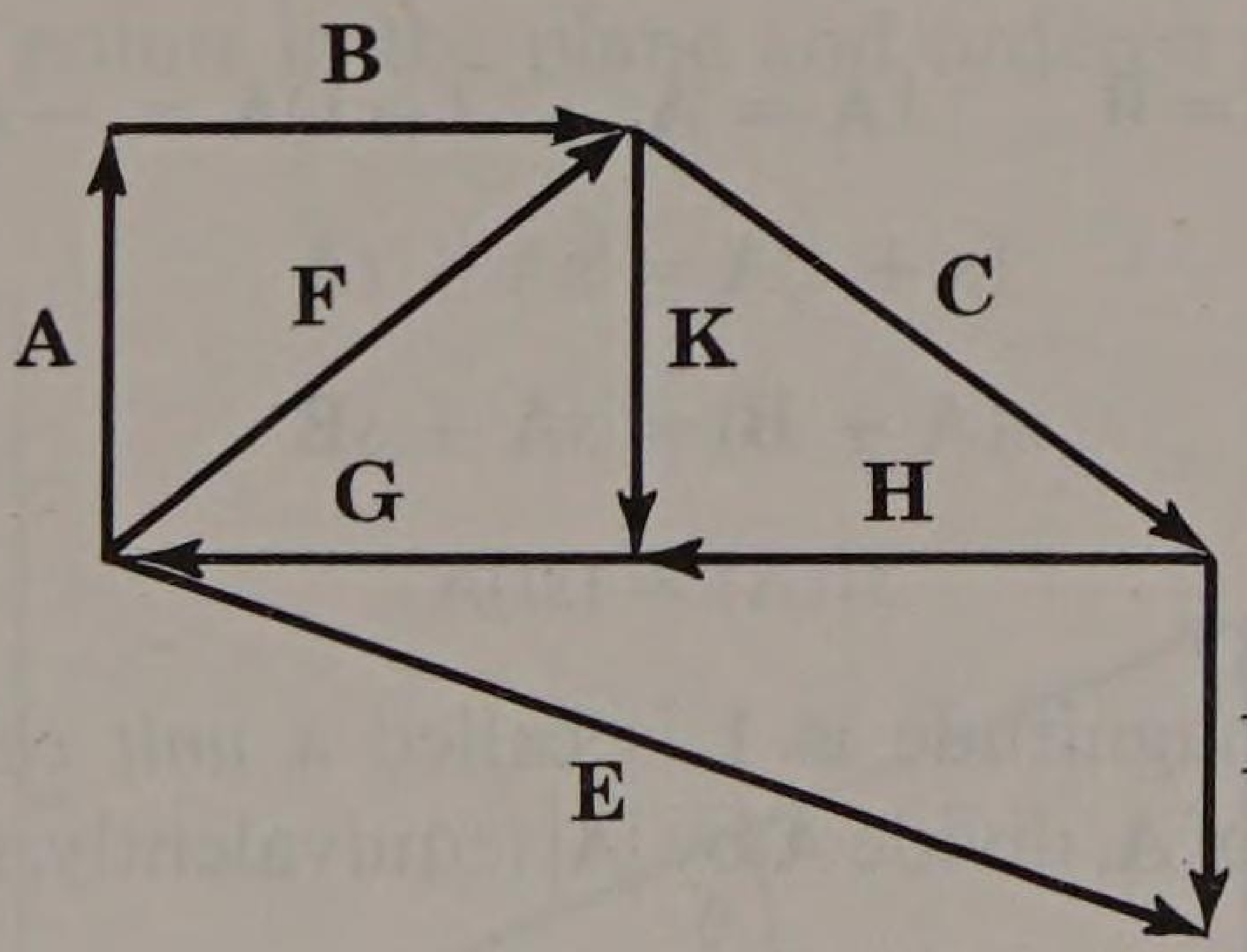

FIGURE 1.5

The following problems refer to Fig. 1.5:

5. Write **C** in terms of **E**, **D**, **F**.
6. Write **G** in terms of **C**, **D**, **E**, **K**.
7. Solve for **x**: **x** + **B** = **F**.
8. Solve for **x**: **x** + **H** = **D** − **E**.

1.3 MULTIPLICATION OF VECTORS BY NUMBERS

The symbol $|\mathbf{A}|$ denotes the *magnitude* of the vector **A**. Although it should not be confused with $|s|$, which denotes (as usual) the absolute value of a number s, it does have many properties that are quite similar. For example, $|\mathbf{A}|$ is never negative, and $|\mathbf{A}| = 0$ if and only if $\mathbf{A} = \mathbf{0}$. Since **A** and $-\mathbf{A}$ have the same magnitude, we can always write $|\mathbf{A}| = |-\mathbf{A}|$ and $|\mathbf{A} - \mathbf{B}| = |\mathbf{B} - \mathbf{A}|$. The "triangle inequality"

$$|\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|$$

is the vector expression of the fact that any side of a triangle does not exceed, in length, the sum of the lengths of the other two sides (Fig. 1.2).

If s is a number and **A** is a vector, $s\mathbf{A}$ is defined to be the vector having magnitude $|s|$ times that of **A** and pointing in the same direction if s is positive or in the opposite direction if s is negative. Any vector $s\mathbf{A}$ is called a *scalar multiple* of **A** (Fig. 1.6).

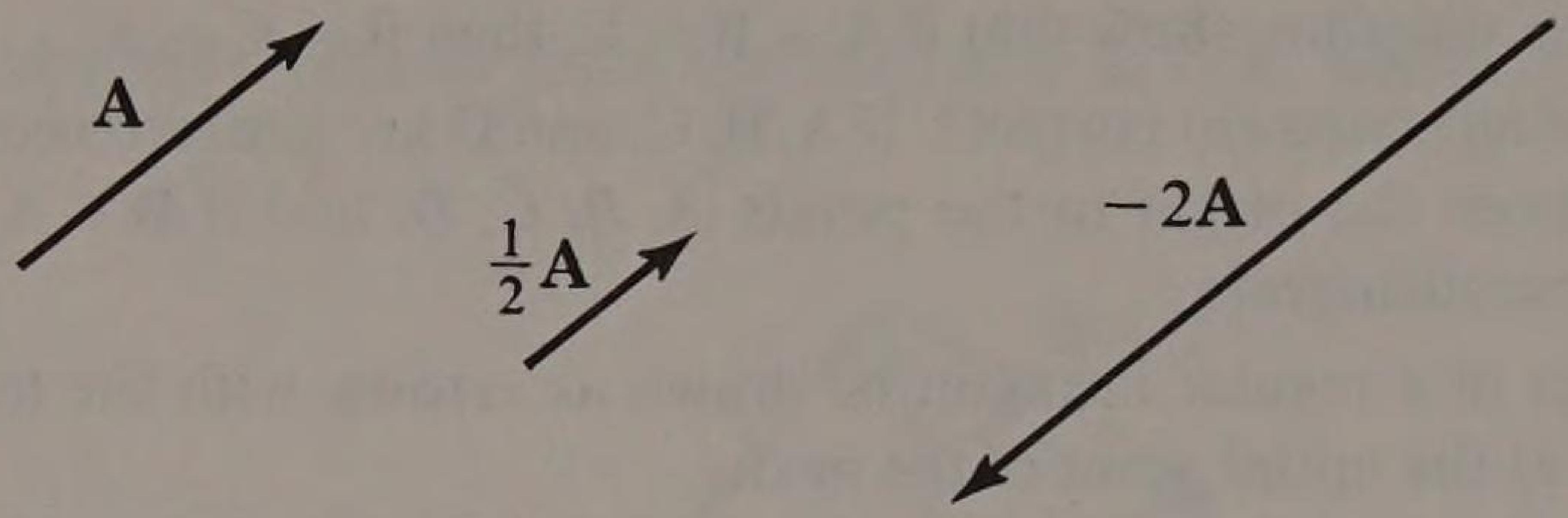

FIGURE 1.6

Here are the fundamental properties of the operation of multiplying vectors by numbers:

$$0\mathbf{A} = \mathbf{0} \qquad 1\mathbf{A} = \mathbf{A} \qquad (-1)\mathbf{A} = -\mathbf{A} \tag{1.1}$$

$$(s + t)\mathbf{A} = s\mathbf{A} + t\mathbf{A} \tag{1.2}$$

$$s(\mathbf{A} + \mathbf{B}) = s\mathbf{A} + s\mathbf{B} \tag{1.3}$$

$$s(t\mathbf{A}) = (st)\mathbf{A} \tag{1.4}$$

A vector whose magnitude is 1 is called a *unit vector*. To get a unit vector in the direction of **A**, divide **A** by $|\mathbf{A}|$ (equivalently, multiply **A** by $|\mathbf{A}|^{-1}$),

$$\left|\frac{\mathbf{A}}{|\mathbf{A}|}\right| = \frac{|\mathbf{A}|}{|\mathbf{A}|} = 1$$

Exercises

1. Is it ever possible to have $|\mathbf{A}| < 0$?
2. If $|\mathbf{A}| = 3$, what is $|4\mathbf{A}|$? What is $|-2\mathbf{A}|$? What can you say about $|s\mathbf{A}|$ if you know that $-2 \leq s \leq 1$?

3. If $\mathbf{A}$ is a nonzero vector, and if $s = |\mathbf{A}|^{-1}$, what is $|-s\mathbf{A}|$?
4. If $\mathbf{B}$ is a nonzero vector, and $s = |\mathbf{A}|/|\mathbf{B}|$, what can you say about $|s\mathbf{B}|$?
5. If $\mathbf{A}$ is a scalar multiple of $\mathbf{B}$, is $\mathbf{B}$ necessarily a scalar multiple of $\mathbf{A}$?
6. If $\mathbf{A} - \mathbf{B} = \mathbf{0}$, is it necessarily true that $\mathbf{A} = \mathbf{B}$?
7. If $|\mathbf{A}| = |\mathbf{B}|$, is it necessarily true that $\mathbf{A} = \mathbf{B}$?
8. You are given a plane in space. How many distinct vectors of unit magnitude are perpendicular to this plane?
9. How many distinct vectors exist, all having unit magnitude, perpendicular to a given line in space?
10. If $\mathbf{A}$ is a nonzero vector, how many distinct scalar multiples of $\mathbf{A}$ will have unit magnitude?
11. Let $\mathbf{A}$ and $\mathbf{B}$ be nonzero vectors represented by arrows with the same initial point to points A and B respectively. Let $\mathbf{C}$ denote the vector represented by an arrow from this same initial point to the midpoint of the line segment AB. Write $\mathbf{C}$ in terms of $\mathbf{A}$ and $\mathbf{B}$.
12. Prove that $|\mathbf{A} - \mathbf{B}| \geq |\mathbf{A}| - |\mathbf{B}|$.
13. Find nonzero scalars a, b, and c such that $a\mathbf{A} + b(\mathbf{A} - \mathbf{B}) + c(\mathbf{A} + \mathbf{B}) = \mathbf{0}$ for every pair of vectors $\mathbf{A}$ and $\mathbf{B}$.

1.4 CARTESIAN COORDINATES

Let us consider a cartesian coordinate system in the plane, obtained by introducing two mutually perpendicular axes, labeled x and y, with the same unit of length on both axes (Fig. 1.7). We assume that the reader is already familiar with this construction, which sets up a one-to-one correspondence between points in the plane and ordered pairs (x,y) of numbers.

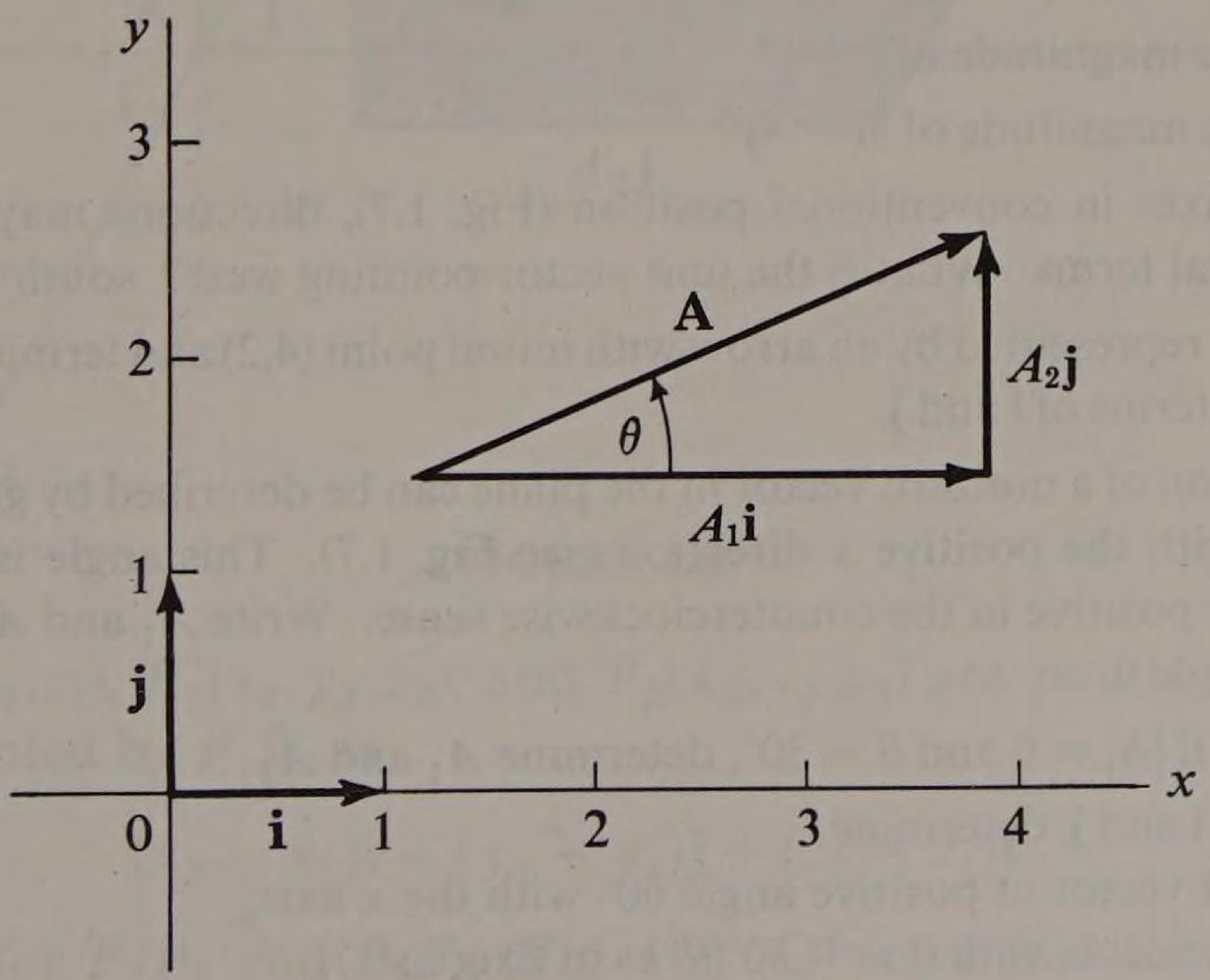

FIGURE 1.7

Let **i** denote the unit vector parallel to the x axis, in the positive x direction, and **j** the unit vector in the positive y direction. Every vector in the plane can be written uniquely in the form

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j}$$

for a suitable choice of numbers A_1 and A_2. These numbers are called the *components* of **A** in the x direction and y direction respectively; the component of a vector in a given direction is the orthogonal projection of the vector in that direction.

The magnitude of **A** can be determined from its components by using the pythagorean theorem (Fig. 1.7),

$$|\mathbf{A}| = \sqrt{A_1{}^2 + A_2{}^2}$$

To determine the components of a vector, *any* directed line segment representing the vector can be used. Thus, if $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are points in the xy plane, the vector represented by the directed line segment P_1P_2 (initial point P_1, terminal point P_2) is $(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$. Any other directed line segment equivalent to P_1P_2 would give the same components.

Example 1.1 The directed line segment extending from (4,6) to (7,11) is equivalent to the directed line segment extending from $(-1,3)$ to (2,8) because both of these directed line segments represent the vector $3\mathbf{i} + 5\mathbf{j}$.

Exercises

1. What is the x component of **i**?
2. What is the x component of **j**?
3. What is the magnitude of $\mathbf{i} + \mathbf{j}$?
4. What is the magnitude of $3\mathbf{i} - 4\mathbf{j}$?
5. With the axes in conventional position (Fig. 1.7), directions may be specified in geographical terms. What is the unit vector pointing west? south? northeast?
6. Vector **A** is represented by an arrow with initial point (4,2) and terminal point $(5,-1)$. Write **A** in terms of **i** and **j**.
7. The direction of a nonzero vector in the plane can be described by giving the angle θ it makes with the positive x direction (see Fig. 1.7). This angle is conventionally taken to be positive in the counterclockwise sense. Write A_1 and A_2 in terms of $|\mathbf{A}|$ and this angle θ.
8. In Fig. 1.7, if $|\mathbf{A}| = 6$ and $\theta = 30°$, determine A_1 and A_2.
9. In terms of **i** and **j**, determine:
   (a) the unit vector at positive angle 60° with the x axis;
   (b) the unit vector with $\theta = -30°$ (θ as in Exercise 7);
   (c) the unit vector having the same direction as $3\mathbf{i} + 4\mathbf{j}$;

(d) the unit vectors having x components equal to $\frac{1}{2}$;
(e) the unit vectors perpendicular to the line $x + y = 0$.

10. Determine $|6\mathbf{i} + 8\mathbf{j}|$, $|-3\mathbf{i}|$, $|\mathbf{i} + s\mathbf{j}|$, $|(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}|$.

11. In terms of **i** and **j**, determine the vector represented by the arrow extending from the origin to the midpoint of the line segment joining (1,4) with (3,8).

1.5 SPACE VECTORS

Throughout most of this book, we shall be concerned with vectors in three-dimensional space. By the introduction of three mutually perpendicular axes, with the same unit of length along all three axes, we obtain the usual cartesian coordinate system. The conventional orientation of axes is shown in Fig. 1.8. Every vector can be expressed in the form $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, where **i**, **j**, and **k** are unit vectors in the positive x, y, and z directions respectively. The numbers A_1, A_2, and A_3 are the *components*, or *orthogonal projections*, of **A** in the x, y, and z directions respectively.

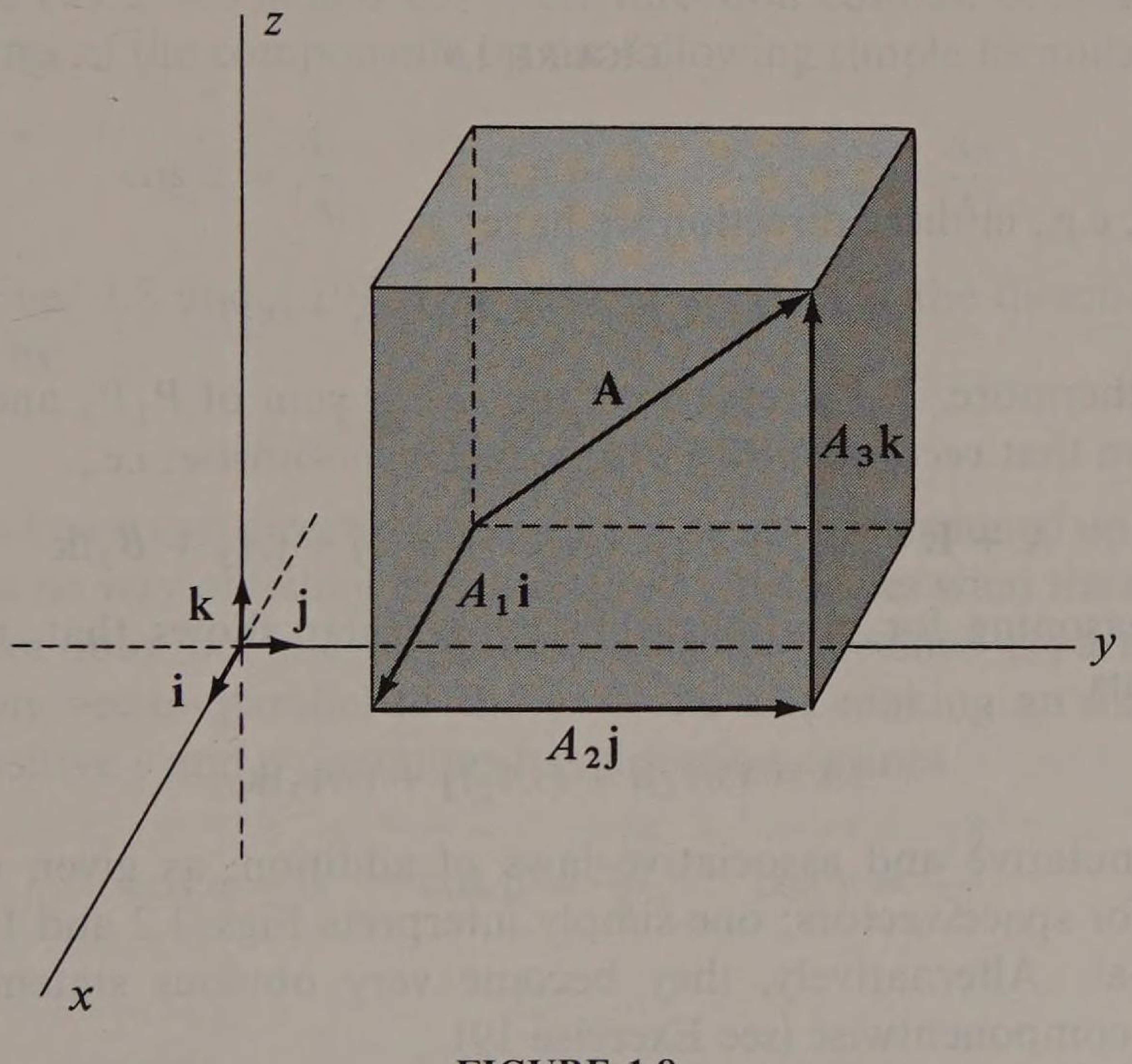

FIGURE 1.8

If $P_1(x_1,y_1,z_1)$, $P_2(x_2,y_2,z_2)$, and $P_3(x_3,y_3,z_3)$ are points in space, the vector represented by P_1P_2 is

$$(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

and similarly for P_2P_3 and P_1P_3 (Fig. 1.9). Observe that the components of P_1P_3 are given by the sums of the corresponding components of P_1P_2

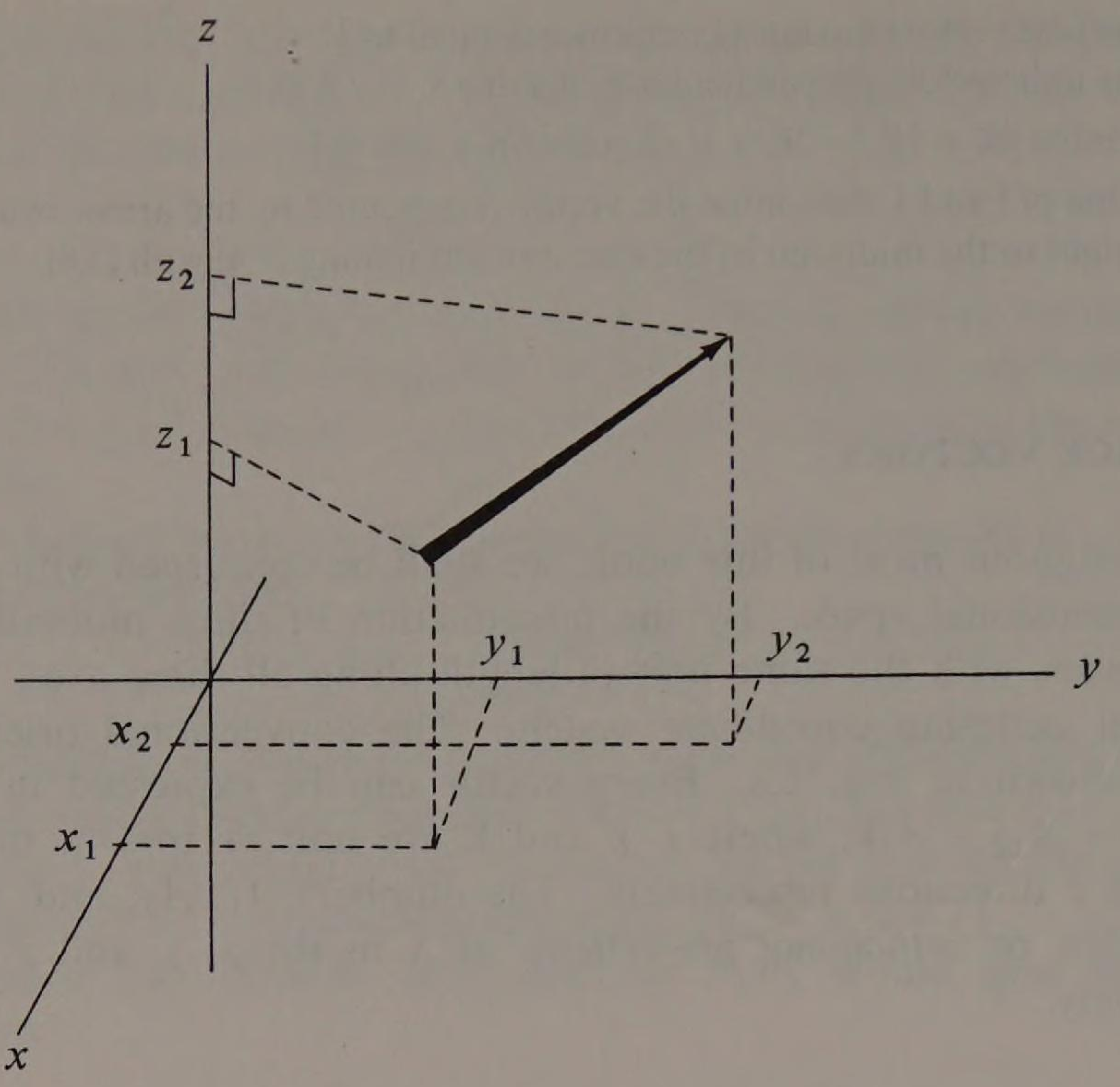

FIGURE 1.9

and P_2P_3; e.g., in the x direction we have

$$x_3 - x_1 = (x_2 - x_1) + (x_3 - x_2)$$

Since, furthermore, P_1P_3 represents the *vector* sum of P_1P_2 and P_2P_3, we have shown that *vector addition proceeds componentwise*; i.e.,

$$\mathbf{A} + \mathbf{B} = (A_1 + B_1)\mathbf{i} + (A_2 + B_2)\mathbf{j} + (A_3 + B_3)\mathbf{k}$$

Similar reasoning for multiplication by a scalar shows that, in terms of components,

$$s\mathbf{A} = (sA_1)\mathbf{i} + (sA_2)\mathbf{j} + (sA_3)\mathbf{k}$$

The commutative and associative laws of addition, as given in Sec. 1.2, are valid for space vectors; one simply interprets Figs. 1.2 and 1.3 as three-dimensional. Alternatively, they become very obvious statements when expressed componentwise (see Exercise 19).

By a double application of the pythagorean theorem, we obtain

$$|\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

An alternate description of a vector in space is obtained by giving its magnitude and direction. We can specify the direction by prescribing the three *direction angles* α, β, and γ between the vector and the positive x, y, and z directions respectively (see Fig. 1.10). Sometimes it is more convenient

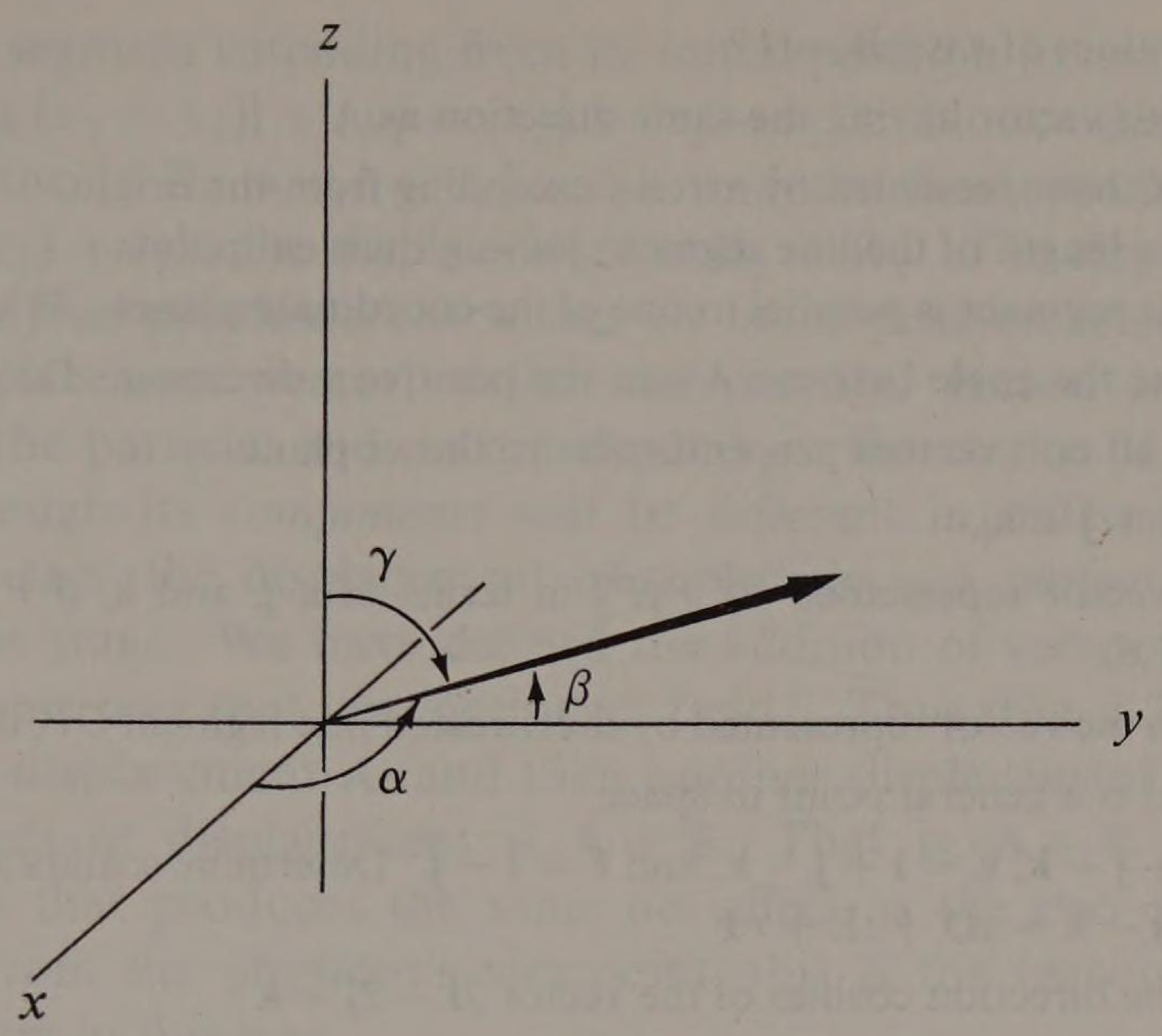

FIGURE 1.10

to prescribe cos α, cos β, and cos γ, the direction cosines, because they are given in terms of the components by the following simple formulae:

$$\cos\alpha = \frac{A_1}{|\mathbf{A}|} \qquad \cos\beta = \frac{A_2}{|\mathbf{A}|} \qquad \cos\gamma = \frac{A_3}{|\mathbf{A}|}$$

(compare Figs. 1.8 and 1.10). It is easy to verify that the direction cosines are related by

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

so that if we know two direction cosines, the third is determined up to its sign.

There is no way of telling from the direction cosines what the magnitude of the vector may be; the magnitude must be specified separately. For example, *any* vector parallel to the yz plane and making an angle of 45° with the positive y and z directions has direction cosines

$$\cos\alpha = 0 \qquad \cos\beta = \frac{\sqrt{2}}{2} \qquad \cos\gamma = \frac{\sqrt{2}}{2}$$

Exercises

In the first seven problems below, let $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{B} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, and $\mathbf{C} = 3\mathbf{i} - 4\mathbf{k}$.

1. Find $|\mathbf{A}|$, $|\mathbf{B}|$, and $|\mathbf{C}|$.

2. Find $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{C}$.

3. Determine $|\mathbf{A} - \mathbf{C}|$.

4. For what values of s is $|s\mathbf{B}| = 1$?

5. Find the unit vector having the same direction as **A**.

6. Let **A** and **C** be represented by arrows extending from the origin.
   (a) Find the length of the line segment joining their endpoints.
   (b) This line segment is parallel to one of the coordinate planes. Which one?

7. Let α denote the angle between **A** and the positive x direction. Determine $\cos\alpha$.

8. Determine all unit vectors perpendicular to the xz plane.

9. Compute $|\mathbf{i} + \mathbf{j} + \mathbf{k}|$.

10. Write the vector represented by P_1P_2 in terms of **i**, **j**, and **k**, if $P_1 = (3,4,7)$ and $P_2 = (4,-1,6)$.

11. Write down the vector represented by the directed line segment OP, if O is the origin and $P(x,y,z)$ is a general point in space.

12. Let $\mathbf{D} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{E} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, and $\mathbf{F} = \mathbf{i} - \mathbf{j}$. Determine scalars s, t, and r, such that $4\mathbf{i} + 6\mathbf{j} - \mathbf{k} = s\mathbf{D} + t\mathbf{E} + r\mathbf{F}$.

13. What are the direction cosines of the vector $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$?

14. Derive the identity $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

15. Give a geometrical description of the locus of all points P for which OP represents a vector with direction cosine $\cos\alpha = \frac{1}{2}$ (O is the origin).

16. How many unit vectors are there for which $\cos\alpha = \frac{1}{2}$ and also $\cos\beta = \frac{1}{2}$? Illustrate with a diagram.

17. **A** is a vector with direction cosines $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ respectively. What are the direction cosines of the reflected image of **A** in the yz plane? (Think of the yz plane as a mirror.)

18. Determine all unit vectors for which $\cos\alpha = \cos\beta = \cos\gamma$.

19. Verify the commutative and associative laws of addition for space vectors by expressing them componentwise.

1.6 DIGRESSION

A first step in solving some problems in mechanics is to choose a coordinate system. For instance, if the problem involves a particle sliding down an inclined plane, it may be convenient to take one of the axes, say the x axis, parallel to the plane, and another axis, say the z axis, perpendicular to the plane. After we have chosen a particular coordinate system, we can speak of the *position vector* of the particle. This is the vector represented by the directed line segment extending from the origin (0,0,0) to the point (x,y,z) where the particle is located, and (in terms of **i**, **j**, and **k**) it is the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Strictly speaking, we should not say "position vector of a particle" because this might give the false impression that it is an intrinsic property of the particle, whereas it also depends on the location of the origin of the coordinate system.

If a particle moves from an initial position (x_1,y_1,z_1) to another position (x_2,y_2,z_2), the *displacement* of the particle is the vector represented by the directed line segment extending from its initial position to its final position. This vector is $(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$. Notice that if the initial position vector is $\mathbf{R}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and the final position vector is $\mathbf{R}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$, the displacement is $\mathbf{R}_2 - \mathbf{R}_1$. *The displacement of a particle is the final position vector minus the initial position vector.*

The displacement vector, unlike the position vector, *is* an intrinsic property of the particle; it does not depend on the choice of a coordinate system (although its *components* will be different in different coordinate systems). In fact, the displacement of a particle is a perfect model for a vector, at this stage. We have defined the addition of vectors so that they add in the same way that displacements "add." Thus (Fig. 1.2) if a particle undergoes a displacement **A**, and then another displacement **B**, it is clear that the resultant displacement is $\mathbf{A} + \mathbf{B}$. That is, $\mathbf{A} + \mathbf{B}$ is the single displacement that produces the same net effect as the two displacements **A** and **B**. From the physicist's viewpoint, this is the reason for defining vector addition in this way.

Occasionally it is helpful to think of vectors as representing displacements, even when no physics is involved. For example, consider Exercise 5 of Sec. 1.2, where we are asked to write **C** in terms of **E**, **D**, and **F**. The answer is $\mathbf{C} = -\mathbf{F} + \mathbf{E} - \mathbf{D}$, which is clear since the net result of the three displacements $-\mathbf{F}$, **E**, and $-\mathbf{D}$ is **C**, as one can see by looking at Fig. 1.5.

Do not get the mistaken impression that when we represent a displacement by a vector **A**, the path of the particle has necessarily been straight. The directed line segment representing a displacement extends directly from the initial position to the final position, but the particle itself may have gone by way of the North Pole!

Forces are also vector quantities. This may *seem* obvious since a force is conveniently represented geometrically by a directed line segment. It is *not* so obvious, however. How do we know that forces "add" in the same way as vectors? We shall simply take the word of the physicists that they do, and direct the interested reader to the laboratory. If $\mathbf{F}_1$ and $\mathbf{F}_2$ are forces acting on a particle, their vector sum $\mathbf{F}_1 + \mathbf{F}_2$ is the single force that would produce the same effect, and it is sometimes called the *resultant* of the two forces. In elementary physics the resultant of two or more forces is usually found in the following manner: one draws a diagram showing the forces, then systematically marks out each force, replacing it by its components along the coordinate axes. The forces along each axis are summed algebraically, so that one has a single force remaining along each of the coordinate axes. The magnitude of the resultant force **F** can then be found by the pythagorean theorem, since the axes are perpendicular. This is discussed in every introductory physics book. Obviously, the process is equivalent to writing each force in terms of **i**, **j**, and **k**, and adding them in the manner of the preceding section.

It is rather surprising that *rotations* in space are *not* vector quantities. Clearly, a rotation can be represented by a directed line segment; the *direction* would be the axis of rotation, and the *length* would be the angle through which the body is rotated. But the result of two successive rotations is not represented by the vector sum of these line segments. In fact, the "sum" of two rotations is not even commutative! A body rotated through 90° about, first, the x direction, then the y direction, will achieve a final position quite different from the one resulting from the rotations performed in the other order (try this with the textbook). In this light, it is even more remarkable that *angular velocity* is, nonetheless, a vector quantity. This matter is discussed in Appendix C.

Exercises

1. A particle moves from (3,7,8) to (5,2,0). Write its displacement in terms of **i**, **j**, and **k**.
2. Write down the position vector of a particle located at the point (1,2,9).
3. The position vector of a moving particle at time t is $\mathbf{R} = 3\mathbf{i} + 4t^2\mathbf{j} - t^3\mathbf{k}$. Find its displacement during the time interval from $t = 1$ to $t = 3$.
4. What is the *magnitude* of the resultant of the following two displacements: 6 miles east, 8 miles north?
5. Strings are tied to a small metal ring and, by an arrangement of pulleys and weights, four forces are exerted on the ring. One force is directed upward with magnitude 3 lb, another is directed east with magnitude 6 lb, and a third is directed north with magnitude 2 lb. The ring is in equilibrium (i.e., it is not moving). What is the magnitude of the fourth force that is counterbalancing the other three?
6. The *center of mass* of a system of n particles is defined by the position vector

$$\mathbf{R}_{\text{cm}} = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2 + \cdots + m_n\mathbf{R}_n}{m_1 + m_2 + \cdots + m_n}$$

where the ith particle is located at $\mathbf{R}_i$ and has mass m_i. The *mass unbalance* of the system, measured at the position $\mathbf{R}$, is defined to be

$$m_1(\mathbf{R}_1 - \mathbf{R}) + m_2(\mathbf{R}_2 - \mathbf{R}) + \cdots + m_n(\mathbf{R}_n - \mathbf{R})$$

Show that the mass unbalance, measured at the center of mass, is zero.

1.7 SOME PROBLEMS IN GEOMETRY

To avoid circumlocution, practically everybody who works with vectors makes no distinction between vectors and directed line segments. It is easier to say "the vector **A**" than to say "the vector represented by the directed line segment **A**." When we do this, it is still important to recognize that the concept of a vector is an *abstraction* from the concept of a directed line segment, in which we ignore the actual location of the directed line segment: we say "$\mathbf{A} = \mathbf{B}$" when we really mean "the directed line segments

A and **B** are equivalent and therefore represent the same vector." If **A** extends from (2,3,4) to (2,3,5), and if **B** extends from (3, –2,8) to (3, –2,9), then we have as vectors $\mathbf{A} = \mathbf{B}$ even though they extend from different points.

What we are saying is that two things are equal when they are really not identical but are only "equivalent" according to some definition. We are already familiar with this in elementary arithmetic. We say the fractions $\frac{2}{3}$ and $\frac{4}{6}$ are "equal" when in fact they are not identical but are only "equivalent" in a certain way. Strictly speaking, we should say that $\frac{2}{3}$ and $\frac{4}{6}$ are fractions that represent the same rational number: as fractions, they are not equal, but they represent the same rational number.

Similarly, if we have two directed line segments **A** and **B**, we may write $\mathbf{A} = \mathbf{B}$ even when the directed line segments are not equal (because they extend from different points) but are equivalent according to the definition given in Sec. 1.1.

With this in mind, we now turn to the practical utility of vector algebra. The simplest applications are in geometry, and will be considered first.

Example 1.2 If the midpoints of the consecutive sides of a quadrilateral are joined by line segments, is the resulting quadrilateral a parallelogram?

Let $PQRS$ be the quadrilateral and $TUVW$ the midpoints of its sides. In the case shown in Fig. 1.11, it certainly appears that $TUVW$ is a parallelogram. Keep in mind, however, that $PQRS$ need not be a plane figure; perhaps S is a point several inches above the plane containing P, Q, and R. In view of this possibility, is $TUVW$ a parallelogram?

Solution Let the sides be made into directed line segments **A**, **B**, **C**, and **D**, as shown in Fig. 1.11. Then one very obvious relationship is

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = \mathbf{0}$$

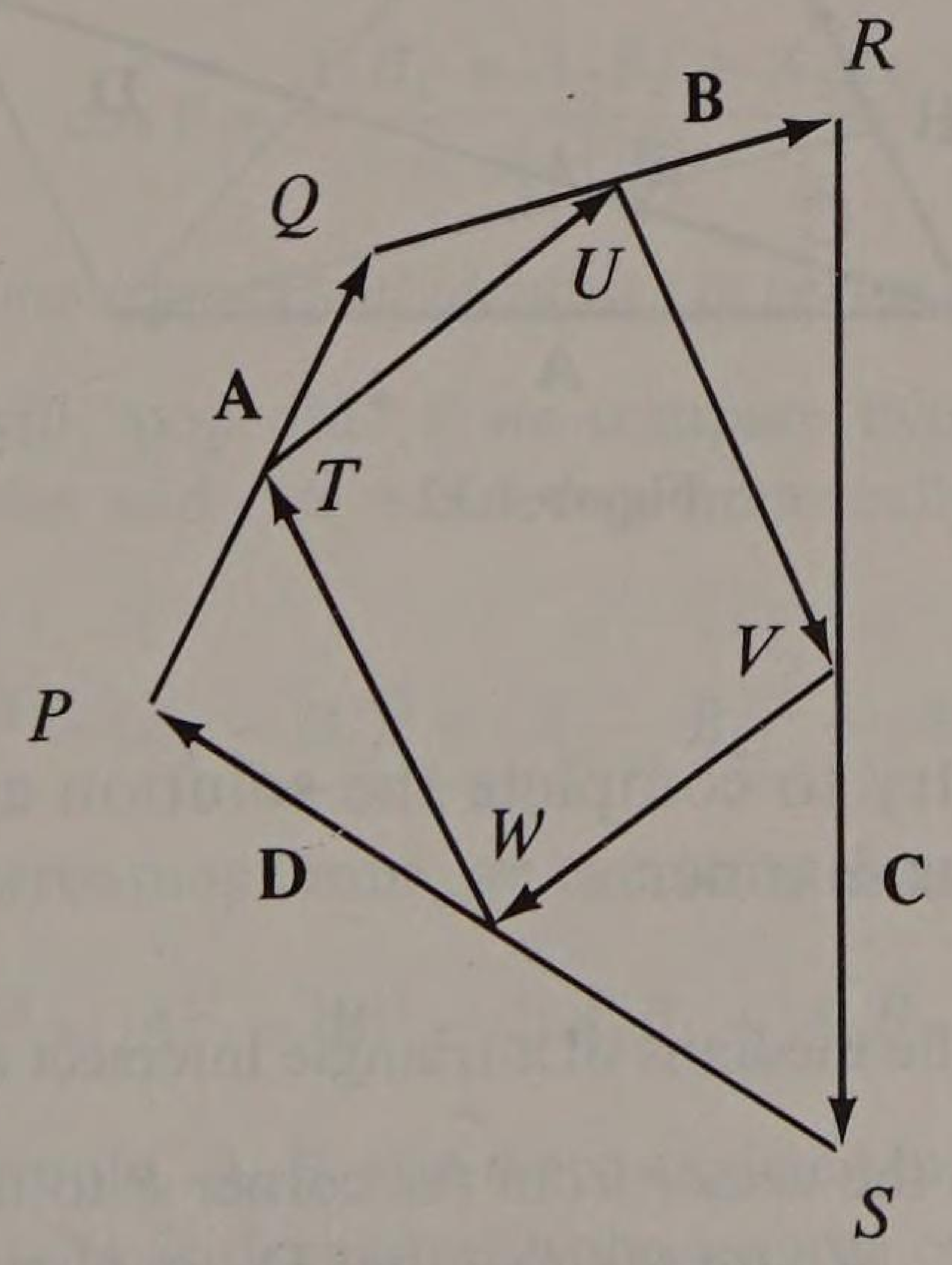

FIGURE 1.11

To conclude that $TUVW$ is a parallelogram, we need to show that $TU = -VW$. From the figure, TU may be expressible in terms of **A** and **B**; in fact, TU equals the "tip half" of **A** plus the "tail half" of **B**. Thus

$$TU = \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{B} = \tfrac{1}{2}(\mathbf{A} + \mathbf{B})$$

Similarly

$$VW = \tfrac{1}{2}(\mathbf{C} + \mathbf{D})$$

But our basic relationship shows that $\mathbf{A} + \mathbf{B} = -(\mathbf{C} + \mathbf{D})$. Thus $TU = -VW$.

Example 1.3 Line segments are drawn from a vertex of a parallelogram to the midpoints of the opposite sides. Show that they trisect a diagonal.

Solution We have diagrammed the situation in Fig. 1.12, labeling certain vectors for convenience. Since the diagonal is $\mathbf{A} + \mathbf{B}$, the problem reduces to showing $\mathbf{C} = \mathbf{D} = \tfrac{1}{3}(\mathbf{A} + \mathbf{B})$. Let us try to express **C** in terms of **A** and **B**. First of all, certainly $\mathbf{C} = s(\mathbf{A} + \mathbf{B})$ for some scalar s. Also, since the tip of **C** lies on the line connecting the tip of **A** to the tip of $\tfrac{1}{2}\mathbf{B}$, we have $\mathbf{C} - \mathbf{A} = t(\tfrac{1}{2}\mathbf{B} - \mathbf{A})$, for some scalar t. If we equate the two expressions for **C**,

$$s(\mathbf{A} + \mathbf{B}) = \mathbf{A} + t(\tfrac{1}{2}\mathbf{B} - \mathbf{A})$$

we derive

$$(s - \tfrac{1}{2}t)\mathbf{B} = (1 - s - t)\mathbf{A}$$

Since **A** and **B** are not parallel, this equation can only be true if the scalars are zero.

$$s - \tfrac{1}{2}t = 0$$

$$1 - s - t = 0$$

Solving, we obtain $s = \tfrac{1}{3}$, so $\mathbf{C} = \tfrac{1}{3}(\mathbf{A} + \mathbf{B})$.

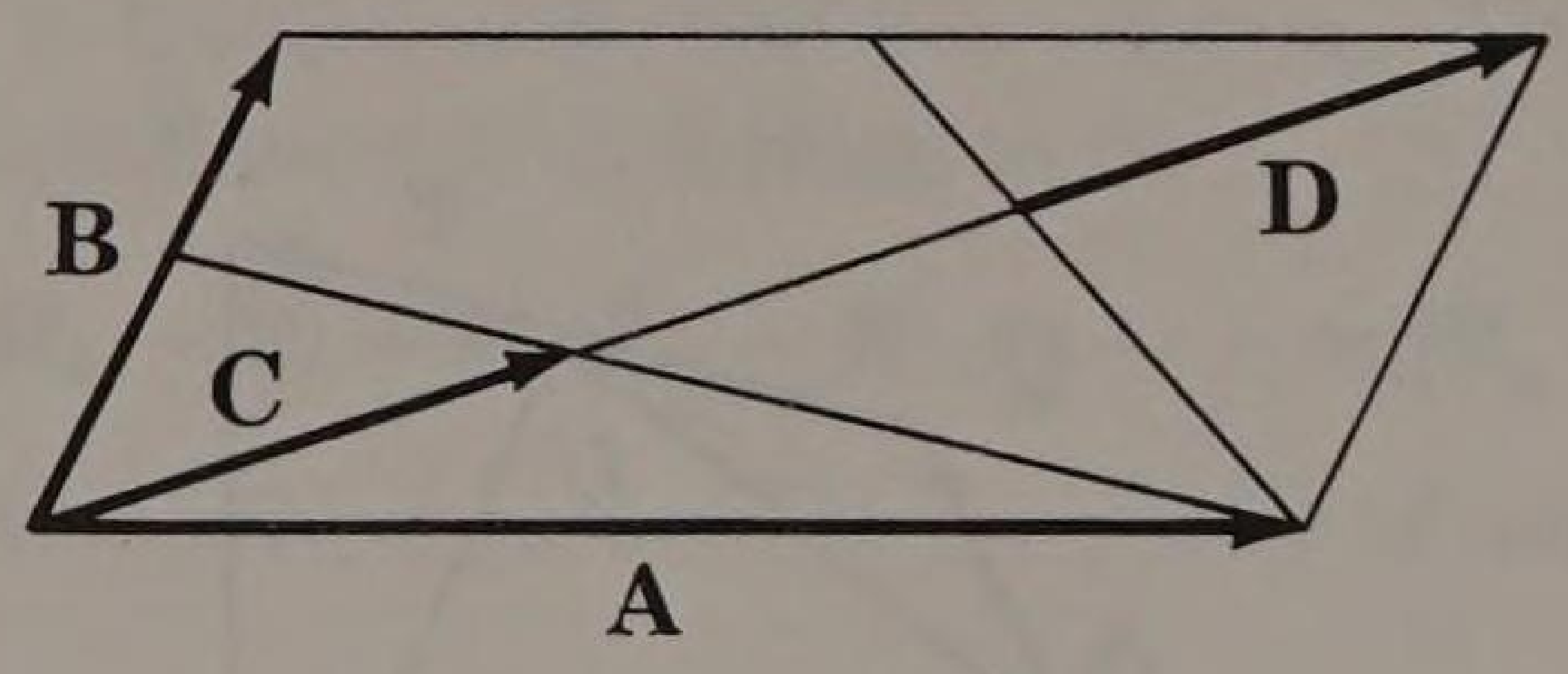

Figure 1.12

The reader should try to complete the solution as an exercise, manipulating **D** in an analogous manner.

Example 1.4 Prove that the medians of a triangle intersect at a single point.

Solution In Fig. 1.13, **D** is the vector from the corner P to the point of intersection of the medians from Q and R. We must show that **D** lies along the median from P, i.e.,

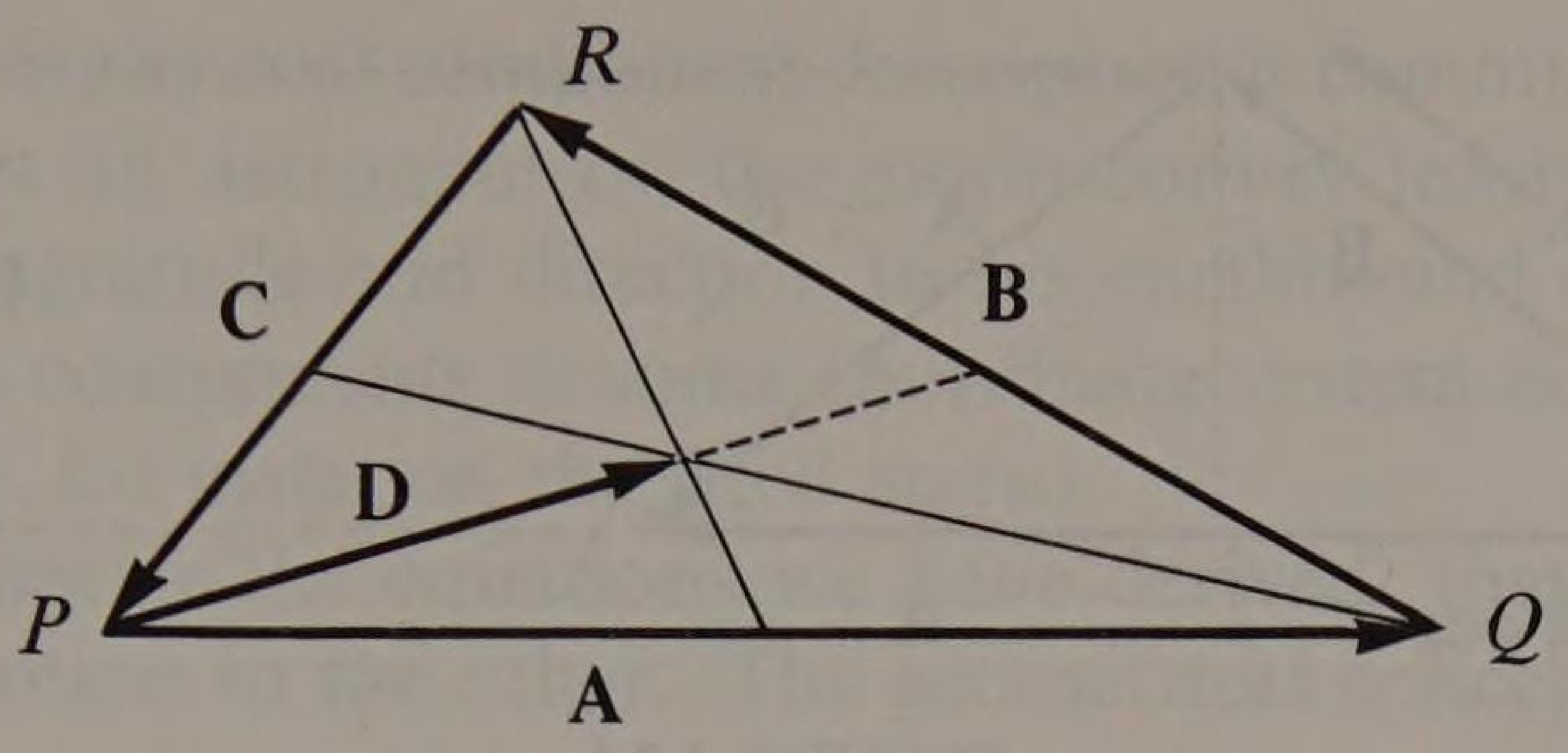

FIGURE 1.13

that it is a multiple of $\mathbf{A} + \frac{1}{2}\mathbf{B}$. The condition that **D** lie along the median from R is expressed as

$$\mathbf{C} + \mathbf{D} = s(\mathbf{C} + \tfrac{1}{2}\mathbf{A})$$

for some number s, while that fact that **D** lies on the median from Q implies that, for some number t,

$$\mathbf{A} - \mathbf{D} = t(\tfrac{1}{2}\mathbf{C} + \mathbf{A})$$

Solving for **D** and equating the expressions, we derive

$$(s + \tfrac{1}{2}t - 1)\mathbf{C} = (1 - t - \tfrac{1}{2}s)\mathbf{A}$$

As in Example 1.3, we conclude that both coefficients must vanish; thus

$$s = t = \tfrac{2}{3}$$

Using this in either equation for **D** and writing **C** in terms of **A** and **B**, we find

$$\mathbf{D} = \tfrac{2}{3}(\mathbf{A} + \tfrac{1}{2}\mathbf{B})$$

which is the form that we sought.

Example 1.5 Let θ denote the angle between two nonzero vectors **A** and **B**. Show that

$$\cos\theta = \frac{A_1B_1 + A_2B_2 + A_3B_3}{|\mathbf{A}|\,|\mathbf{B}|} \tag{1.5}$$

NOTE: *This is one of the most important identities in vector algebra.*

Solution This formula will "pop out" if we compare two expressions for $|\mathbf{A} - \mathbf{B}|^2$, one derived componentwise and one derived geometrically. Using components, we know that

$$|\mathbf{A} - \mathbf{B}|^2 = (A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2$$

Expanding powers and regrouping terms, we can write this as

$$|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2(A_1B_1 + A_2B_2 + A_3B_3)$$

Now for the geometric formula. **A**, **B**, and θ are depicted in Fig. 1.14; also, the perpendicular from the tip of **B** to **A** is drawn, with the lengths of the appropriate segments

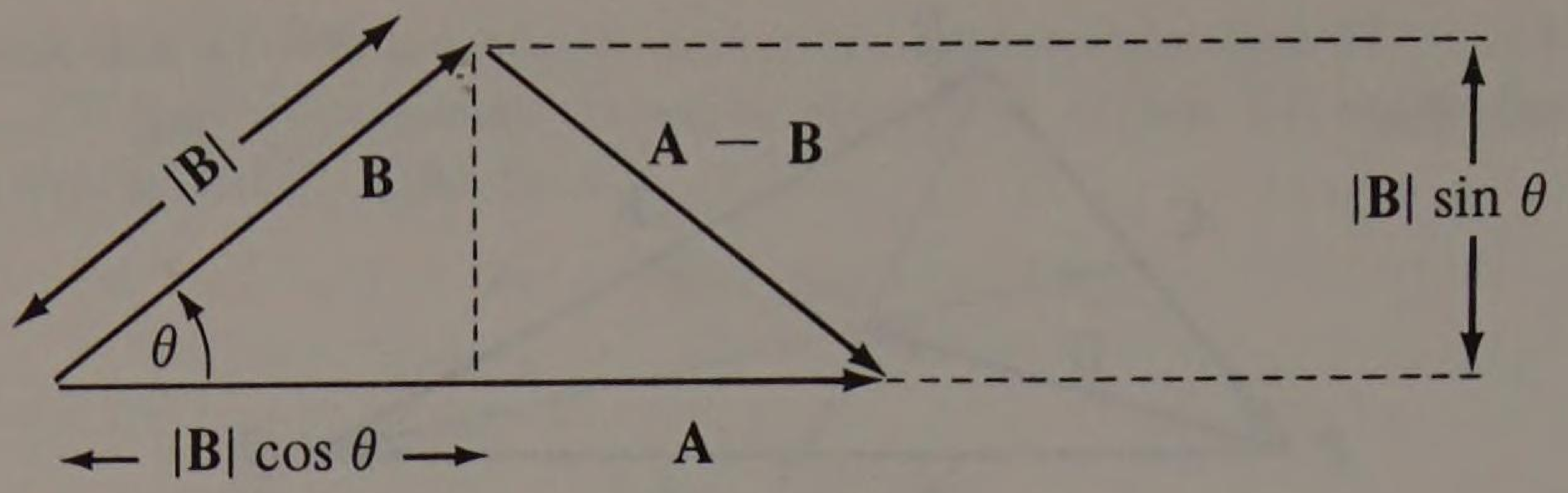

FIGURE 1.14

indicated. We can visualize **A** − **B** as the hypotenuse of a right triangle, and according to Pythagoras

$$\begin{aligned}|\mathbf{A}-\mathbf{B}|^2 &= (|\mathbf{B}|\sin\theta)^2 + (|\mathbf{A}| - |\mathbf{B}|\cos\theta)^2\\ &= |\mathbf{B}|^2(\sin^2\theta + \cos^2\theta) + |\mathbf{A}|^2 - 2|\mathbf{A}|\,|\mathbf{B}|\cos\theta\end{aligned}$$

$$|\mathbf{A}-\mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}|\,|\mathbf{B}|\cos\theta \tag{1.6}$$

Comparing this with the componentwise expression, we conclude

$$|\mathbf{A}|\,|\mathbf{B}|\cos\theta = A_1B_1 + A_2B_2 + A_3B_3 \tag{1.7}$$

which is equivalent to the desired identity.

Incidentally, by referring to Fig. 1.14 the alert reader will recognize Eq. (1.6) as the *law of cosines* from trigonometry.

As an application of this formula, consider the next example.

Example 1.6 Show that the vectors $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 7\mathbf{j} + \mathbf{k}$ are perpendicular.

Solution

$$\cos\theta = \frac{2 - 7 + 5}{\sqrt{30}\,\sqrt{51}} = 0$$

Hence $\theta = 90°$.

SUMMARY—GEOMETRIC AND COORDINATE DESCRIPTIONS

Now is a good time to catch our breath and get an overview of what we have learned. There are two ways of looking at vectors—geometrically, and componentwise. Geometric descriptions are more physical; a vector has magnitude and direction, and relationships are described in terms of lengths and angles. But it is often difficult to compute with these quantities, especially if the problem is three-dimensional and hard to sketch. Thus to solve such problems as finding the resultant of several forces, we introduce a cartesian coordinate system and represent all vectors by their components. Then a vector becomes an ordered triple of numbers. (Another reason for using this rather unphysical component description is communication. How, for instance, does an astronaut on the moon convey information about a quantity with magnitude and direction to his earthbound colleagues? He must describe its components in some coordinate system common to both, as determined by, for instance, the fixed stars.)

Let us summarize the equations we have derived; they tell us how to relate one description to the other. The geometrical concept of length of a vector is computed in terms of components by

$$|\mathbf{A}| = (A_1{}^2 + A_2{}^2 + A_3{}^2)^{1/2}$$

The angle θ between two vectors **A** and **B** is computed from components using Eq. (1.5)

$$\cos\theta = \frac{A_1B_1 + A_2B_2 + A_3B_3}{|\mathbf{A}|\,|\mathbf{B}|}$$

In particular, the direction cosines of **A**, which are the cosines of the angles between **A** and the positive coordinate axes, can be computed by substituting **i**, **j**, or **k** for **B** in the above; thus

$$\cos\alpha = \frac{A_1}{|\mathbf{A}|} \qquad \cos\beta = \frac{A_2}{|\mathbf{A}|} \qquad \cos\gamma = \frac{A_3}{|\mathbf{A}|}$$

Viewed another way, these equations can be used to compute the component description of a vector from its geometric characteristics; we have

$$A_1 = |\mathbf{A}|\cos\alpha \qquad A_2 = |\mathbf{A}|\cos\beta \qquad A_3 = |\mathbf{A}|\cos\gamma$$

Hence the cycle is complete and we are free to exploit whichever description, geometrical or componentwise, is most convenient. Exercises 6 through 10, which follow, illustrate these ideas.

Exercises

1. Imitate the solution of Example 1.2, except instead of proving that $TU = -VW$, prove that $UV = -WT$.
2. Using vector methods, prove directly that if two sides of a quadrilateral are parallel and equal in magnitude, the other two sides are also.
3. By vector methods, show that the line segment joining the midpoints of two sides of a triangle is parallel to the third side, and has length equal to one-half the length of the third side.
4. Show that the diagonals of a parallelogram bisect each other.
5. Construct another proof of the fact that the medians of a triangle intersect at a point, based on the following observation: if **D**, **E**, and **F** are vectors drawn from some fixed point to the corners of the triangle, then

$$\mathbf{D} + \tfrac{3}{2}\left[\tfrac{1}{3}(\mathbf{D} + \mathbf{E} + \mathbf{F}) - \mathbf{D}\right] = \tfrac{1}{2}(\mathbf{E} + \mathbf{F})$$

Verify this algebraically and then interpret it geometrically. [*Hint*: The tip of the vector $\frac{1}{3}(\mathbf{D}+\mathbf{E}+\mathbf{F})$ *is* this point of intersection.]

6. Find the angle between $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $3\mathbf{i}-4\mathbf{k}$.

7. Find the angle between the x axis and $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

8. Find the three angles of the triangle with vertices $(2,-1,1)$, $(1,-3,-5)$, $(3,-4,-4)$.

9. Find the angle between the xy plane and $2\mathbf{i}+2\mathbf{j}-\mathbf{k}$. (Note that $\mathbf{k}$ is perpendicular to the xy plane. You will have to decide what is meant by the angle between a vector and a plane.)

10. Show that $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is perpendicular to the plane $x+y+z=0$. (*Hint*: This plane passes through the origin. Show that $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is perpendicular to every vector extending from the origin to a point in the plane.)

The following simple exercises are inserted here to help you to recall some of the basic ideas of analytic geometry.

11. True or false: $3x-4y+5z=0$ represents a plane passing through the origin.

12. True or false: The yz plane is represented by the equation $x=0$.

13. True or false: The locus of points for which $x=3$ and $y=4$ is a line parallel to the z axis whose distance from the z axis is 5.

14. True or false: $x^2+y^2+z^2=9$ is the equation of a sphere centered at the origin having radius 9.

15. Write down the equation of a sphere centered at the point $(2,3,4)$ having radius 3.

16. Write down an equation for the cylinder concentric with the z axis having radius 2.

17. Do the equations $x=y=z$ represent a *line* or a *plane*?

18. What is the locus of points for which $x^2+z^2=0$?

19. What is the locus of points for which $(x-2)^2+(y+3)^2+(z-4)^2=0$?

20. What geometrical figure is represented by the equation $xyz=0$? (Keep in mind that a product of numbers is zero if, and only if, at least one of the numbers is zero.)

21. What is the distance between the points $(2,3,4)$ and $(5,3,8)$?

22. What is the distance between the point $(3,8,9)$ and the xz plane? (Distance in such cases always means *shortest distance* or *perpendicular distance*.)

23. What is the distance between the point $(0,3,0)$ and the cylinder $x^2+y^2=4$? (I doubt you will find a formula for this in any of your books. Just use some common sense.)

24. The expression x^2+y^2 gives the square of the distance between (x,y,z) and the z axis. In view of this, what figure is represented by $x^2+y^2=z^2$?

25. Do you know what figure is represented by the equation $(x/2)^2+(y/3)^2+(z/4)^2=1$? (If so, you know more analytic geometry than is required to read this book.)

1.8 EQUATIONS OF A LINE

The *position vector of a point* is the vector extending from the origin to the point. Thus the position vector of a point (x,y,z) is the vector $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. This correspondence between points and vectors is the fundamental means whereby problems in analytic geometry can be studied by vector methods.

As an elementary example, let us derive the equations of a line passing through a given point (x_0,y_0,z_0) and parallel to a given nonzero vector $\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ (see Fig. 1.15).

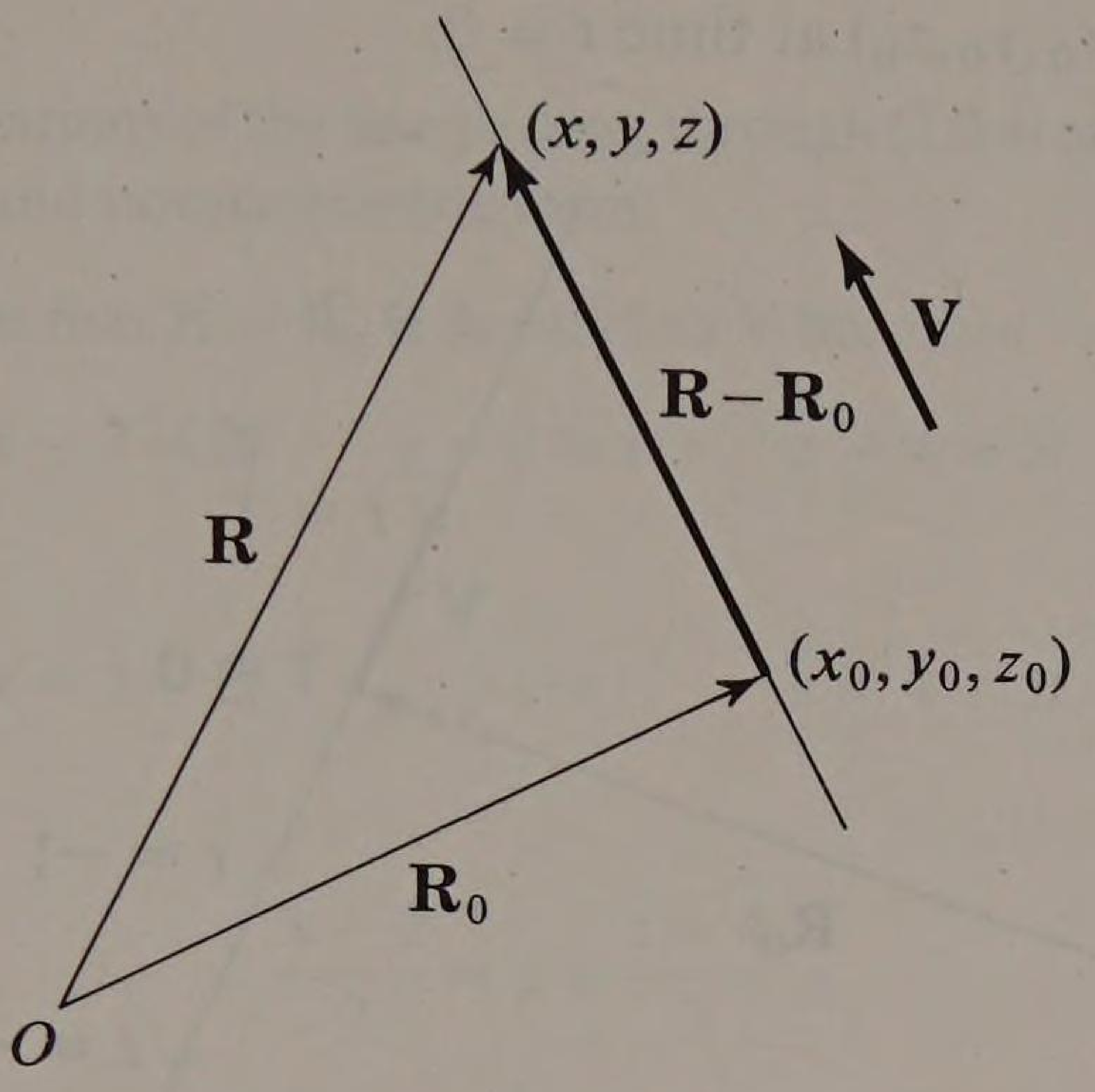

FIGURE 1.15

Let $\mathbf{R}_0$ be the position vector of (x_0,y_0,z_0) and let $\mathbf{R}$ be the position vector of a point (x,y,z). It is not immediately obvious what conditions on the vector $\mathbf{R}$ itself will make the point (x,y,z) lie on the desired line, but the vector from (the tip of) $\mathbf{R}_0$ to (the tip of) $\mathbf{R}$ *must be parallel to* $\mathbf{V}$. This vector, describing $\mathbf{R}$ "relative to $\mathbf{R}_0$", is, of course, $\mathbf{R} - \mathbf{R}_0$. It will be parallel to $\mathbf{V}$ if and only if it equals some scalar multiple of $\mathbf{V}$, so the condition that (x,y,z) be on the line is that $\mathbf{R} - \mathbf{R}_0 = t\mathbf{V}$ for some number t. Rewriting this as $\mathbf{R} = \mathbf{R}_0 + t\mathbf{V}$ and expressing it in terms of the components of the vectors, we obtain

$$\mathbf{R} = \mathbf{R}_0 + t\mathbf{V} \qquad \text{or} \qquad \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} \tag{1.8}$$

A point (x,y,z) is on the line passing through (x_0,y_0,z_0) and parallel to $\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ if and only if its coordinates satisfy all three of the equations (1.8) for some value of the scalar t between $-\infty$ and $+\infty$.

Let us dwell for a moment on the significance of the scalar t. It seems to be somewhat artificial in the description. After all, physically speaking we can draw the line once we know $\mathbf{R}_0$ and $\mathbf{V}$; no other data are needed. The introduction of this element t is just a mathematical device to help us say, *with equations*, that $\mathbf{R} - \mathbf{R}_0$ is parallel to $\mathbf{V}$. Equations (1.8) are called the *parametric form* of the equations of the line, and the "dummy" variable t is called the *parameter*.

An interpretation of the role of t can be gleaned from observing points $\mathbf{R}$ on the line for various values of t. Thus, for instance, if $t = 0$, $\mathbf{R} = \mathbf{R}_0$;

if $t = 1$, $\mathbf{R} = \mathbf{R}_0 + \mathbf{V}$; if $t = -1$, $\mathbf{R} = \mathbf{R}_0 - \mathbf{V}$; other points, in between and beyond, are indicated in Fig. 1.16. If we think of the parameter t as representing time, we can think of Eqs. (1.8) as giving the position of a moving particle at time t. This particle traverses a line parallel to $\mathbf{V}$ and passes through the point (x_0,y_0,z_0) at time $t = 0$.

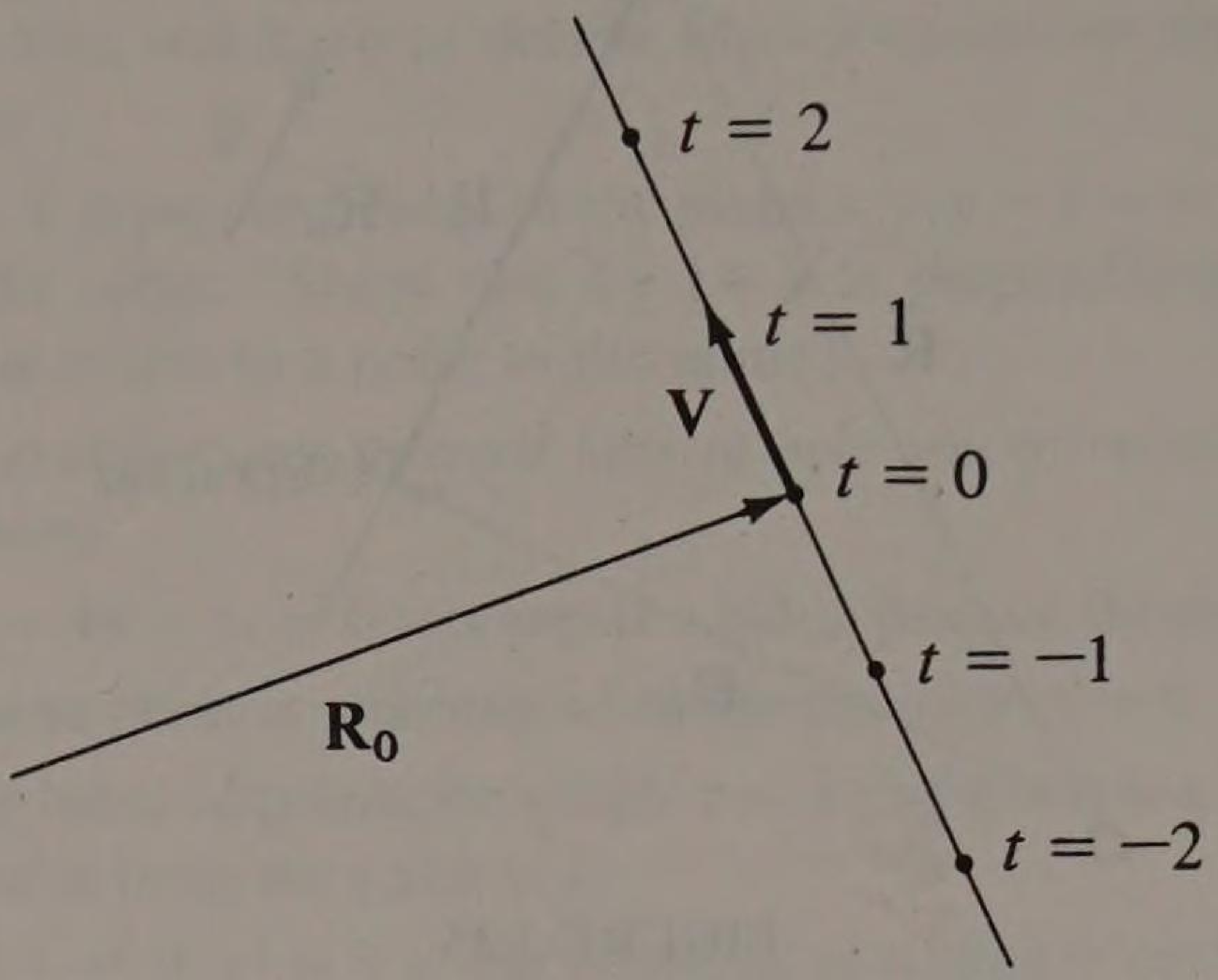

FIGURE 1.16

As far as the line itself is concerned, the scalar multiple t in Eq. (1.8) could be replaced by any scalar function of t, such as $t/2$, $-t$, or t^3, as long as the function takes all values between $-\infty$ and $+\infty$. However, if we wrote

$$\mathbf{R} = \mathbf{R}_0 + t^2\mathbf{V}$$

we would only be adding *positive* multiples of $\mathbf{V}$ to $\mathbf{R}_0$, so we would be generating only "half" of the line (i.e., a *ray*). If we wrote

$$\mathbf{R} = \mathbf{R}_0 + (\sin t)\mathbf{V}$$

we would generate just the *segment* of the line between $\mathbf{R}_0 - \mathbf{V}$ and $\mathbf{R}_0 + \mathbf{V}$ (since $-1 \leq \sin t \leq 1$), and we would be covering this segment infinitely often; interpreting t as time, the particle would oscillate forever from one end of the segment to the other. This same segment could be generated by the original parametric equations, Eqs. (1.8), if we restrict t to the interval $-1 \leq t \leq 1$.

Clearly, the parametric form is not unique. Most people, however, would agree that Eqs. (1.8) are the simplest form. Even so, notice that $\mathbf{R}_0$ could be replaced by any other vector describing a point on the line, and $\mathbf{V}$ could be replaced by any other vector having the same direction.

The parameter t can be eliminated by manipulating Eqs. (1.8). The reader can easily verify that if none of the components of $\mathbf{V}$ are zero, one can derive

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{1.9}$$

This is a nonparametric form, and from it one can immediately read off the components of $\mathbf{V}$ and of $\mathbf{R}_0$. [In using Eq. (1.9), keep in mind the essential feature that the coefficients of x, y, and z are 1's! Also observe that (1.9) represents two equations.]

Example 1.7 Find equations of the line passing through (2,0,4) and parallel to $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, both in parametric and nonparametric form.

Solution The condition that $\mathbf{R} - \mathbf{R}_0$ is parallel to $\mathbf{V}$ becomes

$$x - 2 = 2t \qquad y - 0 = 1t \qquad z - 4 = 3t$$

Thus,

$$x = 2 + 2t \qquad y = t \qquad z = 4 + 3t$$

Nonparametrically,

$$\frac{x-2}{2} = y = \frac{z-4}{3}$$

Example 1.8 Find equations of the line passing through $(0,3,-1)$ parallel to $3\mathbf{i} + 4\mathbf{k}$.

Solution In parametric form we have

$$\begin{aligned} x &= 3t \\ y &= 3 \\ z &= -1 + 4t \end{aligned}$$

For the nonparametric form $b = 0$, so Eq. (1.9) does not make sense. If we eliminate t from the first and third equations above, we find

$$\frac{x}{3} = \frac{z+1}{4}$$

To this equation we append $y = 3$, which is already nonparametric.

Example 1.9 Find a unit vector parallel to the line

$$\frac{x-4}{2} = y - 3 = \frac{z+1}{2}$$

Solution By comparison with (1.9) we have $a = 2$, $b = 1$, and $c = 2$, so a vector parallel to the line is $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Dividing this vector by its own length, we obtain a unit vector $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$. The negative of this vector is also a correct solution.

Example 1.10 Find the point of intersection of the two straight lines

$$\begin{aligned} \mathbf{R} &= 3\mathbf{i} + 2\mathbf{j} + (2\mathbf{i} + \mathbf{j} + \mathbf{k})t \\ \mathbf{R} &= \mathbf{i} - 2\mathbf{k} + (\mathbf{j} + \mathbf{k})t \end{aligned}$$

Solution This is a little deceptive. Although we have used the same letter, t, for the parameter on both lines, we do not imply that at the point of intersection t takes the same values for each of the two lines; in terms of the particle-motion interpretation we are saying that the two paths may intersect, but the individual particles can go through the point of intersection at different times. Thus, we should go to a nonparametric description.

In order that the point (x,y,z) lie on the first line, we must have

$$\frac{x-3}{2} = y - 2 = z$$

The condition for the second line reads

$$x = 1 \qquad y = z + 2$$

These constitute four equations that the three unknowns (x, y, z) must satisfy. If we just consider the first three equations,

$$\frac{x-3}{2} = y - 2 \qquad y - 2 = z \qquad x = 1$$

we find that they have a solution $(x, y, z) = (1, 1, -1)$. We must still check that the fourth equation,

$$y = z + 2$$

is satisfied; otherwise, there is no point of intersection (which is quite possible in space!). In this case it checks, so the point of intersection has $\mathbf{i} + \mathbf{j} - \mathbf{k}$ as its position vector.

Example 1.11 Find the angle between the lines in Example 1.10.

Solution We have already verified that the lines do intersect, so the problem makes sense. The first line is parallel to $2\mathbf{i} + \mathbf{j} + \mathbf{k}$, the second to $\mathbf{j} + \mathbf{k}$. The angle between these vectors satisfies [see Eq. (1.5)]

$$\cos\theta = \frac{2(0) + 1(1) + 1(1)}{(2^2 + 1^2 + 1^2)^{\frac{1}{2}}(0^2 + 1^2 + 1^2)^{\frac{1}{2}}} = \frac{2}{12^{\frac{1}{2}}} = \frac{1}{3^{\frac{1}{2}}}$$

Therefore,

$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

Exercises

1. Find parametric equations of the line passing through the origin parallel to $3\mathbf{i} + 7\mathbf{k} - 2\mathbf{j}$.
2. Find the equations of the line parallel to the z axis passing through the point (1,2,3).
3. Find equations of the line perpendicular to the yz plane, passing through (1,2,3).
4. Find the two unit vectors parallel to the line

$$\frac{x-1}{3} = \frac{y+2}{4} \qquad z = 9$$

5. Find two unit vectors parallel to the line $x = 2y = 3z + 3$. These equations can be written in form (1.9) as follows:

$$x = \frac{y}{\frac{1}{2}} = \frac{z+1}{\frac{1}{3}}$$

6. Find two unit vectors parallel to the line represented by the equations $x + y = 1$, $x - 3z = 5$. [*Hint*: Rewrite in form (1.9).]

7. Find equations of the line passing through the origin and parallel to the line

$$x - 3 = \frac{y+2}{4} = 1 - z$$

8. Find equations of the line passing through the points (3,4,5) and (3,4,7).

9. Find equations of the line passing through the points (1,4,−1) and (2,2,7).

10. By vector methods, find the cosine of the angle between the lines

$$\frac{x-1}{3} = \frac{y-0.5}{2} = z \quad \text{and} \quad x = y = z$$

11. Find the angle between the two intersecting lines

$$\frac{x-1}{3} = \frac{y-3}{4} = \frac{z}{5} \quad \text{and} \quad \frac{x-1}{2} = 3 - y = 2z$$

12. Let A and B be two points with position vectors **A** and **B**, respectively. Show that the line passing through these points may be represented by the vector equation

$$\mathbf{R} = s\mathbf{A} + t\mathbf{B} \qquad s + t = 1 \tag{1.10}$$

13. Solve Exercise 9 by making use of Eq. (1.10).

14. (*Points of Division*) If the points A, B, and P are collinear, P is said to divide the segment AB in the ratio λ when the segments AP and PB are related by

$$AP = \lambda(PB) \tag{1.11}$$

(a) For what values of λ does P lie between A and B? to the left of A? to the right of B?

(b) Show that, relative to an origin O, Eq. (1.11) can be written

$$OP = \frac{OA + \lambda(OB)}{1 + \lambda}$$

Relate this to Eq. (1.10).

(c) If P and P' divide AB internally and externally in the same numerical ratios $\pm\lambda$, show that A and B divide PP' internally and externally in the ratios $\pm(1-\lambda)/(1+\lambda)$.

15. Find the point(s) of intersection of the following pairs of straight lines.

(a) $\mathbf{R} = (5\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})t + 7\mathbf{i} + 6\mathbf{j} + 8\mathbf{k}$ and
$\mathbf{R} = (6\mathbf{i} + 4\mathbf{j} + 6\mathbf{k})t + 8\mathbf{i} + 6\mathbf{j} + 9\mathbf{k}$

(b) $\mathbf{R} = (3\mathbf{i} + 2\mathbf{j} + \mathbf{k})t + 2\mathbf{k}$ and
$\mathbf{R} = (6\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})t + 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

(c) $\mathbf{R} = (3\mathbf{i} - \mathbf{j} + \mathbf{k})t$ and
$\mathbf{R} = (-6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})t + 2\mathbf{i}$
(d) $\mathbf{R} = (\mathbf{i} + \mathbf{j} + \mathbf{k})t$ and
$\mathbf{R} = (\mathbf{i} + \mathbf{j} - 3\mathbf{k})t - \mathbf{i} + \mathbf{j}$

1.9 SCALAR PRODUCTS

The *scalar product* of two vectors is the number

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta \tag{1.12}$$

where θ denotes the angle between the vectors. Although **A** and **B** are vectors, $\mathbf{A} \cdot \mathbf{B}$ is a number. The scalar product is also called the *dot product* or the *inner product*. From Fig. 1.17, we identify $|\mathbf{B}| \cos\theta$ as the component of **B** parallel to **A**; i.e., the length of the orthogonal projection of **B** in the direction of **A**, with the appropriate sign. Thus, we can interpret $\mathbf{A} \cdot \mathbf{B}$ as

(*length of* **A**)(*signed component of* **B** *along* **A**)

Since the definition is symmetric in **A** and **B**, it can equally well be interpreted as

(*length of* **B**)(*signed component of* **A** *along* **B**)

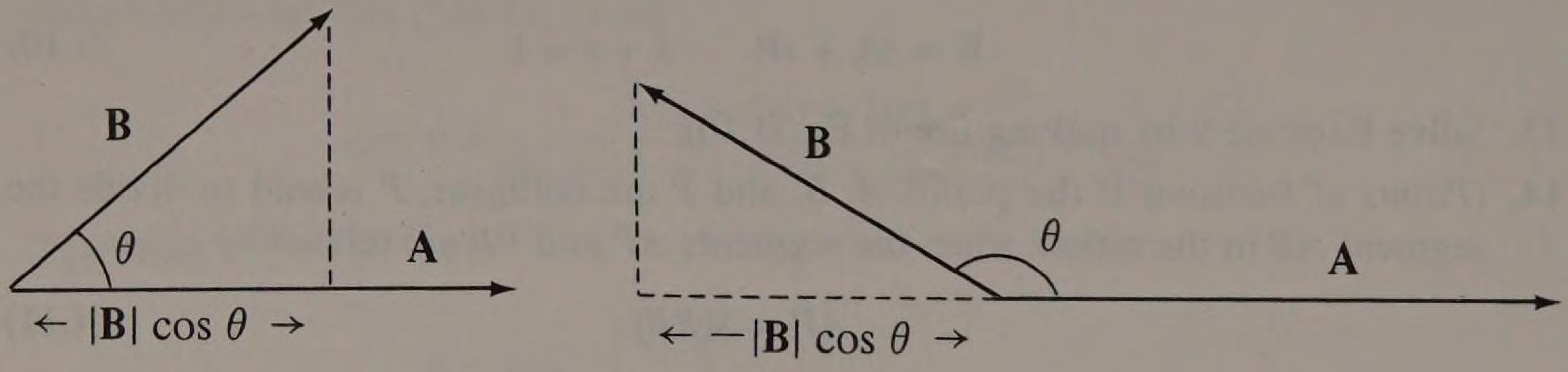

FIGURE 1.17

In a few simple cases the scalar product of two vectors is easily computed directly from this definition. For example, the scalar product of the vectors shown in Fig. 1.18 is $9\sqrt{3}$.

If either **A** or **B** is the zero vector, we have $|\mathbf{A}| = 0$ or $|\mathbf{B}| = 0$, so by (1.12) it follows that $\mathbf{A} \cdot \mathbf{B} = 0$. (We ignore the fact that θ is not defined in this case.)

On the other hand, it is possible to have $\mathbf{A} \cdot \mathbf{B} = 0$ even though both

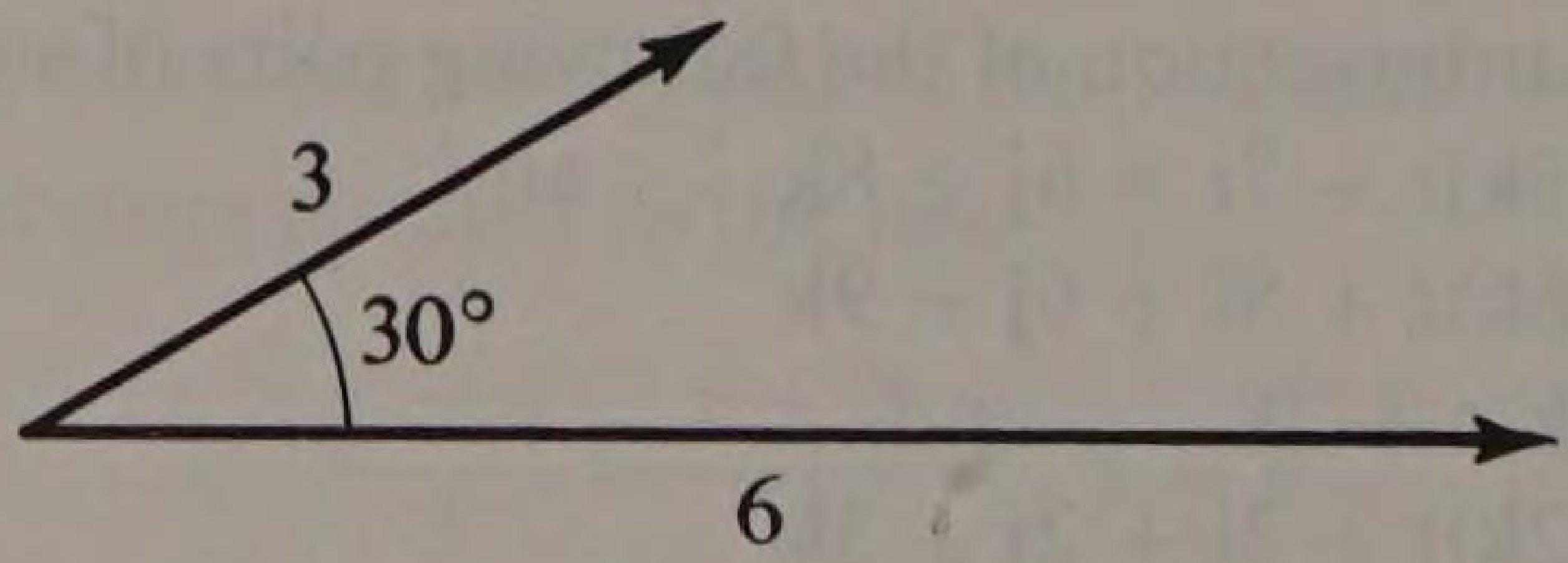

FIGURE 1.18

A and **B** are nonzero vectors. For example, if **A** and **B** are perpendicular, then $\cos\theta = \cos 90^\circ = 0$ and hence $\mathbf{A}\cdot\mathbf{B} = 0$.

Recall that in Sec. 1.7 we derived a component expression for the right hand side of (1.12), namely Eq. (1.7):

$$|\mathbf{A}|\,|\mathbf{B}|\cos\theta = A_1B_1 + A_2B_2 + A_3B_3 \tag{1.7}$$

Combining this with (1.12) gives us

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 \tag{1.13}$$

Thus we have two important formulas for the scalar product. Equation (1.12) describes $\mathbf{A}\cdot\mathbf{B}$ in terms of geometric concepts and provides a visualization, while Eq. (1.13) gives the componentwise description and is useful for computations. Memorize both formulas now. They are important.

Example 1.12 Find the scalar product of $4\mathbf{i} - 5\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Solution $(4)(1) + (-5)(2) + (-1)(3) = -9$. (The negative sign indicates that the angle between the vectors must be greater than 90°.)

Example 1.13 Find the angle between the vectors $\mathbf{A} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{B} = 3\mathbf{i} + 4\mathbf{j}$.

Solution We have $|\mathbf{A}| = 3$ and $|\mathbf{B}| = 5$. Using (1.13), we see that $\mathbf{A}\cdot\mathbf{B} = 14$. Substituting these values in (1.12), we solve to get $\theta = \cos^{-1} 14/15$.

Example 1.14 If **F** is a constant force acting through a displacement **D**, the work done by **F** is defined to be the product of the magnitude of the displacement with the component of the force in the direction of the displacement. In vector notation,

$$\textit{Work} = \mathbf{F}\cdot\mathbf{D}$$

The following properties of the scalar product are easily verified from (1.13):

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$
$$(s\mathbf{A} + \mathbf{B})\cdot\mathbf{C} = s\mathbf{A}\cdot\mathbf{C} + \mathbf{B}\cdot\mathbf{C}$$
$$\mathbf{A}\cdot(s\mathbf{B} + \mathbf{C}) = s\mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$
$$|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A}$$

Example 1.15 (*A Maximum Principle*) Let there be given a nonzero vector **D**, and let **n** denote a unit vector. Then $|\mathbf{n}| = 1$ and $\mathbf{D}\cdot\mathbf{n} = |\mathbf{D}|\,|\mathbf{n}|\cos\theta = |\mathbf{D}|\cos\theta$. This will be a maximum when $\cos\theta = 1$, i.e., when $\theta = 0$. Thus we have derived the following maximum principle, which will be useful to us in later sections:

> *The unit vector* **n** *making* $\mathbf{D}\cdot\mathbf{n}$ *a maximum is the unit vector pointing in the same direction as* **D**.

Example 1.16 The scalar product can be used to express components along the axes, of course; thus the component of **D** in the x direction is $\mathbf{D}\cdot\mathbf{i}$, and so forth. In fact, for any vector **D** we can write

$$\mathbf{D} = (\mathbf{D}\cdot\mathbf{i})\mathbf{i} + (\mathbf{D}\cdot\mathbf{j})\mathbf{j} + (\mathbf{D}\cdot\mathbf{k})\mathbf{k} \tag{1.14}$$

As another example of the use of the scalar product, consider the following problem. One is given two vectors, **A** and **B**, and one wishes to decompose **B** into a vector parallel to **A** plus a vector perpendicular to **A**. In other words, one wishes to find expressions for the vectors $\mathbf{B}_{\|}$ and $\mathbf{B}_{\perp}$ in Fig. 1.19. Clearly, the (signed) length of $\mathbf{B}_{\|}$ is $\mathbf{B} \cdot \mathbf{A}/|\mathbf{A}|$. To construct a vector of this length in the direction of **A**, we take the unit vector along **A** and multiply by this scalar. Since $\mathbf{A}/|\mathbf{A}|$ is the unit vector, we have the following simple formula:

$$\mathbf{B}_{\|} = \frac{\mathbf{B} \cdot \mathbf{A}}{|\mathbf{A}|} \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{B} \cdot \mathbf{A}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A}$$

Having computed $\mathbf{B}_{\|}$, $\mathbf{B}_{\perp}$ is just the rest of **B**:

$$\mathbf{B}_{\perp} = \mathbf{B} - \mathbf{B}_{\|} = \mathbf{B} - \frac{\mathbf{B} \cdot \mathbf{A}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A}$$

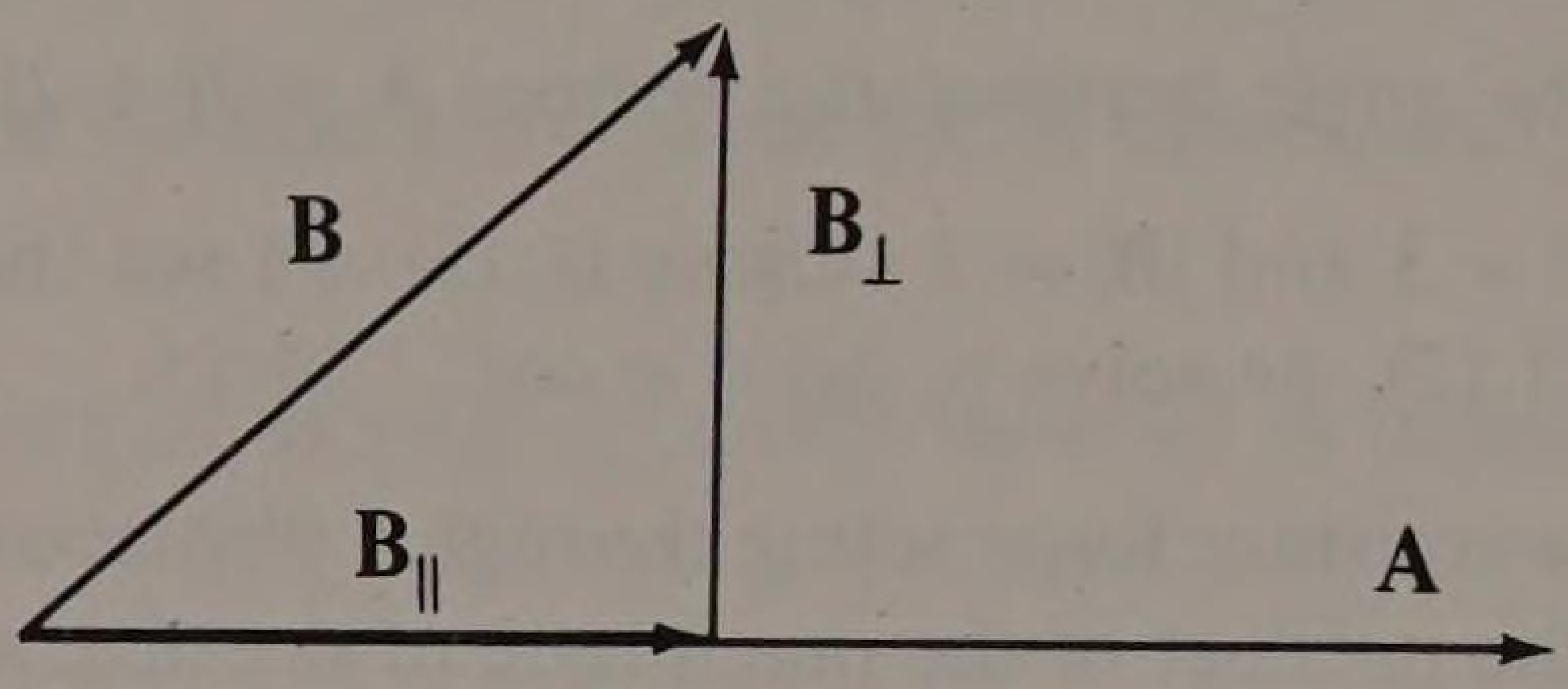

FIGURE 1.19

Example 1.17 Resolve the vector $6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ into vectors parallel and perpendicular to $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Solution The parallel vector is

$$\frac{6+2-2}{1+1+1}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

The perpendicular vector is

$$6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} - 2(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 4\mathbf{i} - 4\mathbf{k}$$

Example 1.18 Find a formula for the mirror image, $\mathbf{V}'$, of a vector **V**, reflected in a plane mirror with unit normal **n**. (See Fig. 1.20*a*.)

Solution In Fig. 1.20*b*, we have drawn representatives of **n**, **V**, and $\mathbf{V}'$ with a common tail. The dotted lines illustrate the fact that **V** and $\mathbf{V}'$ have the same component perpendicular to **n**, but the parallel components are opposite. To obtain $\mathbf{V}'$ from **V**, we must subtract this parallel component *twice*. Hence, keeping in mind that **n** was given as a *unit* normal,

$$\mathbf{V}' = \mathbf{V} - 2\mathbf{V}_{\|} = \mathbf{V} - 2(\mathbf{V} \cdot \mathbf{n})\mathbf{n}$$

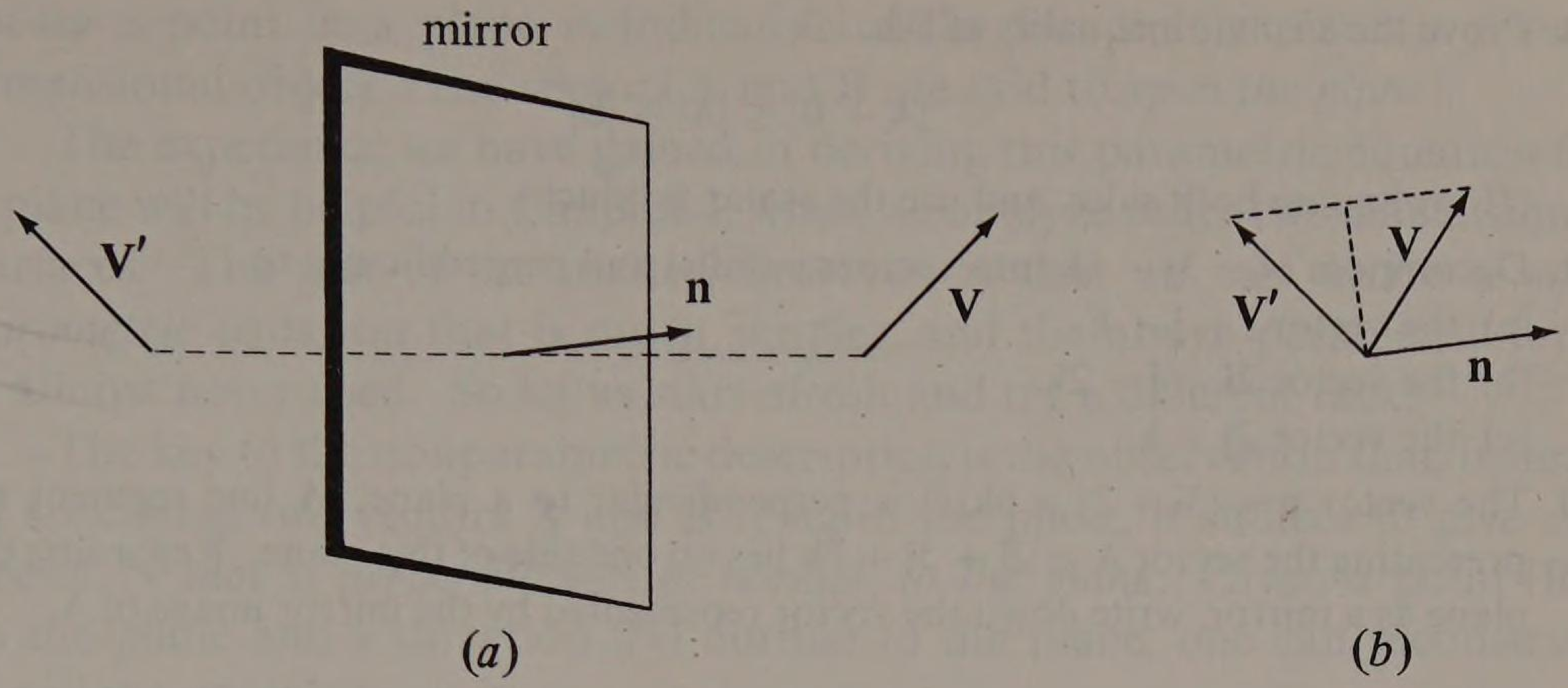

FIGURE 1.20

Exercises

1. Find the scalar product of $3\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}$ with $5\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.
2. Find the scalar product of $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ with $4\mathbf{i} - 3\mathbf{k} + 9\mathbf{j}$.
3. Find the scalar product of $3\mathbf{i} + 4\mathbf{j}$ with $5\mathbf{j} - 10\mathbf{k}$.
4. Determine the angle between $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $3\mathbf{i} - 4\mathbf{j}$.
5. Find the angle between $2\mathbf{i}$ and $3\mathbf{i} + 4\mathbf{j}$.
6. A force $\mathbf{F} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ acts through a displacement $\mathbf{D} = -2\mathbf{i} + \mathbf{j} - \mathbf{k}$. Find the work done.
7. Find the component of $8\mathbf{i} + \mathbf{j}$ in the direction of $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.
8. Find the component of $\mathbf{i} + \mathbf{j} + \mathbf{k}$ in the direction of $\mathbf{i} + \mathbf{j}$.
9. Find the component of the force $5\mathbf{i} + 7\mathbf{j} - \mathbf{k}$ in the direction of the displacement PQ, where $P(3,0,1)$ and $Q(4,4,4)$ are points in space.
10. Find the vector in the same direction as $\mathbf{i} + \mathbf{j}$ whose component in the direction of $2\mathbf{i} - 4\mathbf{k}$ is unity.
11. If $\mathbf{A} \cdot \mathbf{A} = 0$ and $\mathbf{A} \cdot \mathbf{B} = 0$ what can you conclude about the vector $\mathbf{B}$?
12. By interpreting $2x + 3y + 4z$ as a scalar product, show that $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ is perpendicular to the plane $2x + 3y + 4z = 0$.
13. If $\mathbf{A}$ is a fixed nonzero vector, interpret geometrically $(\mathbf{R} - \mathbf{A}) \cdot \mathbf{R} = 0$,
    (a) in the plane, $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$,
    (b) in space, $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
14. If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, and θ is the angle between them, find $\frac{1}{2}|\mathbf{u} - \mathbf{v}|$ in terms of θ.
15. Let $\mathbf{A} = (\cos\phi)\mathbf{i} + (\sin\phi)\mathbf{j}$ and $\mathbf{B} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$. Draw these vectors in the xy plane. By interpreting the scalar product $\mathbf{A} \cdot \mathbf{B}$ geometrically, prove that $\cos(\phi - \theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$.
16. Prove, by vector methods, that the median from the vertex angle of an isosceles triangle is perpendicular to the base.
17. Prove the parallelogram equality, i.e., the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of its sides.

18. Prove the triangle inequality of Sec. 1.3,

$$|\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|$$

(*Hint*: Square both sides, and use the scalar product.)

19. Decompose $6\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$ into vectors parallel and perpendicular to
   (a) the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
   (b) the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.
   (c) the vector $2\mathbf{j} - \mathbf{k}$.

20. The vector $\mathbf{n} = (3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})/7$ is perpendicular to a plane. A line segment representing the vector $\mathbf{A} = 2\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$ lies on one side of this plane. Regarding the plane as a mirror, write down the vector represented by the mirror image of **A**.

1.10 EQUATIONS OF A PLANE

Recall that in Sec. 1.8 we specified a straight line by giving a point on the line and a vector parallel to the line. By analogy, then, we might specify a plane by giving a point (x_0,y_0,z_0) in the plane, and *two* vectors **A** and **B** parallel to the plane. Of course, **A** and **B** must not be parallel to each other. Introducing the position vectors $\mathbf{R}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ and $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we seek the condition on **R** guaranteeing that (x,y,z) lies in the plane. It is not immediately obvious how **R** depends on $\mathbf{R}_0$, **A**, and **B**, but, clearly, the "relative vector" $\mathbf{R} - \mathbf{R}_0$ must lie in the plane (more precisely, it has a representative that lies in the plane, Fig. 1.21); hence it can be expressed as a combination of **A** and **B**. Thus we have

$$\mathbf{R} - \mathbf{R}_0 = s\mathbf{A} + t\mathbf{B}$$

for some scalars s and t, each taking values between $-\infty$ and $+\infty$.

These scalars s and t play a similar role to the single parameter t in the equation [Eq. (1.8)] for a straight line. The need for *two* parameters to

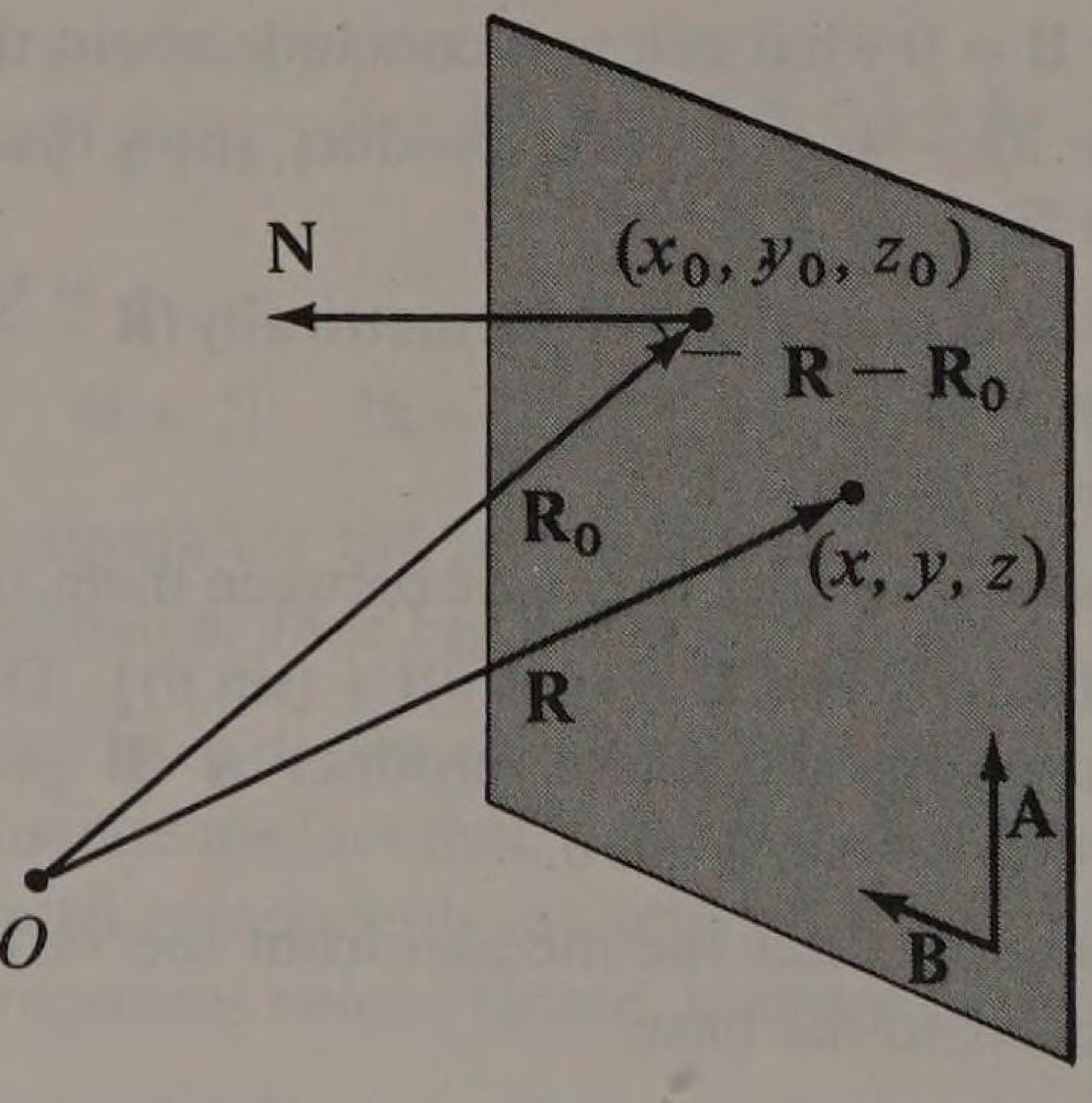

FIGURE 1.21

locate a point in a plane is indicative of the fact that a plane is a *two-dimensional* object. The vectors **A** and **B** are said to *span the plane*.

The experience we have gained in deriving this parametric equation for a plane will be helpful in Chapter 4, when we analyze other two-dimensional surfaces. The fact of the matter, however, is that we can derive a *nonparametric* equation that is much simpler, and the above parametric form is almost never used. So let us start afresh and try a different tack.

The key to the nonparametric description is the observation that, instead of specifying *two* vectors **A** and **B** lying in the plane, it suffices to give *one* vector **N** *that is perpendicular, or normal, to the plane.* Given a point ($\mathbf{R}_0$) in the plane and a direction (**N**) normal to the plane, one can reconstruct the plane unambiguously.

The condition that **R** is the position vector to a point in the plane can be expressed by saying that the "relative vector" $\mathbf{R} - \mathbf{R}_0$, which lies in the plane as before, is perpendicular to **N** (see Fig. 1.21 again). According to the previous section, this condition can be written

$$(\mathbf{R} - \mathbf{R}_0) \cdot \mathbf{N} = 0 \tag{1.15}$$

Conversely, if (1.15) is satisfied, then $\mathbf{R} - \mathbf{R}_0$ is perpendicular to **N**. This ensures that **R** is the position vector of a point in the plane.

Hence (1.15) is a vector equation describing the plane. In terms of the components of $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, it becomes

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{1.16}$$

Lumping the constant terms, this can be written

$$ax + by + cz = d \tag{1.17}$$

where $d = ax_0 + by_0 + cz_0$.

Example 1.19 Find an equation of the plane passing through $(1,3,-6)$ perpendicular to the vector $3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$.

Solution By (1.16) we can write the equation down at once: $3(x - 1) - 2(y - 3) + 7(z + 6) = 0$. This can be simplified to $3x - 2y + 7z = -45$.

Example 1.20 Find an equation of the plane passing through $(1,2,3)$ perpendicular to the line

$$\frac{x-1}{4} = \frac{y}{5} = \frac{z+5}{6}$$

Solution We recall from Sec. 1.8 that we can find a *vector* parallel to the given line by reading off the coefficients in the denominators: $4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$. This vector is perpendicular to the desired plane, and therefore the equation of the plane is $4(x - 1) + 5(y - 2) + 6(z - 3) = 0$.

To be logically complete we should show that any equation of the form (1.17) does represent a plane, with normal $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ (assumed nonzero). This is straightforward: Let $\mathbf{R}_0$ be the position vector of some point satisfying (1.17), as, for instance, $(d/c)\mathbf{k}$. Then if $\mathbf{R}$ also satisfies (1.17), we have $\mathbf{R} \cdot \mathbf{N} = d = \mathbf{R}_0 \cdot \mathbf{N}$, so $(\mathbf{R} - \mathbf{R}_0) \cdot \mathbf{N} = 0$, and we have recovered the form (1.15).

Example 1.21 Find a unit vector perpendicular to the plane $2x + y - 2z = 7$.

Solution Reading off the coefficients, we see that $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is perpendicular to the plane. Its magnitude is 3, so the desired unit vector is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$. The negative of this vector is also a correct answer.

Example 1.22 Find the angle between the two planes $3x + 4y = 0$ and $2x + y - 2z = 5$.

Solution The desired angle equals the angle between the normals $\mathbf{N}_1 = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{N}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. By the methods of Sec. 1.9,

$$\cos\theta = \frac{\mathbf{N}_1 \cdot \mathbf{N}_2}{|\mathbf{N}_1||\mathbf{N}_2|} = \frac{6+4}{(5)(3)} = \frac{2}{3}$$

The desired angle is approximately 48°.

Example 1.23 In books on analytic geometry it is shown that the distance between an arbitrary point (x_1,y_1,z_1) and the plane $ax + by + cz = d$ is given by the expression

$$\frac{|ax_1 + by_1 + cz_1 - d|}{(a^2 + b^2 + c^2)^{\frac{1}{2}}}$$

Derive this expression by vector methods.

Solution Let $\mathbf{R}_0$ be the position vector of a point in the plane, and let $\mathbf{R}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. The desired distance is the absolute value (distance is never negative!) of the component of $\mathbf{R}_1 - \mathbf{R}_0$ in the direction of $\mathbf{N}$. Hence this distance is

$$\frac{|(\mathbf{R}_1 - \mathbf{R}_0) \cdot \mathbf{N}|}{|\mathbf{N}|} = \frac{|\mathbf{R}_1 \cdot \mathbf{N} - d|}{|\mathbf{N}|}$$

which, written out in terms of components, is the expression given above.

Example 1.24 Find the distance between the parallel planes $x + y + z = 5$ and $x + y + z = 10$.

Solution Take an arbitrary point in the first plane, say (1,1,3), and find its distance to the second plane by the expression derived in Example 1.23. We obtain

$$\frac{|5 - 10|}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$$

Exercises

1. Find unit vectors normal to the planes
   (a) $2x + y + 2z = 8$
   (b) $4x - 4z = 0$
   (c) $-y + 6z = 0$
   (d) $x = 5$
   (e) $y = z + 2$
   (f) $x = y$

2. Find an equation of the plane through the origin perpendicular to $2\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}$.
3. Find an equation of the plane perpendicular to **D** and through P, where
$$\mathbf{D} = 10\mathbf{i} - 10\mathbf{j} + 5\mathbf{k}$$
and P is $(1,1,-3)$.
4. Find a plane crossing through $(1,3,3)$, parallel to the plane $3x + y - z = 8$.
5. Is it possible to find a plane perpendicular to both **i** and **j**?
6. By vector methods find the distance from the point $(3,4,7)$ to the plane $2x - y - 2z = 4$.
7. Find the distances between the pairs of planes
   (a) $x + 2y + 3z = 5$ and $x + 2y + 3z = 19$
   (b) $x + y = 4$ and $x + y = 10$
   (c) $x = 5$ and $x = 7$ (no calculations needed here!)
8. Determine $\cos\theta$, where θ is the angle between the planes $x + y + z = 0$ and $x = 0$.
9. By vector methods, show that the line $x = y = \frac{1}{3}(z + 2)$ is parallel to the plane $2x - 8y + 2z = 5$.
10. By vector methods find the angle between the line $x = y = 2z$ and the plane $x + y + z = 0$.
11. Find the angle between the plane $x + y + z = 21$ and the line $x - 1 = y + 2 = 2z + 3$.
12. Find the equation of a line in the xy plane perpendicular to the vector $3\mathbf{i} - \mathbf{j}$.
13. Find the distance between the lines $x + y = 0$ and $x + y = 5$ in the xy plane.
14. Find a line in the xy plane parallel to $3x + 2y = 4$ passing through the point $(3,1)$.
15. Write the equation of the plane containing the lines
$$x = y = \frac{4 - z}{4}$$
and
$$2x = 2 - y = z$$
16. We are given two distinct parallel planes, and we are told the distance between the planes is d. A vector **v** is perpendicular to the planes and its magnitude is $1/d$. The planes intersect the y axis in the points $(0,1,0)$ and $(0,4,0)$ respectively. What is the y component of **v**? (There are two possible answers, depending on the two possible directions of **v**.)
17. Find the intersection of the following geometric objects:
   (a) the plane $3x + 2y - z = -9$ and the line $\frac{1}{2}x = y - 2 = -\frac{1}{4}(z - 1)$
   (b) the plane $x + y + 2z = 6$ and the line $-x = 2y = 4z + 1$
   (c) the plane $3x - y + z = 3$ and the plane $2x + z = 0$
   (d) the plane $x - y + 2z = 4$ and the plane $-2x + 2y - 4z = 1$.

1.11 ORIENTATION

In working in the xy plane, it is conventional to take the positive x direction to the right and the positive y direction upward. Angles are then taken to be *positive* in the *counterclockwise* direction.

When working with planes in space, there is no generally accepted convention for determining the positive sense for angles. The choice is quite arbitrary. Given any plane in space, we may arbitrarily decree in which direction we shall consider angles to be positive. The plane is then said to be oriented.

One way of orienting a plane is as follows. Let **A** and **B** be nonzero vectors, not parallel, represented by arrows in the given plane. Let these arrows extend from the same point. Let **A** be rotated through the smallest angle possible to coincide in direction with **B**. The sense of this rotation is then said to be "positive" and the plane is thereby oriented. *The plane is oriented by giving the vectors* **A**, **B** *in that order.*

For example, the usual orientation of the xy plane is obtained by giving the vectors **i**, **j** in that order. By a 90° rotation the direction of **i** can be made to coincide with that of **j**, and this rotation has the conventional "positive" sense. We obtain the same orientation by giving the vectors $\mathbf{i} + \mathbf{j}$ and **j** in that order (Fig. 1.22). On the other hand, if we specified the orientation by giving **j**, **i** in that order, we would obtain the opposite orientation, whereby angles would be measured positive in the clockwise sense (which is not conventional, but is perfectly satisfactory).

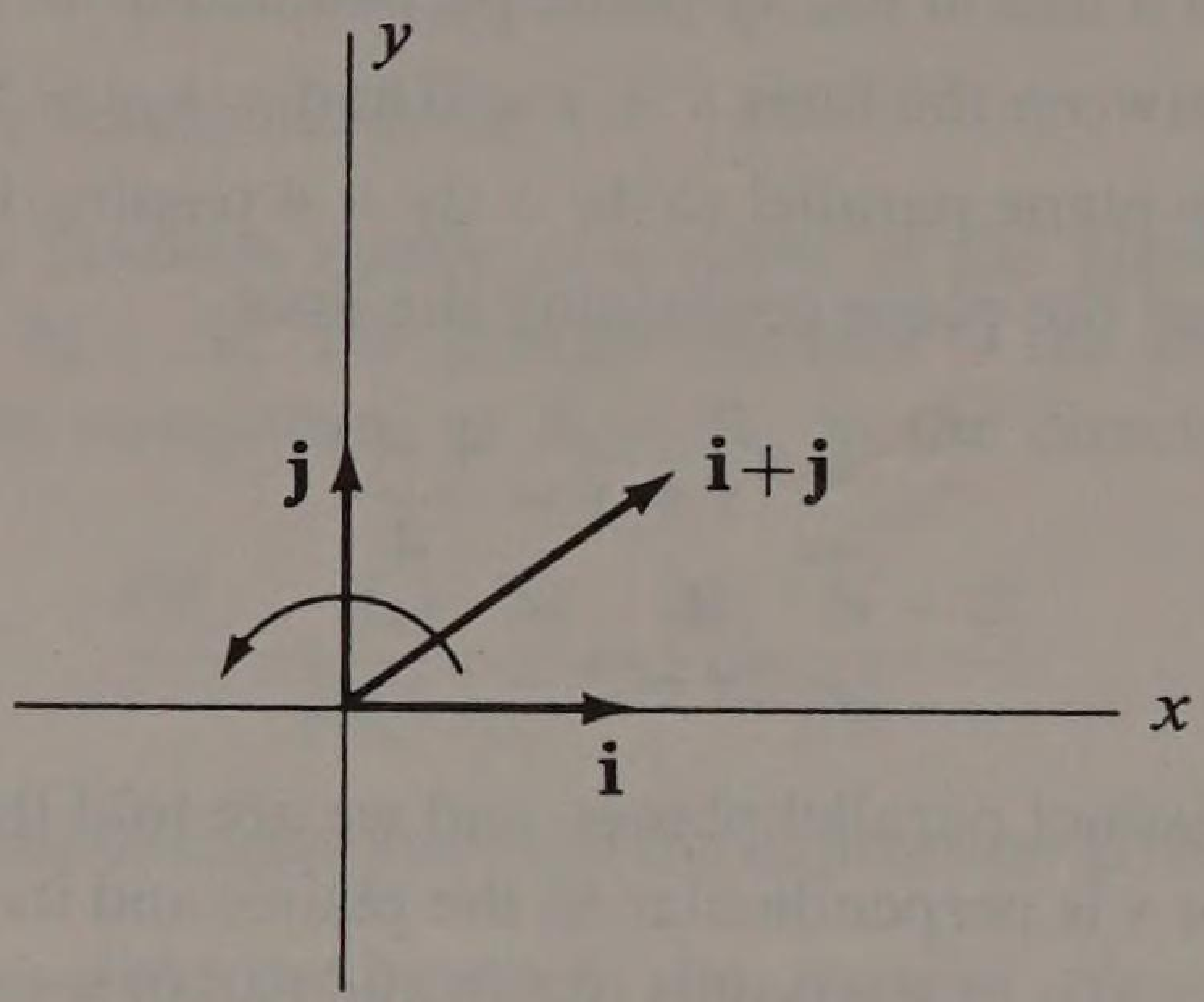

FIGURE 1.22

Another way of orienting a plane is as follows. Let there be given a single vector that is not parallel to the plane. Let this vector be represented by an arrow that has its initial point in the plane. Then the terminal point of the arrow will be on one side of the plane, which we call (arbitrarily) the *positive* side. We now take the positive sense for angles in the plane to be such that a right-handed screw with head parallel to the plane and shank perpendicular to the plane would advance in the direction of the positive side of the plane if rotated in the positive sense. (This is independent of the way in which the screw points.) Alternatively, if we imagine the right hand grasping the given vector, with thumb pointing in the direction of the arrowhead, the fingers will curl around the shank of the arrow in the positive sense of rotation in the plane.

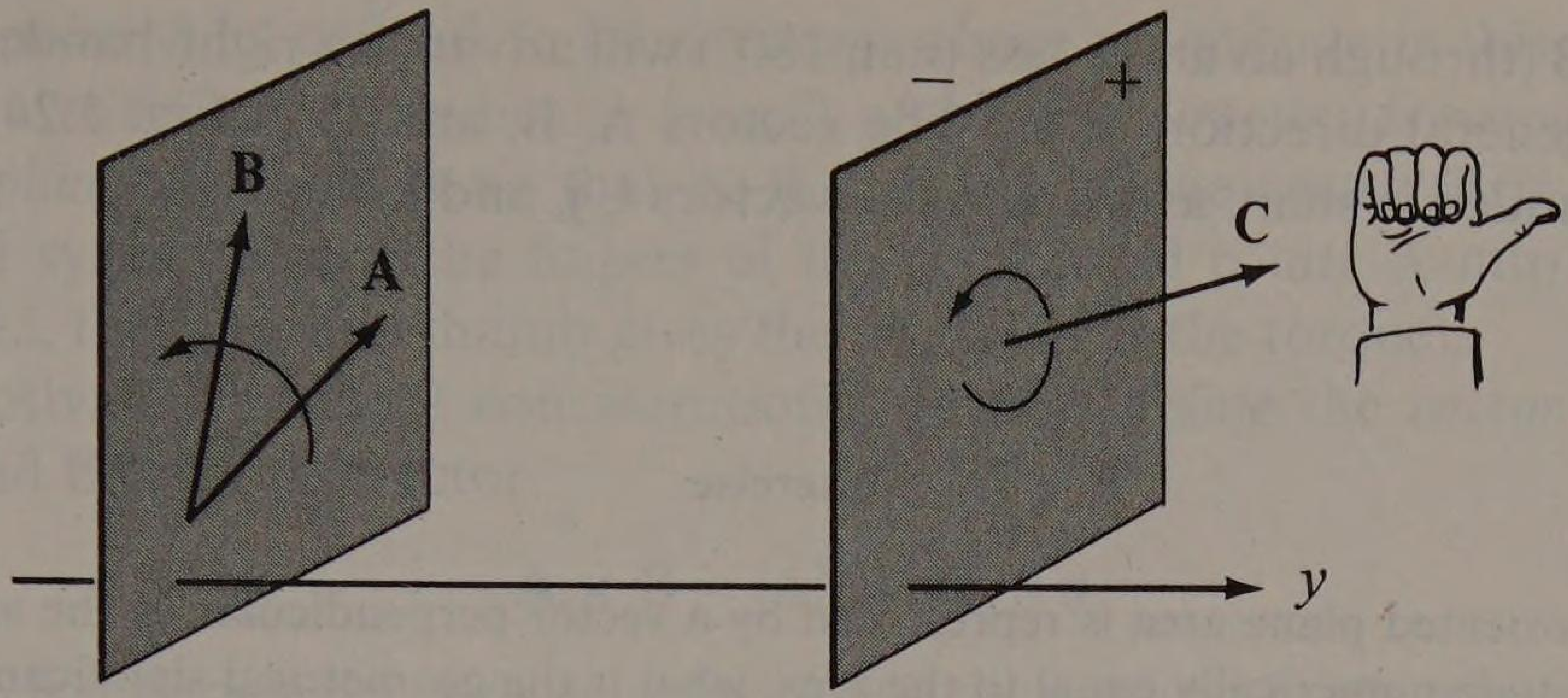

FIGURE 1.23

In Fig. 1.23 both methods of orienting a plane are illustrated for planes perpendicular to the y axis. At the left, the plane is oriented by prescribing two vectors in the plane, **A** and **B**, in that order. On the right, the same orientation is achieved by prescribing a vector **C** extending from a point in the plane.

Now let **A**, **B**, and **C** be nonzero vectors, not all parallel to the same plane, represented by arrows with initial points at the origin (Fig. 1.24). The vectors **A** and **B** determine a plane passing through the origin. If the orientation of this plane, as determined by **A**, **B** in that order, is identical with its orientation as determined by **C**, we say that **A**, **B**, and **C** in that order form a *right-handed system.* One reason for this terminology is that if the thumb and first two fingers of the right hand are held so they are mutually perpendicular, the thumb, forefinger, and second finger form such a system. Another reason is that if **A**, **B**, and **C**, in that order, form a *right-handed* system, the rotation of

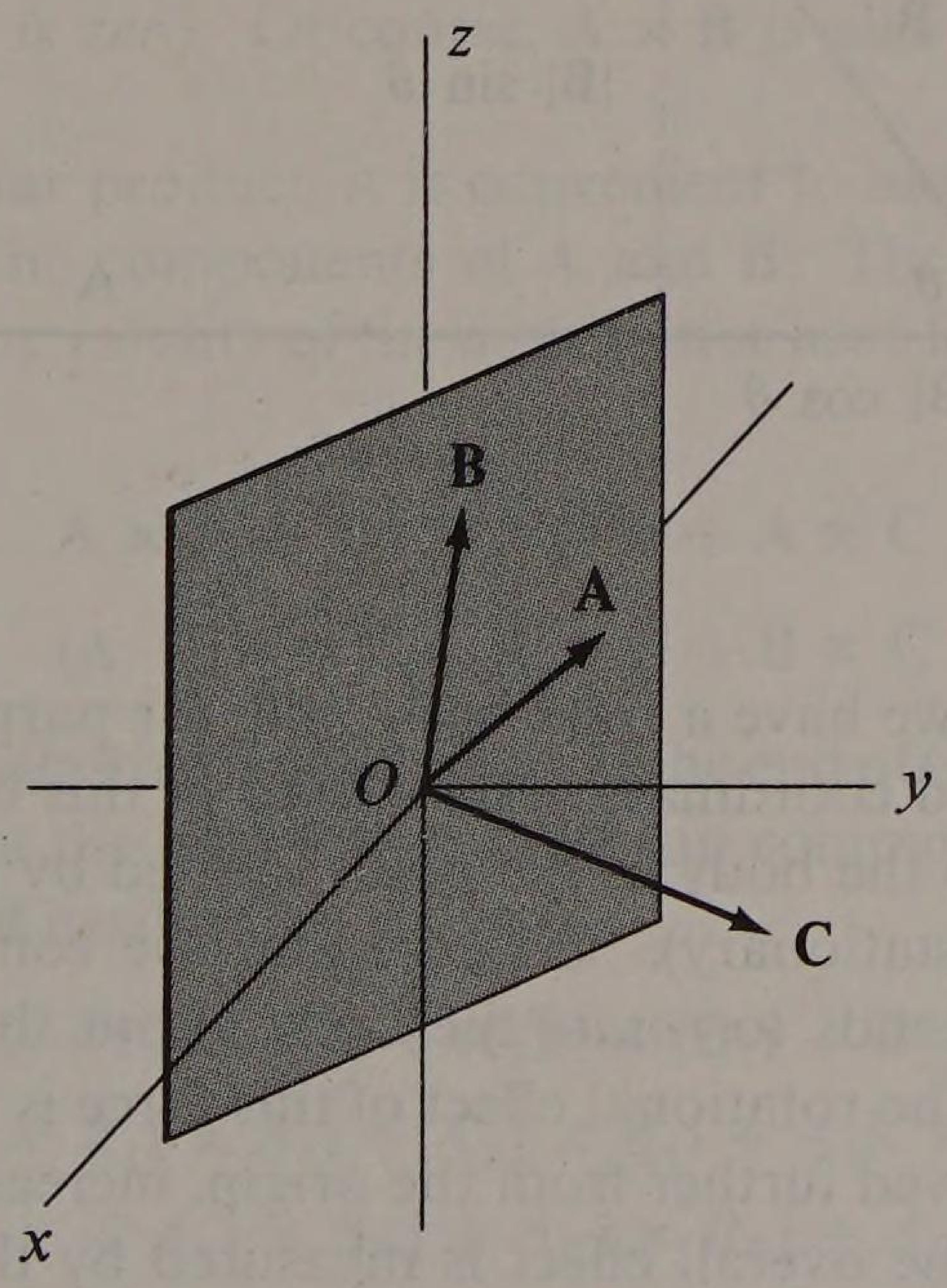

FIGURE 1.24

A into **B** (through an angle less than 180°) will advance a right-handed screw in the general direction of **C**. The vectors **A**, **B**, and **C** of Fig. 1.24 form a right-handed system, as do also the vectors **i**, **j**, and **k**.

Exercise

1. If an oriented plane area is represented by a vector perpendicular to the area, with magnitude numerically equal to the area, what is the geometrical significance of the components of the vector?

1.12 VECTOR PRODUCTS

We have seen that the scalar product of two vectors **A** and **B** can be interpreted as the length of **A** times the component of **B** parallel to **A**; in mechanics, it expresses the work done by a force **B** exerted through a displacement **A**, and it is also a very useful tool in analytic geometry. So we are naturally led to explore the possible advantages of defining another kind of product, given by the length of **A** times the component of **B** *perpendicular* to **A** (i.e., $|\mathbf{B}| \sin \theta$ in Fig. 1.25). Mechanics again lends a provocative interpretation to this operation.

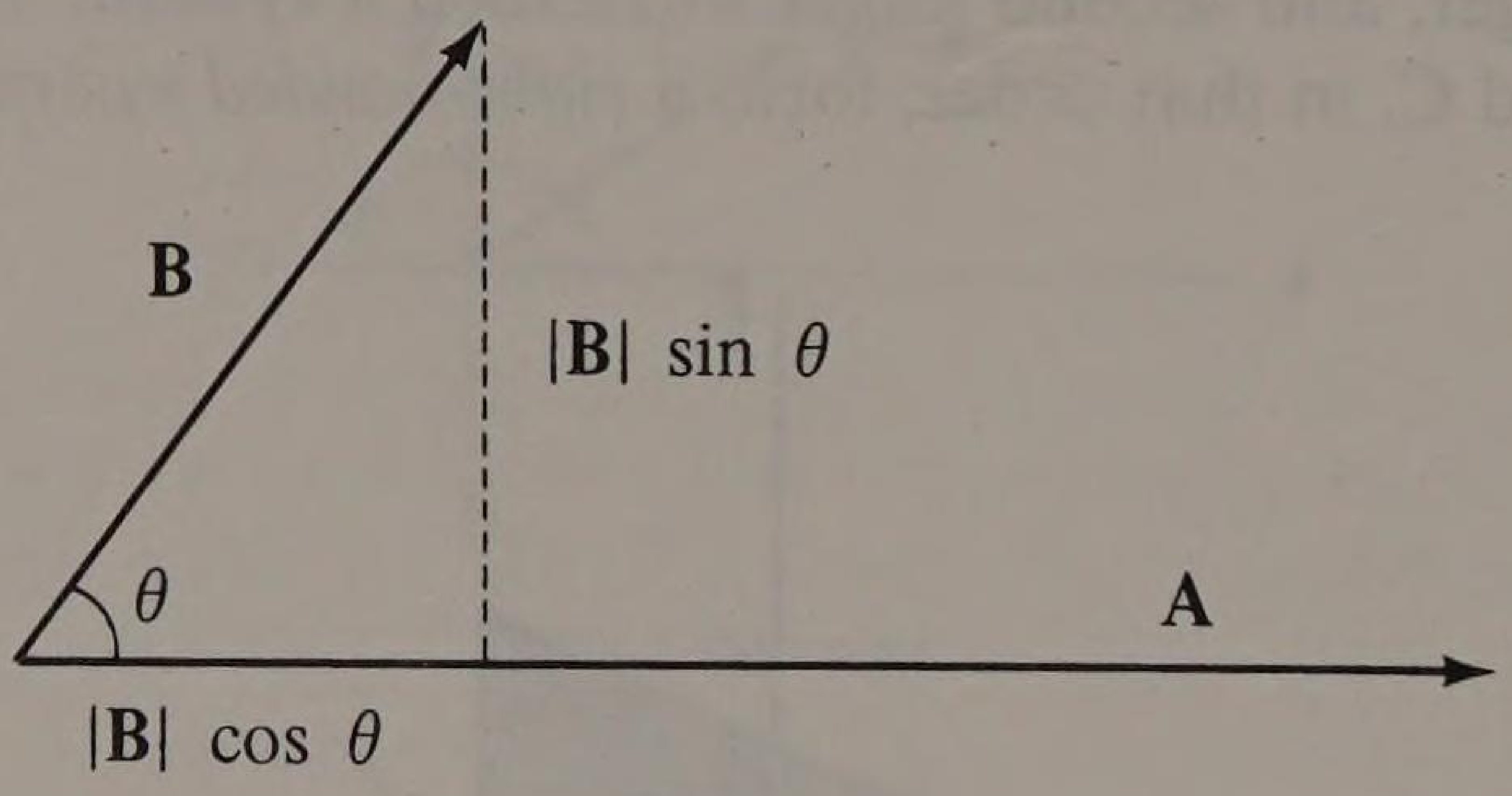

FIGURE 1.25

Let us suppose we have a rigid body and, for purposes of reference, we define a right-handed coordinate system fixed in this body. We interpret **B** as a force applied to the body at the point located by the vector **A** (relative to the origin, held stationary). Observe that the component of this force *perpendicular* to **A** tends to *rotate* the body about the axis normal to the plane of **A** and **B**. The rotational effect of this force is enhanced if the point of application is moved further from the origin, increasing the "leverage" of the force. In fact, the overall effect is measured by the vector product we just proposed. Consequently, in physics the *torque* due to the force **B** applied at the point **A** is defined to be a vector whose magnitude is this product ("lever arm times perpendicular force"), and *whose direction is perpendicular to the plane of* **A** *and* **B**, so that **A**, **B**, and the torque vector form a right-handed system (i.e., if the fingers of the right hand rotate **A** into **B** as in Fig. 1.23, the extended thumb gives the direction of the torque).

Motivated by these considerations, we now define the *vector product* of **A** and **B** to be the vector

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \sin \theta\, \mathbf{n}$$

where θ is the angle between the vectors, and the unit vector **n** is perpendicular to both **A** and **B**, with **A**, **B**, and **n** forming a right-handed system (see Fig. 1.26). Sometimes **A** × **B** is called the *cross product.*

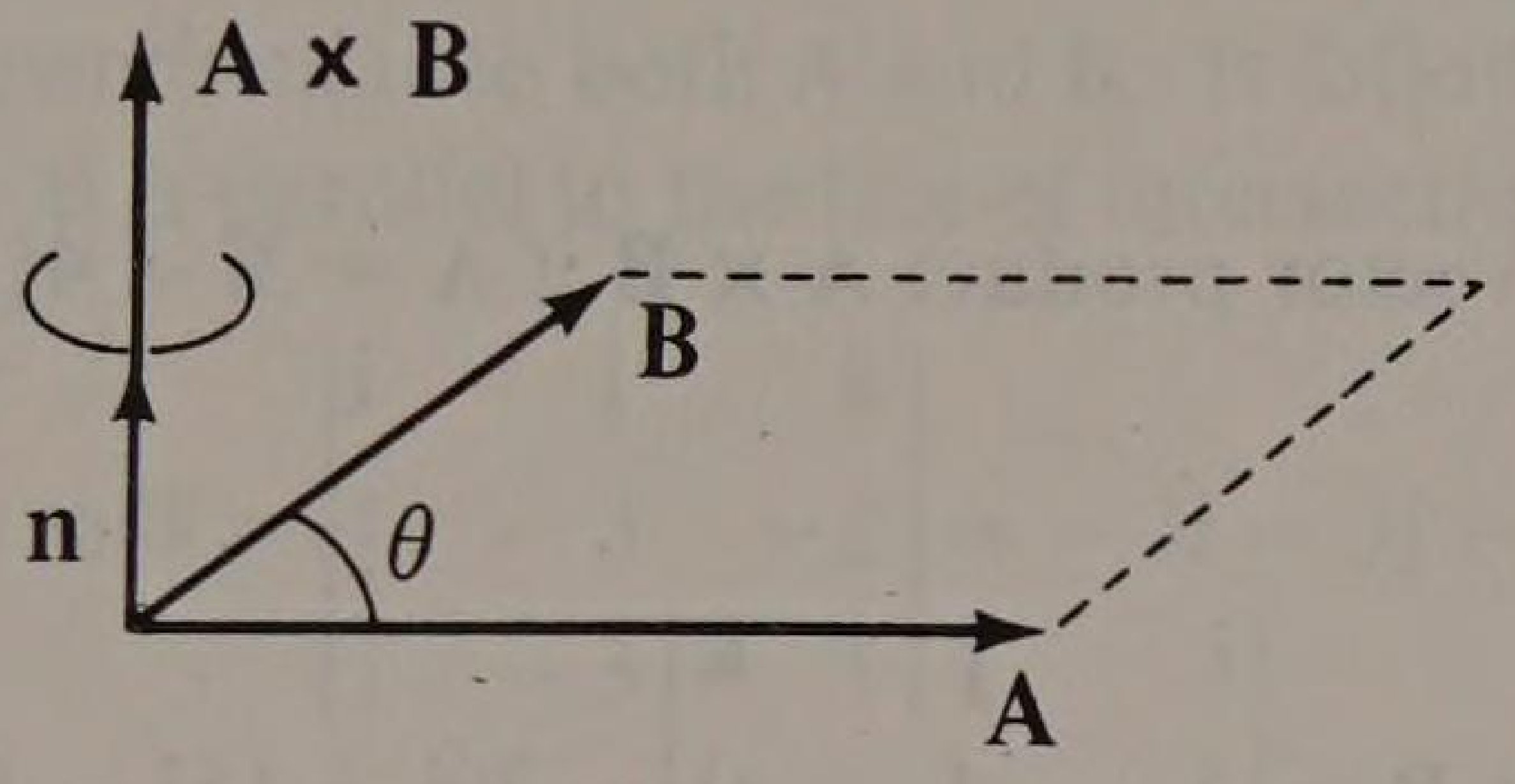

FIGURE 1.26

Notice that $|\mathbf{A} \times \mathbf{B}|$ is the area of the parallelogram determined by **A** and **B** (computed as base times height). Observe further that because of the rule determining the direction of **n**, we have

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

From these geometric considerations we see that *if two vectors are parallel, their vector product is zero.* Of course, **A** × **B** is also zero if either **A** or **B** is zero.

As with the scalar product, it is convenient to have a representation of **A** × **B** in terms of the components of **A** and **B**. The derivation of such a formula hinges on the validity of the distributive laws for the vector product, that is

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1.18}$$

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C} \tag{1.19}$$

The proof of the distributive laws appears at the end of this section as optional reading. If we accept the laws for now, we can compute the componentwise expression for **A** × **B** easily:

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \times (B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) \\ &= A_1\mathbf{i} \times B_1\mathbf{i} + A_2\mathbf{j} \times B_1\mathbf{i} + A_3\mathbf{k} \times B_1\mathbf{i} \\ &\quad + A_1\mathbf{i} \times B_2\mathbf{j} + A_2\mathbf{j} \times B_2\mathbf{j} + A_3\mathbf{k} \times B_2\mathbf{j} \\ &\quad + A_1\mathbf{i} \times B_3\mathbf{k} + A_2\mathbf{j} \times \mathrm{B}_3\mathbf{k} + A_3\mathbf{k} \times \mathrm{B}_3\mathbf{k}\end{aligned}$$

The vector products in this expression are easy to evaluate from the definition; $A_1\mathbf{i} \times B_1\mathbf{i} = \mathbf{0}$, $A_2\mathbf{j} \times B_1\mathbf{i} = -A_2B_1\mathbf{k}$, and so forth. Thus, we finally arrive at the componentwise expression for the vector product

$$\mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k} \quad (1.20)$$

This formula may be conveniently memorized in determinant form:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \quad (1.20')$$

This symbolic determinant is interpreted to be the vector whose x, y, and z components are the cofactors respectively of the first, second, and third entries in the first row.

Example 1.25 Find the vector product $\mathbf{A} \times \mathbf{B}$ if $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = \mathbf{i} + 5\mathbf{k} - 2\mathbf{j}$.

Solution

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 0 \\ 1 & -2 & 5 \end{vmatrix} = 20\mathbf{i} - 15\mathbf{j} - 10\mathbf{k}$$

For convenience, we list the algebraic properties of the vector product here:

$$\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$$
$$(s\mathbf{A} + \mathbf{B}) \times \mathbf{C} = s(\mathbf{A} \times \mathbf{C}) + (\mathbf{B} \times \mathbf{C})$$
$$\mathbf{A} \times (s\mathbf{B} + \mathbf{C}) = s(\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$$

Example 1.26 Find two unit vectors perpendicular to both $\mathbf{A} = 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Solution We have seen that $\mathbf{A} \times \mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. We have

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -3 \\ 1 & 3 & 1 \end{vmatrix} = 11\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$$

The length of this vector is $9\sqrt{2}$. The desired *unit* vector is therefore

$$\mathbf{n} = \frac{11}{9\sqrt{2}}\mathbf{i} - \frac{5}{9\sqrt{2}}\mathbf{j} + \frac{4}{9\sqrt{2}}\mathbf{k}$$

If we had taken $\mathbf{B} \times \mathbf{A}$ instead we would have obtained the negative of this vector. The two answers are

$$\pm\left(\frac{11\sqrt{2}}{18}\mathbf{i} - \frac{5\sqrt{2}}{18}\mathbf{j} + \frac{2\sqrt{2}}{9}\mathbf{k}\right)$$

Example 1.27 Find the area of the parallelogram determined by $\mathbf{A} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = 5\mathbf{k} - 6\mathbf{j}$.

Solution

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ 0 & -6 & 5 \end{vmatrix} = -13\mathbf{i} - 5\mathbf{j} - 6\mathbf{k}$$

$$|\mathbf{A} \times \mathbf{B}| = \sqrt{13^2 + 5^2 + 6^2} = \sqrt{230}$$

which is the desired area.

Example 1.28 Find the equations of the line passing through $(3,2,-4)$ parallel to the line of intersection of the two planes $x + 3y - 2z = 8, x - 3y + z = 0$.

Solution Observe that $\mathbf{A} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$ are the normals to the planes, and $\mathbf{A} \times \mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. It follows that $\mathbf{A} \times \mathbf{B}$ is parallel to both planes. Hence $\mathbf{A} \times \mathbf{B}$ is parallel to the line of intersection. We have

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 1 & -3 & 1 \end{vmatrix} = -3\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$$

Equations of the desired line are

$$\frac{x-3}{-3} = \frac{y-2}{-3} = \frac{z+4}{-6}$$

or, equivalently,

$$x - 3 = y - 2 = \frac{z+4}{2}$$

Now consider a rigid body rotating about a fixed axis with constant angular speed ω. The *angular velocity* is represented by a vector $\boldsymbol{\omega}$ of magnitude ω extending along the axis of rotation with sense determined by the right-hand rule: if the fingers of the right hand are wrapped about the axis in the direction of rotation, the thumb points in the direction of $\boldsymbol{\omega}$ (Fig. 1.27).

Let us assume that the origin O is on the axis of rotation, and let $\mathbf{R}$ denote the position vector of a particle in the body. Then the velocity $\mathbf{v}$ of the particle is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R} \tag{1.21}$$

To see this, we first note that $|\mathbf{R}| \sin \theta$ is the distance of the particle from the axis of rotation, so $\mathbf{v}$ has magnitude $\omega|\mathbf{R}| \sin \theta$. Moreover, the velocity $\mathbf{v}$ is necessarily perpendicular to both $\mathbf{R}$ and $\boldsymbol{\omega}$, and the sense of $\boldsymbol{\omega}$ is such that $\mathbf{v}$ equals $\boldsymbol{\omega} \times \mathbf{R}$ rather than $\mathbf{R} \times \boldsymbol{\omega}$, as we see from Fig. 1.27.

Example 1.29 A rigid body rotates with constant angular velocity ω about the line $x = y/2 = z/2$. Find the speed of a particle at the instant it passes through the point $(2,3,5)$.

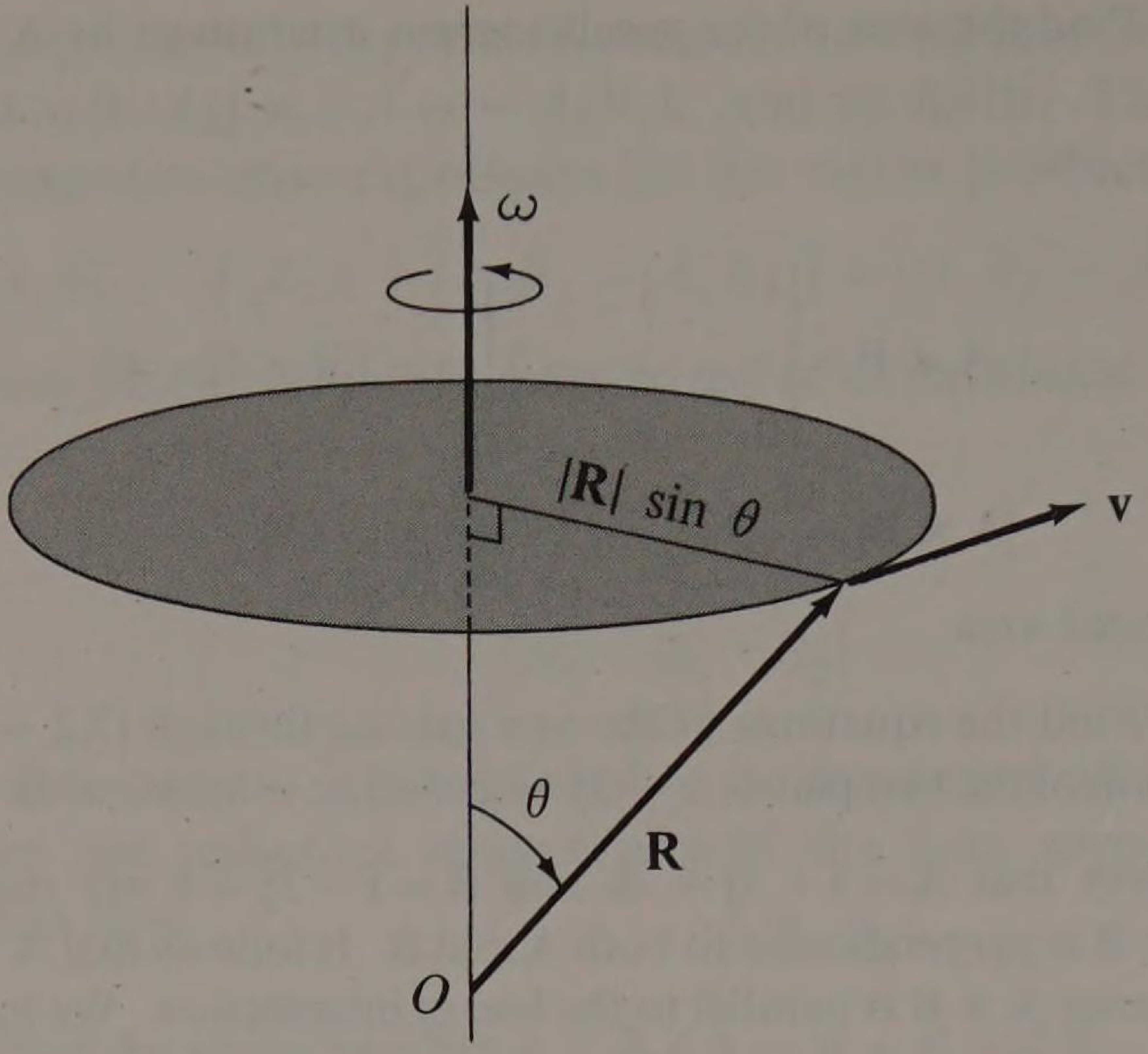

FIGURE 1.27

Solution The vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ is parallel to the axis. A unit vector parallel to the axis is $\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$. Therefore

$$\boldsymbol{\omega} = \pm\omega(\tfrac{1}{3}\mathbf{i} + \tfrac{2}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k})$$

(The statement of the problem leaves the sign ambiguous.) The velocity is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R} = \pm\omega\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 2 & 3 & 5 \end{vmatrix} = \pm\,\omega(\tfrac{4}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{1}{3}\mathbf{k})$$

The speed is

$$|\mathbf{v}| = \omega(\tfrac{16}{9} + \tfrac{1}{9} + \tfrac{1}{9})^{1/2} = \sqrt{2}\omega$$

OPTIONAL READING: THE PROOF OF THE DISTRIBUTIVE LAWS

Observe that we only have to prove Eq. (1.18); (1.19) will then follow since

$$\begin{aligned}(\mathbf{A} + \mathbf{B}) \times \mathbf{C} &= -\mathbf{C} \times (\mathbf{A} + \mathbf{B}) \\ &= -(\mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B}) \\ &= \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}\end{aligned}$$

We begin by proving (1.18) in the special case where **B** and **C** are both perpendicular to **A**; then, of course, (**B** + **C**) is also. In this case it follows from the definition of the vector product that **A** × **B**, for instance, is a vector that can be formed from the vector **B** by multiplying its length by the factor |**A**|, and rotating it counterclockwise through 90° about **A** as an axis. In

Fig. 1.28, think of the vector **A** as perpendicular to the page, pointing to the reader. Then **B**, **C**, and **B** + **C** all lie in the plane of the page, as do the "rescaled and rotated" vectors $\mathbf{A} \times \mathbf{B}$, $\mathbf{A} \times \mathbf{C}$, and $\mathbf{A} \times (\mathbf{B} + \mathbf{C})$.

Now Eq. (1.18) makes a claim about the sum of vectors; geometrically, it can be interpreted as saying that $\mathbf{A} \times (\mathbf{B} + \mathbf{C})$ is the diagonal of the parallelogram whose sides are $\mathbf{A} \times \mathbf{B}$ and $\mathbf{A} \times \mathbf{C}$. This can be seen by considering the similar triangles resulting from the equal angles and proportional sides in Fig. 1.28.

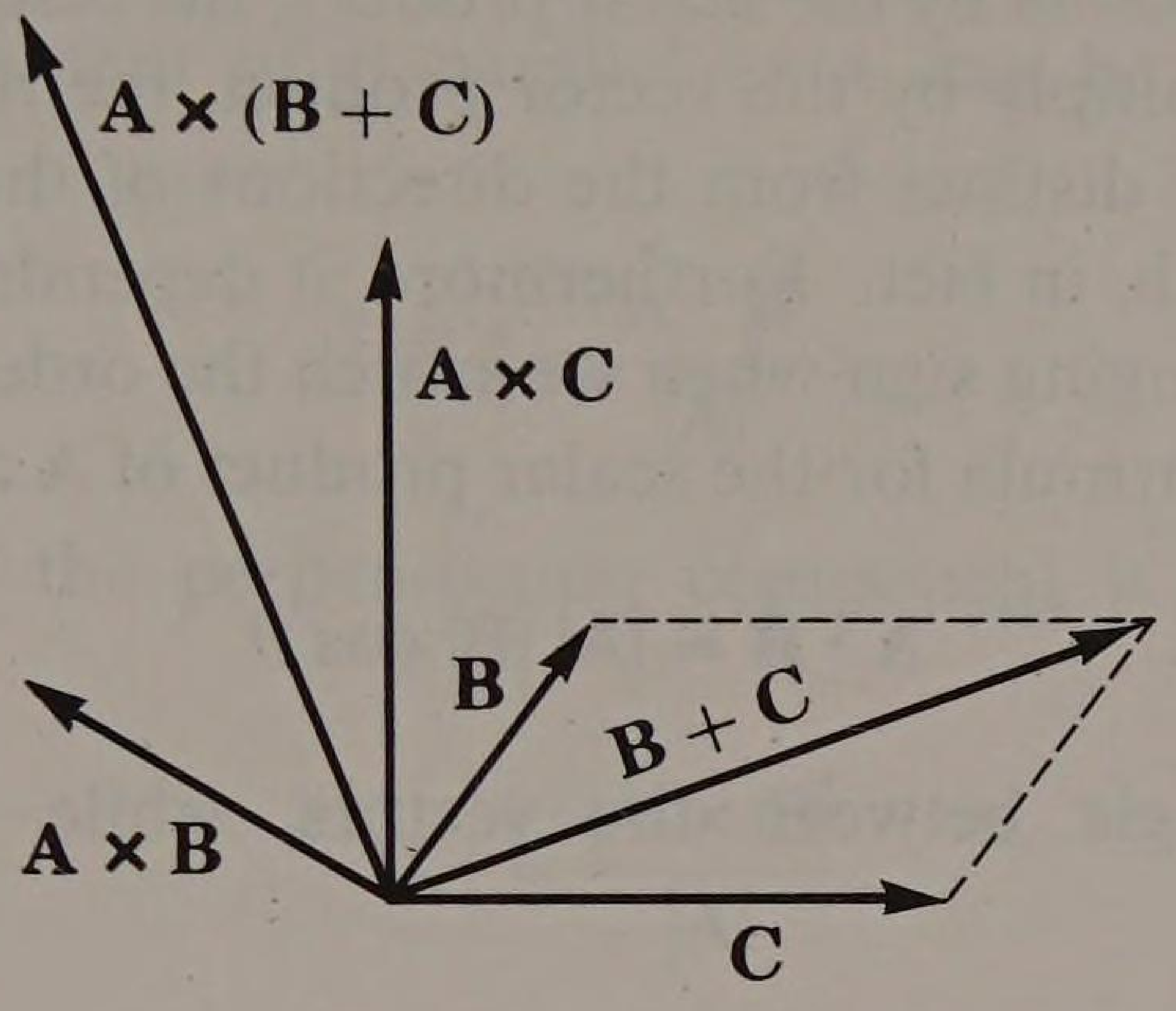

FIGURE 1.28

To prove Eq. (1.18) in the general case, with no assumptions about the directions of the vectors, we resolve **B** and **C** into their vector components parallel and perpendicular to **A**, as in Sec. 1.10 (recall Fig. 1.19):

$$\mathbf{B} = \mathbf{B}_{\|} + \mathbf{B}_{\perp} \qquad \mathbf{C} = \mathbf{C}_{\|} + \mathbf{C}_{\perp}$$

Then it follows from the definition of vector product that

$$\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{B}_{\perp} \qquad \mathbf{A} \times \mathbf{C} = \mathbf{A} \times \mathbf{C}_{\perp}$$

(Think this over: Neither the direction nor magnitude of $\mathbf{A} \times \mathbf{B}$ is changed if we replace **B** by $\mathbf{B}_{\perp}$). Furthermore, it is easy to see that the identity

$$\mathbf{B} + \mathbf{C} = (\mathbf{B}_{\|} + \mathbf{C}_{\|}) + (\mathbf{B}_{\perp} + \mathbf{C}_{\perp})$$

resolves the sum **B** + **C** into vector components parallel and perpendicular to **A**, and therefore

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times (\mathbf{B}_{\perp} + \mathbf{C}_{\perp})$$

Since we have proved the validity of Eq. (1.18) for vectors perpendicular to **A**, its general validity is seen as follows:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times (\mathbf{B}_{\perp} + \mathbf{C}_{\perp}) = \mathbf{A} \times \mathbf{B}_{\perp} + \mathbf{A} \times \mathbf{C}_{\perp} = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

(Another proof is outlined in Supplementary Problems 22 and 23 at the end of this chapter.)

SUMMARY—MULTIPLYING VECTORS

Now we have defined all the essential elements of vector algebra; let us review their interpretations and applications.

We started by learning how to add two vectors; the sum has the usual algebraic properties of commutativity and associativity, and it is compatible with scalar multiplication.

The effect of multiplying two vectors is rather more involved. We have defined two kinds of multiplication, and they have quite different properties. If we multiply two vectors by the scalar product, the result is not a vector—it is a scalar. If we multiply by the vector product, the result is a vector, but its direction is quite distinct from the directions of the original vectors—perpendicular to both, in fact. Furthermore, it depends on the order of the original vectors, changing sign when we switch the order.

The geometric formula for the scalar product of **A** and **B** is

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos \theta$$

where θ is the angle between the vectors, while the componentwise expression is

$$\mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$

For the vector product, the geometric formula is

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \sin \theta\, \mathbf{n}$$

where **n** is the unit vector perpendicular to **A** and **B** so that **A**, **B**, and **n** form a right-hand system; and the componentwise expression is most conveniently represented

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

From the geometric formulae we saw that a zero scalar product was a test for orthogonality, while a zero vector product was an indication of parallelism.

Example 1.30 Derive the nonparametric equations for the straight line passing through $\mathbf{R}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, parallel to $\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, using the vector product.

Solution Recall that in Sec. 1.8 we observed that $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ would be the position vector of a point on the line if $\mathbf{R} - \mathbf{R}_0$ was parallel to **V**. Setting the vector product equal to zero

$$(\mathbf{R} - \mathbf{R}_0) \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x - x_0 & y - y_0 & z - z_0 \\ a & b & c \end{vmatrix} = \mathbf{0}$$

we derive the equations

$$(x - x_0)b = (y - y_0)a$$
$$(y - y_0)c = (z - z_0)b$$
$$(x - x_0)c = (z - z_0)a$$

which are equivalent to Eqs. (1.9).

The geometric interpretations of the scalar and vector products can be visualized with the aid of the by-now familiar Fig. 1.29. We see that the length of the component of **B** parallel to **A** can be computed from the scalar product:

$$|\mathbf{B}_{||}| = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|}$$

while the length of the perpendicular component is computed from the vector product:

$$|\mathbf{B}_{\perp}| = \frac{|\mathbf{A} \times \mathbf{B}|}{|\mathbf{A}|}$$

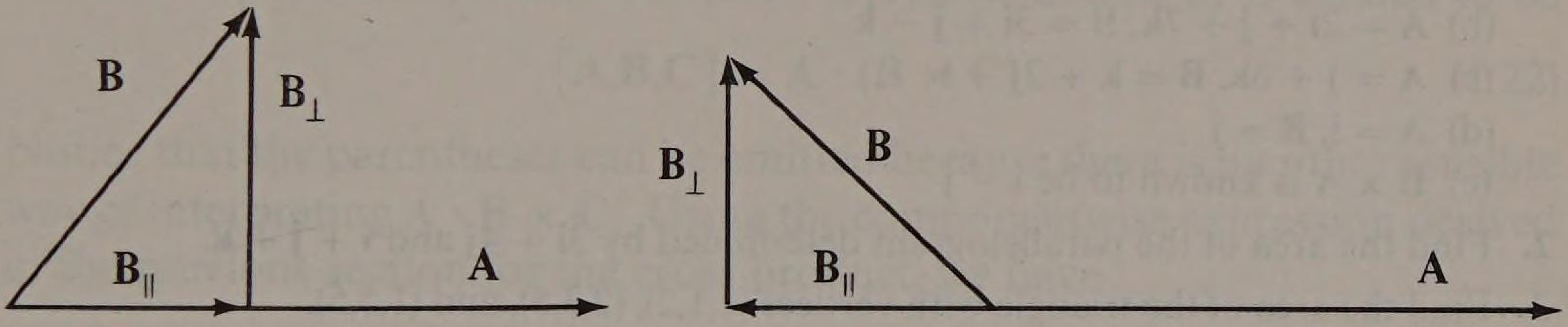

FIGURE 1.29

To express the *vectors* $\mathbf{B}_{||}$ and $\mathbf{B}_{\perp}$, we use a unit vector in the direction of **A**:

$$\mathbf{B}_{||} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|} \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A}$$

while $\mathbf{B}_{\perp}$ can be computed as the difference

$$\mathbf{B}_{\perp} = \mathbf{B} - \mathbf{B}_{||}$$

Example 1.31 Derive an expression for $\mathbf{B}_{\perp}$ directly in terms of **A** and **B**.

Solution Clearly we need an expression for a vector in the direction of $\mathbf{B}_{\perp}$. The key here is to analyze the vector $(\mathbf{A} \times \mathbf{B}) \times \mathbf{A}$.

Referring to Fig. 1.29, we see that $\mathbf{A} \times \mathbf{B}$ points toward the reader, perpendicular to the page. Now taking the vector product of this with **A**, we find that the resulting vector falls back in the plane of **A** and **B**, and in the direction of $\mathbf{B}_\perp$! Keeping in mind that the angle between $\mathbf{A} \times \mathbf{B}$ and **A** is 90°, we compute the length:

$$\begin{aligned} |(\mathbf{A} \times \mathbf{B}) \times \mathbf{A}| &= |\mathbf{A} \times \mathbf{B}|\,|\mathbf{A}| \sin 90^\circ \\ &= (|\mathbf{A}|\,|\mathbf{B}| \sin \theta)\,|\mathbf{A}|\,(1) \\ &= |\mathbf{A}|^2\,|\mathbf{B}| \sin \theta \end{aligned}$$

Since $|\mathbf{B}_\perp| = |\mathbf{B}| \sin \theta$, we have

$$\mathbf{B}_\perp = \frac{(\mathbf{A} \times \mathbf{B}) \times \mathbf{A}}{|\mathbf{A}|^2} = \frac{(\mathbf{A} \times \mathbf{B}) \times \mathbf{A}}{\mathbf{A} \cdot \mathbf{A}}$$

The parallel–perpendicular decomposition of **B** can thus be expressed

$$\mathbf{B} = \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}}\mathbf{A} + \frac{(\mathbf{A} \times \mathbf{B}) \times \mathbf{A}}{\mathbf{A} \cdot \mathbf{A}}$$

Exercises

1. Find $\mathbf{A} \times \mathbf{B}$ where
   (a) $\mathbf{A} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{B} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$
   (b) $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + 7\mathbf{k}$, $\mathbf{B} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$
   (c) $\mathbf{A} = \mathbf{j} + 6\mathbf{k}$, $\mathbf{B} = \mathbf{k} + 2\mathbf{j} - \mathbf{i}$
   (d) $\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{j}$
   (e) $\mathbf{B} \times \mathbf{A}$ is known to be $\mathbf{i} - \mathbf{j}$
2. Find the area of the parallelogram determined by $3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
3. Find the area of the triangle with vertices (1,1,2), (2,3,5), and (1,5,5).
4. Find $\mathbf{A} \times \mathbf{B}$ if $\mathbf{A} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$. What is the geometrical significance of this answer?
5. Find a unit vector perpendicular to both $3\mathbf{i} + \mathbf{j}$ and $2\mathbf{i} - \mathbf{j} - 5\mathbf{k}$.
6. By vector methods, find the equations of the line through (2,3,7) parallel to the line of intersection of the planes $2x + y + z = 0$, $x - y + 7z = 0$.
7. Find equations of a line perpendicular to the lines $x = y = z$, $x = 2y = 3z$, passing through the origin.
8. Compute $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ and also $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$, given that $\mathbf{A} = 2\mathbf{i} + 2\mathbf{j}$, $\mathbf{B} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, and $\mathbf{C} = 8\mathbf{i}$. Does the associative law hold for vector products?
9. By vector methods, determine the equation of the plane determined by points (2,0,1), (1,1,3), and (4,7,−2).
10. Find a unit vector in the plane of the vectors $\mathbf{A} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{B} = \mathbf{j} + 2\mathbf{k}$, perpendicular to the vector $\mathbf{C} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.
11. By taking the vector cross product of $(\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}$ and $(\cos \psi)\mathbf{i} + (\sin \psi)\mathbf{j}$ and interpreting geometrically, derive a well-known trigonometric identity.
12. If **A**, **B**, and **C** are vectors from the origin to points *A*, *B*, and *C* respectively, show that $(\mathbf{A} \times \mathbf{B}) + (\mathbf{B} \times \mathbf{C}) + (\mathbf{C} \times \mathbf{A})$ is perpendicular to the plane *ABC*. [*Hint*: Consider $(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})$.]

13. Find the distance from the point (5,7,14) to the line passing through (2,3,8) and (3,6,12). (*Hint*: Use a vector cross product.)
14. Find r and s if $(2\mathbf{i} + 6\mathbf{j} - 27\mathbf{k}) \times (\mathbf{i} + r\mathbf{j} + s\mathbf{k}) = \mathbf{0}$.
15. Given that $\mathbf{A} \cdot \mathbf{B} = 0$ and $\mathbf{A} \times \mathbf{B} = \mathbf{0}$, what can you conclude about the vectors **A** and **B**?
16. Given that **A** and **B** are parallel to the yz plane, that $|\mathbf{A}| = 2$, $|\mathbf{B}| = 4$, and $\mathbf{A} \cdot \mathbf{B} = 0$, What can you say about $\mathbf{A} \times \mathbf{B}$?
17. (a) Do the lines $x/3 = y/2 = z/2$ and $x/5 = y/3 = (z - 4)/2$ intersect?
    (b) Find equations for a line perpendicular to both of these lines.
    (c) What is the distance between these lines?
18. If $\boldsymbol{\omega}$ points in the direction of $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and the body rotates about an axis through the origin with angular velocity $10\sqrt{3}$ rad/sec, find the locus of points having speed 20 ft/sec. What does this locus represent?
19. Supply the missing details of the proof of the distributive law for vector products.
20. If **u**, **v**, and **w** are mutually perpendicular unit vectors and $\mathbf{u} \times \mathbf{v} = \mathbf{w}$, show that $\mathbf{v} = \mathbf{w} \times \mathbf{u}$ and $\mathbf{u} = \mathbf{v} \times \mathbf{w}$.

1.13 TRIPLE SCALAR PRODUCTS

The *triple scalar product* of three vectors, **A**, **B**, and **C**, is defined to be

$$[\mathbf{A},\mathbf{B},\mathbf{C}] = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \tag{1.22}$$

Notice that the parentheses can be omitted because there is no other sensible way of interpreting $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$. Using the componentwise expression derived in the previous section for the cross product, we have

$$\begin{aligned}[\mathbf{A},\mathbf{B},\mathbf{C}] = {} & A_1B_2C_3 - A_1B_3C_2 + A_2B_3C_1 \\ & - A_2B_1C_3 + A_3B_1C_2 - A_3B_2C_1\end{aligned} \tag{1.23}$$

Alternatively, from the determinant expression for the cross product we can express $[\mathbf{A},\mathbf{B},\mathbf{C}]$ in the form

$$[\mathbf{A},\mathbf{B},\mathbf{C}] = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \tag{1.23'}$$

The triple scalar product has a geometric interpretation. Consider the parallelepiped with **A**, **B**, and **C** as coterminal edges, as in Fig. 1.30. The base of this solid is a parallelogram whose area is given, as we saw previously, by $|\mathbf{B} \times \mathbf{C}|$. Its height is the length of the component of **A** perpendicular to the base, which can be regarded as the component of **A** *parallel* to $\mathbf{B} \times \mathbf{C}$, or $|\mathbf{A}| \cos \theta$, as shown in Fig. 1.30. To be precise we should say this height is the *magnitude* of $|\mathbf{A}| \cos \theta$, because $\cos \theta$ would be negative if **A** pointed to the opposite side of the plane of **B** and **C**, i.e., if **A**, **B**, and **C** formed a left-handed system. Thus we see that the volume of the parallelepiped, computed

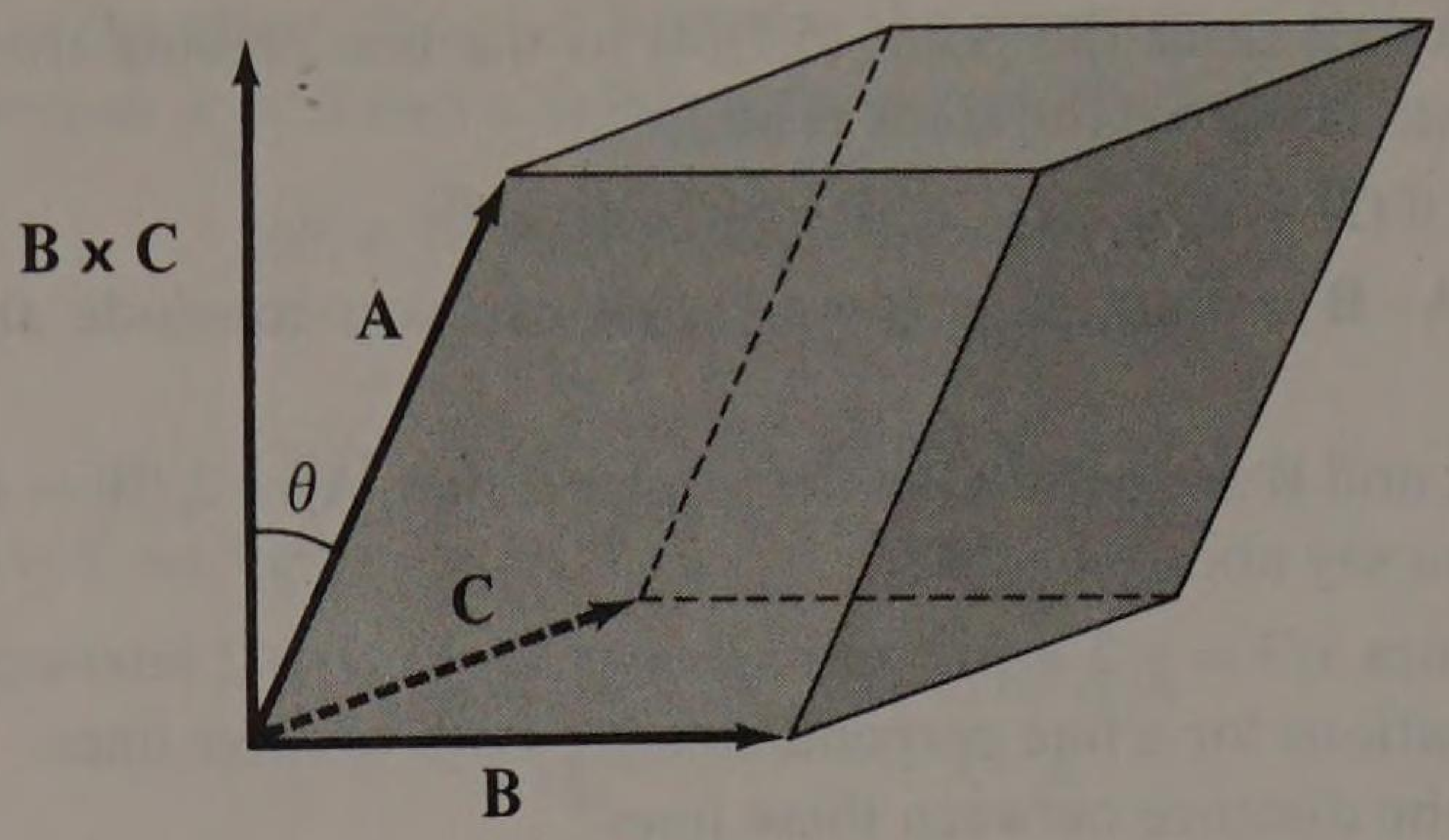

FIGURE 1.30

as base area times height, equals the magnitude of $|\mathbf{B} \times \mathbf{C}| \cos\theta\,|\mathbf{A}|$. But this is precisely $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$, the triple scalar product! Summarizing, we can say that *the volume of the parallelepiped with coterminal edges* **A**, **B**, **C** *is given, up to sign, by* $[\mathbf{A},\mathbf{B},\mathbf{C}]$. Furthermore, $[\mathbf{A},\mathbf{B},\mathbf{C}]$ *is positive if and only if* **A**, **B**, *and* **C** *form a right-handed system.*

Example 1.32 Compute $[\mathbf{A},\mathbf{B},\mathbf{C}]$ if $\mathbf{A} = 2\mathbf{i} + \mathbf{k}$, $\mathbf{B} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{C} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

Solution

$$[\mathbf{A},\mathbf{B},\mathbf{C}] = [2\mathbf{i} + \mathbf{k}, 3\mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{i} + \mathbf{j} + 4\mathbf{k}]$$

$$= \begin{vmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 8 + 3 - 1 - 2 = 8$$

Example 1.33 Compute $[\mathbf{i}, \mathbf{j}, \mathbf{i} + 2\mathbf{j}]$.

Solution

$$[\mathbf{i}, \mathbf{j}, \mathbf{i} + 2\mathbf{j}] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 0$$

(The vectors are coplanar, so the parallelepiped has zero volume.)

We now list some properties of the triple scalar product that can be verified from Eq. (1.23). They will be familiar to students who have studied determinants.

First, notice that the absolute value of the triple scalar product does not depend on the order of the vectors, but the sign changes whenever two of the vectors are switched:

$$[\mathbf{A},\mathbf{B},\mathbf{C}] = -[\mathbf{B},\mathbf{A},\mathbf{C}] = [\mathbf{B},\mathbf{C},\mathbf{A}] \tag{1.24}$$

This shows that the position of the dot and cross can be changed freely, because

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = [\mathbf{A},\mathbf{B},\mathbf{C}] = [\mathbf{C},\mathbf{A},\mathbf{B}] = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} \tag{1.25}$$

Second, the triple scalar product is linear in each of its factors:

$$\begin{aligned}[s\mathbf{A} + \mathbf{B}, \mathbf{C}, \mathbf{D}] &= s[\mathbf{A},\mathbf{C},\mathbf{D}] + [\mathbf{B},\mathbf{C},\mathbf{D}] \\ [\mathbf{A}, s\mathbf{B} + \mathbf{C}, \mathbf{D}] &= s[\mathbf{A},\mathbf{B},\mathbf{D}] + [\mathbf{A},\mathbf{C},\mathbf{D}] \\ [\mathbf{A}, \mathbf{B}, s\mathbf{C} + \mathbf{D}] &= s[\mathbf{A},\mathbf{B},\mathbf{C}] + [\mathbf{A},\mathbf{B},\mathbf{D}]\end{aligned} \tag{1.26}$$

Third, we have the obvious identity

$$[\mathbf{i},\mathbf{j},\mathbf{k}] = 1 \tag{1.27}$$

Clearly, if any two of the vectors **A**, **B**, or **C** are equal, the triple scalar product will be zero (the parallelepiped will have zero volume). Furthermore, if any one of the three vectors is replaced by the sum of that one vector with a linear combination of the other two, the triple scalar product is unchanged. For example, if we replace **A** by $\mathbf{A} + s\mathbf{B} + t\mathbf{C}$, where s and t are any numbers whatsoever, then $[\mathbf{A} + s\mathbf{B} + t\mathbf{C}, \mathbf{B}, \mathbf{C}] = [\mathbf{A},\mathbf{B},\mathbf{C}]$. The proof is easy:

$$[\mathbf{A} + s\mathbf{B} + t\mathbf{C}, \mathbf{B}, \mathbf{C}] = [\mathbf{A},\mathbf{B},\mathbf{C}] + s[\mathbf{B},\mathbf{B},\mathbf{C}] + t[\mathbf{C},\mathbf{B},\mathbf{C}]$$

and the last two terms are zero.

It is interesting to notice that these properties make it possible to evaluate any scalar triple product without using Eqs. (1.23) or (1.23′). For example, let $\mathbf{A} = \mathbf{i} + 3\mathbf{j}$, $\mathbf{B} = \mathbf{i} + \mathbf{k}$, and $\mathbf{C} = -\mathbf{k}$; then

$$\begin{aligned}[\mathbf{A},\mathbf{B},\mathbf{C}] &= [\mathbf{i} + 3\mathbf{j}, \mathbf{i} + \mathbf{k}, -\mathbf{k}] \\ &= [\mathbf{i}, \mathbf{i} + \mathbf{k}, -\mathbf{k}] + [3\mathbf{j}, \mathbf{i} + \mathbf{k}, -\mathbf{k}] \\ &= [\mathbf{i},\mathbf{i},-\mathbf{k}] + [\mathbf{i},\mathbf{k},-\mathbf{k}] + [3\mathbf{j},\mathbf{i},-\mathbf{k}] + [3\mathbf{j},\mathbf{k},-\mathbf{k}] \\ &= -[\mathbf{i},\mathbf{i},\mathbf{k}] - [\mathbf{i},\mathbf{k},\mathbf{k}] - 3[\mathbf{j},\mathbf{i},\mathbf{k}] - 3[\mathbf{j},\mathbf{k},\mathbf{k}] \\ &= -3[\mathbf{j},\mathbf{i},\mathbf{k}] = 3[\mathbf{i},\mathbf{j},\mathbf{k}] = 3\end{aligned}$$

As a final note, let us show how the scalar triple product can be used to relate the parametric and nonparametric equations of a plane derived in Sec. 1.10. The parametric equation was based on specifying a point in the plane with position vector $\mathbf{R}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, and two vectors **A** and **B** parallel to the plane. Clearly $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ will be the position vector of a point in the plane if the parallelepiped formed by $\mathbf{R} - \mathbf{R}_0$, **A**, and **B** is flat, i.e., has zero volume. Hence, the equation for this plane can be expressed

$$[\mathbf{R} - \mathbf{R}_0, \mathbf{A}, \mathbf{B}] = 0$$

Inserting the definition (1.22) for the triple scalar product, we identify $\mathbf{A} \times \mathbf{B}$ as being a vector **N** normal to the plane, and we have

$$(\mathbf{R} - \mathbf{R}_0) \cdot \mathbf{N} = 0$$

This agrees with Eq. (1.15), the nonparametric equation for the plane.

Exercises

1. Find the triple scalar product $[\mathbf{A},\mathbf{B},\mathbf{C}]$ given that
   (a) $\mathbf{A} = 2\mathbf{i}$, $\mathbf{B} = 3\mathbf{j}$, $\mathbf{C} = 5\mathbf{k}$
   (b) $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{B} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{C} = 5\mathbf{k} - \mathbf{j}$
   (c) $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{C} = 2\mathbf{i} + 3\mathbf{k}$
   (d) $\mathbf{A} = \mathbf{k}$, $\mathbf{B} = \mathbf{i}$, $\mathbf{C} = \mathbf{j}$
2. Find the volume of the parallelepiped whose coterminal edges are arrows representing the vectors $3\mathbf{i} + 4\mathbf{j}$, $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $5\mathbf{k}$.
3. Find the volume of the parallelepiped with coterminal edges AB, AC, and AD, where $A = (3,2,1)$, $B = (4,2,1)$, $C = (0,1,4)$, and $D = (0,0,7)$.
4. Find the volume of the tetrahedron with coterminal edges representing the vectors $\mathbf{i} + \mathbf{j}$, $\mathbf{i} - \mathbf{j}$, $2\mathbf{k}$. Illustrate with a sketch. (*Note*: The volume of the tetrahedron is one sixth the volume of the parallelepiped having the same coterminal edges.)
5. Find the area of the parallelogram in the plane with vertices at (0,0), (1,1), (3,4), (4,5). (*Hint*: Convert this to a three-dimensional problem, finding the volume of the parallelepiped with this parallelogram as base, taking the third edge to be of unit length along the z axis.)
6. Find the equation of the plane passing through the origin parallel to the vectors $\mathbf{A} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$.
7. Find the equation of the plane passing through $(3,4,-1)$ parallel to the vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{B} = \mathbf{i} - 3\mathbf{k}$.
8. (a) Show that the vectors $\mathbf{i} - \mathbf{j}$, $\mathbf{j} - \mathbf{k}$, $\mathbf{k} - \mathbf{i}$ are parallel to a plane.
   (b) Find an equation of the plane passing through the origin that is parallel to these three vectors.
9. Consider

$$\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k} \qquad \mathbf{B} = \mathbf{i}$$
$$\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$$

   (a) If $C_1 = 1$, $C_2 = 2$, find C_3 to make the three vectors coplanar.
   (b) If $C_2 = -1$ and $C_3 = 1$, show that no value of C_1 can be found to make the three vectors coplanar.
   (c) Discuss the geometrical reason for the result in part (b).
10. Find the altitude of a parallelepiped determined by $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, if the base is taken to be the parallelogram determined by $\mathbf{A}$ and $\mathbf{B}$, and if

$$\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$
$$\mathbf{B} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$
$$\mathbf{C} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

   (*Hint*: Think of the geometrical interpretation of $[\mathbf{A},\mathbf{B},\mathbf{C}]/|\mathbf{A} \times \mathbf{B}|$.)
11. Sketch the vectors $\mathbf{A} = \mathbf{i} + \mathbf{j}$, $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, and $\mathbf{C} = \mathbf{i} + 3\mathbf{k}$. Determine from your sketch whether or not $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in that order form a right-handed system. Check by computing the sign of $[\mathbf{A},\mathbf{B},\mathbf{C}]$.
12. What can you conclude about nonzero vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, given that $|(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}| + |(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{D}| = 0$?

13. (a) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be mutually perpendicular unit vectors, forming a right-handed system. Show that the vector $\mathbf{A} = \mathbf{i} \times \mathbf{u} + \mathbf{j} \times \mathbf{v} + \mathbf{k} \times \mathbf{w}$ makes the same angle with $\mathbf{i}$ that it does with $\mathbf{u}$.

(b) Find a vector extending along the axis of the rotation that carries $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ into $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ respectively.

14. Show that an arbitrary vector $\mathbf{V}$ can be expressed in terms of any three noncoplanar vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, according to

$$\mathbf{V} = \frac{[\mathbf{V},\mathbf{B},\mathbf{C}]}{[\mathbf{A},\mathbf{B},\mathbf{C}]}\mathbf{A} + \frac{[\mathbf{V},\mathbf{C},\mathbf{A}]}{[\mathbf{A},\mathbf{B},\mathbf{C}]}\mathbf{B} + \frac{[\mathbf{V},\mathbf{A},\mathbf{B}]}{[\mathbf{A},\mathbf{B},\mathbf{C}]}\mathbf{C} \tag{1.28}$$

(*Hint*: We know that $\mathbf{V}$ can be expressed as $a\mathbf{A} + b\mathbf{B} + c\mathbf{C}$; to find a, take the scalar product of $\mathbf{V}$ with $\mathbf{B} \times \mathbf{C}$.)

15. (*Review*) State which of the following have meaning. Do not evaluate. Assume $\mathbf{B} \neq 0$.

(a) $\mathbf{A} \times 5\mathbf{B}$
(b) $[\mathbf{A}, 3\mathbf{B}, \mathbf{C} - \mathbf{D}]$
(c) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$
(d) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \cdot \mathbf{D})$
(e) $(\mathbf{A} \cdot \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})$
(f) $(\mathbf{A} \cdot \mathbf{B}) \times (\mathbf{C} \cdot \mathbf{D})$
(g) $\mathbf{A} \times [(\mathbf{B} \cdot \mathbf{C})\mathbf{D}]$
(h) $\mathbf{A}/|\mathbf{B}|$
(i) $\mathbf{A}/\mathbf{B}$

1.14 VECTOR IDENTITIES

Of the following identities, the first is the most important because the other three can be derived from it fairly easily.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{1.29}$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A} \tag{1.30}$$

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{A},\mathbf{C},\mathbf{D}]\mathbf{B} - [\mathbf{B},\mathbf{C},\mathbf{D}]\mathbf{A} \tag{1.31}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{1.32}$$

In formula (1.29), if $\mathbf{V} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is not the zero vector, then it must be perpendicular to $\mathbf{B} \times \mathbf{C}$. Since $\mathbf{B} \times \mathbf{C}$ is itself perpendicular to both $\mathbf{B}$ and $\mathbf{C}$, it follows that $\mathbf{V}$ must be in the plane of $\mathbf{B}$ and $\mathbf{C}$, and since they are nonzero vectors that are not parallel (otherwise $\mathbf{V}$ would be the zero vector), $\mathbf{V}$ must be a linear combination of $\mathbf{B}$ and $\mathbf{C}$. Thus $\mathbf{V} = m\mathbf{B} + n\mathbf{C}$ for suitable scalars m and n. The fact that $m = \mathbf{A} \cdot \mathbf{C}$ and $n = -\mathbf{A} \cdot \mathbf{B}$ is not obvious, of course. The actual verification of (1.29) can be accomplished by working out the componentwise expression for each side of the equality. We leave this laborious computation to the energetic reader. (Or he can read Sec. 1.15.)

We suggest the following device for memorizing (1.29). As we observed, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ must be expressible as a linear combination of $\mathbf{B}$ and $\mathbf{C}$. If the student can only remember that the coefficients in this expression are scalar products of the other two vectors, and that the terms have opposite signs, he will be able to write

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \pm[(\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}]$$

To get the proper sign, use the familiar vectors **i**, **j**, and **k**; thus

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} = \pm[(\mathbf{i} \cdot \mathbf{j})\mathbf{i} - (\mathbf{i} \cdot \mathbf{i})\mathbf{j}]$$

so the plus sign is correct. [This also works for formula (1.30), of course.]

Formula (1.30) is easily proved by observing

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{C} \times (\mathbf{A} \times \mathbf{B})$$

and using (1.29) for the right-hand side.

To derive (1.31), let $\mathbf{U} = \mathbf{C} \times \mathbf{D}$, whence

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{U} = (\mathbf{A} \cdot \mathbf{U})\mathbf{B} - (\mathbf{B} \cdot \mathbf{U})\mathbf{A} = [\mathbf{A},\mathbf{C},\mathbf{D}]\mathbf{B} - [\mathbf{B},\mathbf{C},\mathbf{D}]\mathbf{A}$$

To derive (1.32),

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{U} = [\mathbf{A},\mathbf{B},\mathbf{U}] &= \mathbf{A} \cdot (\mathbf{B} \times \mathbf{U}) = \mathbf{A} \cdot [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] \\ &= \mathbf{A} \cdot [(\mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{B} \cdot \mathbf{C})\mathbf{D}] \\ &= (\mathbf{B} \cdot \mathbf{D})(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D})\end{aligned}$$

The reader is advised to attach a permanent bookmark in this section, as an aid in referring to the identities in the future.

Exercises

1. Derive the identity

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{A},\mathbf{B},\mathbf{D}]\mathbf{C} - [\mathbf{A},\mathbf{B},\mathbf{C}]\mathbf{D}$$

2. Derive the identity

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) = [\mathbf{A},\mathbf{B},\mathbf{C}]^2$$

3. Derive the identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$$

4. Verify formula (1.29) by working out the componentwise expression.
5. If the vector $\boldsymbol{\omega}$ in Fig. 1.27 is constant, then the acceleration of a particle with position vector **R** is $\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$. Simplify this expression.
6. Are any of the following identities generally valid for vectors?
   (a) $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}$
   (b) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$
   (c) $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$ if and only if $\mathbf{B} = \mathbf{C}$
   (d) $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ if and only if $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$
7. Simplify: $|\mathbf{A} \times \mathbf{B}|^2 + (\mathbf{A} \cdot \mathbf{B})^2 - |\mathbf{A}|^2\,|\mathbf{B}|^2$

1.15 OPTIONAL READING: TENSOR NOTATION

The use of distinguished symbols such as **A**, **A** × **B**, etc., to denote vectors and vector operations provides an excellent and often suggestive shorthand for expressing laws in geometry and physics. However, when we ultimately come down to the actual computations of a concrete problem, these expressions must be dealt with componentwise. Furthermore, the verification (and discovery!) of some of the more complicated vector identities such as those appearing in the previous section is often accomplished most efficiently by dealing with the components. In this section we shall introduce some notation which often facilitates this process; it is widely known as *tensor notation.* Although we do not intend to discuss tensors themselves here, we see no reason to designate the notational system by anything other than its proper name.

The boldface vector symbol **A** suggests, as we have said, a quantity with magnitude and direction; this quantity is equally well represented by three numbers, A_1, A_2, and A_3, the components of the vector. Every statement about the vector is actually a statement about its components. Thus $\mathbf{A} = \mathbf{B}$ means $A_1 = B_1$, $A_2 = B_2$, and $A_3 = B_3$; briefly,

$$A_i = B_i \qquad (i = 1,2,3) \tag{1.33}$$

Expressed simply, the basic idea in tensor notation is to try to write all vector equations in component form, but using dummy subscripts such as i in Eq. (1.33), rather than explicitly writing out the equation for the first component, then the second, then the third. (Whether or not this is always possible will not be discussed here. For now, we will be satisfied with using tensor notation *when we can.*) We will indicate the components of a vector **A** by A_i, or $(\mathbf{A})_i$ if it is more convenient; we shall regard the parenthetical phrase "($i = 1,2,3$)" as understood, and delete it. Let's try some examples.

The expression of the fact that vectors add componentwise becomes

$$(\mathbf{A} + \mathbf{B})_i = A_i + B_i$$

That is, the ith component of $\mathbf{A} + \mathbf{B}$ is the sum of the ith components of **A** and of **B**. Scalar multiplication is expressed

$$(s\mathbf{A})_i = sA_i$$

The associative law for vectors, expressed componentwise, merely reduces to the associative law for *numbers*:

$$\begin{aligned}[(\mathbf{A} + \mathbf{B}) + \mathbf{C}]_i &= (\mathbf{A} + \mathbf{B})_i + C_i \\ &= (A_i + B_i) + C_i \\ &= A_i + (B_i + C_i) \\ &= [\mathbf{A} + (\mathbf{B} + \mathbf{C})]_i\end{aligned}$$

The condition that **R** lie on the line through the tip of **V** and parallel to **W** is expressed

$$R_i = V_i + tW_i$$

For the scalar product, we have

$$\mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$

This can be compacted by using the Greek letter Σ to denote summation. For any set of n numbers $\{a_l\}$ ($l = 1, 2, \ldots, n$), we abbreviate

$$a_1 + a_2 + \cdots + a_n$$

by the expression

$$\sum_{l=1}^{n} a_l$$

Thus our scalar product becomes

$$\mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^{3} A_iB_i \tag{1.34}$$

The componentwise expression of the cross product is a bit complicated. Observe that each component of $\mathbf{A} \times \mathbf{B}$ is a sum of products of components of $\mathbf{A}$ times components of $\mathbf{B}$. If we (conceptually) form all the products $\{A_jB_k\}$, we can say that the ith component of $\mathbf{A} \times \mathbf{B}$ is a linear combination of these with coefficients $+1$, -1, or 0 (if the term doesn't actually appear). So by defining ε_{ijk} appropriately, we can write

$$(\mathbf{A} \times \mathbf{B})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk}A_jB_k \tag{1.35}$$

ε_{ijk} is the coefficient of A_jB_k in the ith component of $\mathbf{A} \times \mathbf{B}$. Comparison of this with expression (1.20) in Sec. 1.12 shows

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (ijk) \text{ is either (123), (231), or (312)} \\ -1 & \text{if } (ijk) \text{ is either (321), (213), or (132)} \\ 0 & \text{otherwise} \end{cases} \tag{1.36}$$

In fact, ε_{ijk} is the coefficient of $\lambda_i\mu_j\eta_k$ in the determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix}$$

a fact which we could have anticipated by comparing the expression (1.35) with the determinant formula for the cross product in Sec. 1.12.

A few observations about the symbol ε_{ijk} are in order.

(*i*) $\varepsilon_{ijk} = 0$ if any of the subscripts are equal.
(*ii*) $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$, i.e., the subscripts can be permuted cyclically.
(*iii*) $\varepsilon_{ijk} = -\varepsilon_{jik}$, i.e., the sign changes if two subscripts are switched.

Of course, the scalar product $\mathbf{A} \cdot \mathbf{B}$ is also composed of products of $\mathbf{A}$'s components with $\mathbf{B}$'s components, and if the expression (1.34) weren't so simple already, we would write it

$$\mathbf{A}\cdot\mathbf{B}=\sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}A_iB_j$$

where

$$\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}\tag{1.37}$$

The effect of δ_{ij} in an expression is simple if the subscripts are summed; since $\delta_{ij}=0$ unless $i=j$, one can merely drop the δ and substitute i for j. Thus,

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}A_iB_j=\sum_{j=1}^{3}A_jB_j\qquad(=\mathbf{A}\cdot\mathbf{B})$$

For this reason, δ_{ij} is sometimes called the *substitution tensor*. It's also known as the "Kronecker delta."

Example 1.34 Show that the triple scalar product can be computed as a determinant.

Solution In the expression $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$, we first use tensor notation for the scalar product:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\sum_{i=1}^{3}A_i(\mathbf{B}\times\mathbf{C})_i$$

Now using (1.35) for the vector product,

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}&=\sum_{i=1}^{3}A_i\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}B_jC_k\\&=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}A_iB_jC_k\end{aligned}\tag{1.38}$$

As we observed earlier, this is the expansion of the determinant

$$\begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}$$

Notice that every time we have used the summation symbol Σ, the subscript over which we were summing occurred *twice* in the term expressing the addend; i is repeated in (1.34), j and k are repeated in (1.35), etc. This happens so often that the following convention is used in tensor notation: *whenever a subscript appears more than once in a single term, it is understood that this particular term is to be summed over all values* (1,2, and 3) *of the repeated subscript.* Scalar products are thus written A_iB_i, and the ith component of $\mathbf{A}\times\mathbf{B}$ is $\varepsilon_{ijk}A_jB_k$. In fact, we have $\delta_{ii}=3$. *Exceptions to this rule must be explicitly indicated.*

The manipulation of expressions involving more than one cross product is aided by the following identity:

$$\varepsilon_{ikm}\varepsilon_{psm}=\delta_{ip}\delta_{ks}-\delta_{is}\delta_{kp}\tag{1.39}$$

(Observe that m is summed over.) To prove this, we notice that the right hand side is zero unless it has the form $1 - 0$ or $0 - 1$. Thus

$$\delta_{ip}\delta_{ks} - \delta_{is}\delta_{kp} = \begin{cases} 1 & \text{if } i = p \text{ and } k = s \text{ but } i \neq s \text{ (or } k \neq p) \\ -1 & \text{if } i = s \text{ and } k = p \text{ but } i \neq p \text{ (or } k \neq s) \\ 0 & \text{otherwise} \end{cases}$$

On the left of (1.39), ε is zero unless all subscripts are different; in which case, $i \neq k$ and $p \neq s$ and m must be different from i, p, k, or s. So, there is actually only one (at most) nonzero term in the sum over values of m! The product is $+1$ if (ikm) is a cyclic permutation of (psm), which can only happen if $i = p$ and $k = s$; and -1 results if (ikm) is in the opposite order as (psm), which requires $i = s$ and $k = p$. Comparing these conditions, we see that the left and right-hand sides of (1.39) are equal.

Example 1.35 Simplify $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

Solution The ith component is

$$\begin{aligned} \varepsilon_{ijk}A_j(\mathbf{B} \times \mathbf{C})_k &= \varepsilon_{ijk}A_j\varepsilon_{klm}B_lC_m \\ &= \varepsilon_{ijk}\varepsilon_{klm}A_jB_lC_m \end{aligned}$$

Remember that we are summing over repeated subscripts. First we sum over k. Since the other terms do not depend on k, we can compute $\varepsilon_{ijk}\varepsilon_{klm}$; this the same as $\varepsilon_{ijk}\varepsilon_{lmk}$ which, by Eq. (1.39), is $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$. So the above expression equals

$$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_jB_lC_m$$

Now sum out the substitution tensors one at a time. Summing over m we get

$$\delta_{il}A_jB_lC_j - \delta_{jl}A_jB_lC_i$$

and, summing over l,

$$A_jC_jB_i - A_jB_jC_i$$

Identifying the scalar products, we recognize that this is the ith component of $(\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$. We have proved formula (1.29)!

Example 1.36 Simplify $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})$.

Solution The ith component is

$$\varepsilon_{ijk}(\mathbf{A} \times \mathbf{B})_j(\mathbf{C} \times \mathbf{D})_k = \varepsilon_{ijk}\varepsilon_{jmn}A_mB_n\varepsilon_{kpq}C_pD_q$$

If we sum over j first, only the first two factors are involved. Rewriting them as $\varepsilon_{kij}\varepsilon_{mnj}$, we use (1.39) to transform the expression to

$$(\delta_{km}\delta_{in} - \delta_{kn}\delta_{im})A_mB_n\varepsilon_{kpq}C_pD_q$$

Summing over the subscripts of the substitution tensors is easy, yielding

$$A_kB_i\varepsilon_{kpq}C_pD_q - A_iB_k\varepsilon_{kpq}C_pD_q$$

Now we identify $\varepsilon_{kpq}A_kC_pD_q$ as the triple scalar product, recalling Eq. (1.38); similarly for $\varepsilon_{kpq}B_kC_pD_q$. So we are left with

$$[\mathbf{A},\mathbf{C},\mathbf{D}]B_i - [\mathbf{B},\mathbf{C},\mathbf{D}]A_i$$

which is the ith component of $[\mathbf{A},\mathbf{C},\mathbf{D}]\mathbf{B} - [\mathbf{B},\mathbf{C},\mathbf{D}]\mathbf{A}$. We have "discovered" formula (1.31).

Exercises

1. Simplify $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.
2. Simplify $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$.
3. Simplify $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A})$.

Supplementary Problems

1. If the vector $\mathbf{V} = 2\mathbf{i} + 3\mathbf{j}$ represents the segment AB, and the midpoint of AB is (2,1), find A and B.
2. If $\mathbf{V}$ is a unit vector in the xy plane making an angle of 30° with the positive y axis, express $\mathbf{V}$ in terms of $\mathbf{i}$ and $\mathbf{j}$ (*two solutions*).
3. Derive a formula for a vector that bisects the angle between two vectors $\mathbf{A}$ and $\mathbf{B}$.
4. Determine s and t so that $\mathbf{C} - s\mathbf{A} - t\mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, given that

$$\mathbf{A} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$$
$$\mathbf{B} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$$
$$\mathbf{C} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$$

5. Prove that $|\mathbf{A}|\mathbf{B} + |\mathbf{B}|\mathbf{A}$ is orthogonal to $|\mathbf{A}|\mathbf{B} - |\mathbf{B}|\mathbf{A}$, for any vectors $\mathbf{A}$ and $\mathbf{B}$.
6. Consider the cube in Fig. 1.31. Find the angles between:
   (a) the face diagonals AB and AC,
   (b) the principal diagonal AD and the face diagonal AB, and
   (c) the principal diagonal AD and the edge AE.
7. Let

$$\mathbf{A} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$
$$\mathbf{B} = 4\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$
$$\mathbf{C} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

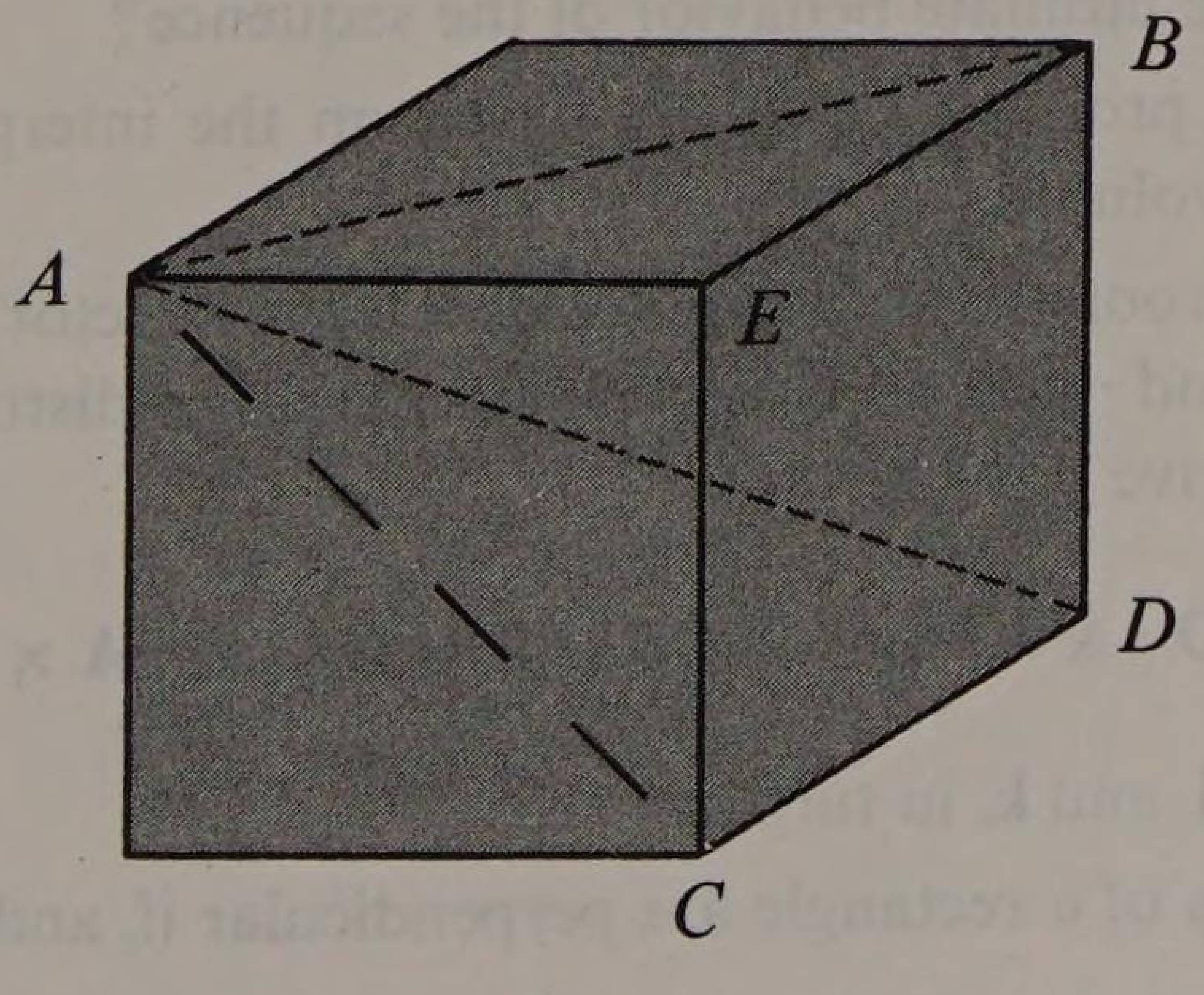

FIGURE 1.31

Find $\mathbf{A}\times\mathbf{B}$, $[\mathbf{A},\mathbf{B},\mathbf{C}]$, $|\mathbf{A}\times\mathbf{B}|$, and the distance from the tip of $\mathbf{C}$ to the plane through the origin spanned by $\mathbf{A}$ and $\mathbf{B}$.

8. Given $\mathbf{A}\neq\mathbf{0}$, is it true that $\mathbf{A}\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{C}$ and $\mathbf{A}\times\mathbf{B}=\mathbf{A}\times\mathbf{C}$ imply $\mathbf{B}=\mathbf{C}$?

9. Prove, for any vector $\mathbf{A}$,

$$\mathbf{i}\times(\mathbf{i}\times\mathbf{A})+\mathbf{j}\times(\mathbf{j}\times\mathbf{A})+\mathbf{k}\times(\mathbf{k}\times\mathbf{A})=-2\mathbf{A}$$

10. Prove: if $\mathbf{A}+\mathbf{B}+\mathbf{C}=\mathbf{0}$, then $\mathbf{A}\times\mathbf{B}=\mathbf{B}\times\mathbf{C}=\mathbf{C}\times\mathbf{A}$. Interpret geometrically.

11. Simplify $[\mathbf{A}\times(\mathbf{A}\times\mathbf{B})]\times\mathbf{A}\cdot\mathbf{C}$.

12. Express $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ as the sum of a vector parallel to $\mathbf{V}$, plus a vector perpendicular to $\mathbf{V}$, with $\mathbf{V}=2\mathbf{i}+4\mathbf{j}-2\mathbf{k}$.

13. Suppose the line ℓ_1 passes through the points $(5,1,-2)$ and $(2,-3,1)$, and the line ℓ_2 passes through $(3,8,1)$ and $(-3,0,7)$. Are these lines perpendicular, parallel, coincident, or none of these?

14. Find the point(s) of intersection of the lines

$$\ell_1\colon\quad \mathbf{R}=2\mathbf{i}+3\mathbf{j}+3\mathbf{k}+t(\mathbf{i}-2\mathbf{j}+5\mathbf{k})$$

$$\ell_2\colon\quad \frac{x+3}{2}=\frac{y+1}{3}=-z$$

15. What is the distance from the origin to the plane intersecting the x, y, and z axes at $x=a$, $y=b$, and $z=c$, respectively?

16. Find the distance between $(1,2,3)$ and the plane $2x-2y+z=4$.

17. Find the distance between the planes $2x+y+z=2$ and $2x+y+z=4$.

18. Under what conditions can one find a unique vector $\mathbf{X}$ that solves both equations

$$\mathbf{A}\times\mathbf{X}=\mathbf{B},\qquad \mathbf{C}\cdot\mathbf{X}=s$$

19. Prove the following theorem of Desargues. Given two (nondegenerate) triangles ABC and DEF with the property that the line through AD, the line through BE, and the line through CF have a point in common; moreover, let the lines through AB and DE intersect at P, the lines through BC and EF intersect at Q, and the lines through AC and DF intersect at R. Then P, Q, and R are collinear.

20. Prove the converse of Desargues' theorem in Exercise 19.

21. Starting with arbitrary $\mathbf{A}_1$ and $\mathbf{B}$, define the sequence of vectors $\mathbf{A}_n$ by $\mathbf{A}_{n+1}=\mathbf{B}\times\mathbf{A}_n$. What is the ultimate behavior of the sequence?

22. Devise a geometric proof of Eq. (1.25) based on the interpretation of the triple scalar product as a volume.

23. Construct another proof of the distributive law for the vector product, based on the interchange of $\times$ and $\cdot$ (see previous exercise) and the distributivity of the scalar product. (*Hint*: Derive the identity

$$\mathbf{D}\cdot\mathbf{A}\times(\mathbf{B}+\mathbf{C})=\mathbf{D}\cdot\mathbf{A}\times\mathbf{B}+\mathbf{D}\cdot\mathbf{A}\times\mathbf{C}$$

and then let $\mathbf{D}$ be $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, in turn.)

24. Prove: the diagonals of a rectangle are perpendicular if, and only if, the rectangle is a square.

25. Describe the set of points located by **R** such that

$$(\mathbf{R} - \mathbf{a}) \cdot (\mathbf{R} + \mathbf{a}) = 0$$

where **a** is fixed. (*Hint*: Draw a diagram.)

26. Given two nonintersecting lines

$$\frac{x}{2} = y = \frac{z-1}{3} \qquad \frac{x}{3} = y = z$$

find points P on the first line and Q on the second so that PQ is perpendicular to both lines.

27. Prove: the sum of the squares of the sides of any quadrilateral, minus the sum of the squares of the two diagonals, equals four times the square of the distance between the midpoints of the diagonals.

CHAPTER 2

Vector Functions of a Single Variable

2.1 DIFFERENTIATION

The theory of vector functions parallels that of real-valued functions. A vector-valued function $\mathbf{F}(t)$ is a rule that associates a vector $\mathbf{F}$ with each real number t in some set, usually an interval $(t_1 \leq t \leq t_2)$ or a collection of intervals. For example, $\mathbf{F}(t) = (1/t)\mathbf{i}$ is defined for $-\infty < t < 0$ and $0 < t < \infty$.

The concept of a limit can be applied to vector functions. The expression

$$\lim_{t \to t_0} \mathbf{F}(t) = \mathbf{A} \tag{2.1}$$

means that, given any positive number ε, no matter how small, one can find a positive number δ such that $|\mathbf{F}(t) - \mathbf{A}| < \varepsilon$ whenever $0 < |t - t_0| < \delta$.

This has a simple intuitive meaning. It means that the magnitude of $\mathbf{F}(t)$ is approaching the magnitude of $\mathbf{A}$, and that (if $\mathbf{A}$ is nonzero) the angle between them is approaching zero (see Fig. 2.1). Equivalently, the *components* of $\mathbf{F}(t)$ are approaching the *components* of $\mathbf{A}$.

The definition just given is identical to that given in calculus books for real-valued functions, except that the expression $|\mathbf{F}(t) - \mathbf{A}|$ now refers to the magnitude of a vector rather than to the absolute value of a number.

A vector function $\mathbf{F}$ is said to be *continuous* at t_0 if

$$\lim_{t \to t_0} \mathbf{F}(t) = \mathbf{F}(t_0) \tag{2.2}$$

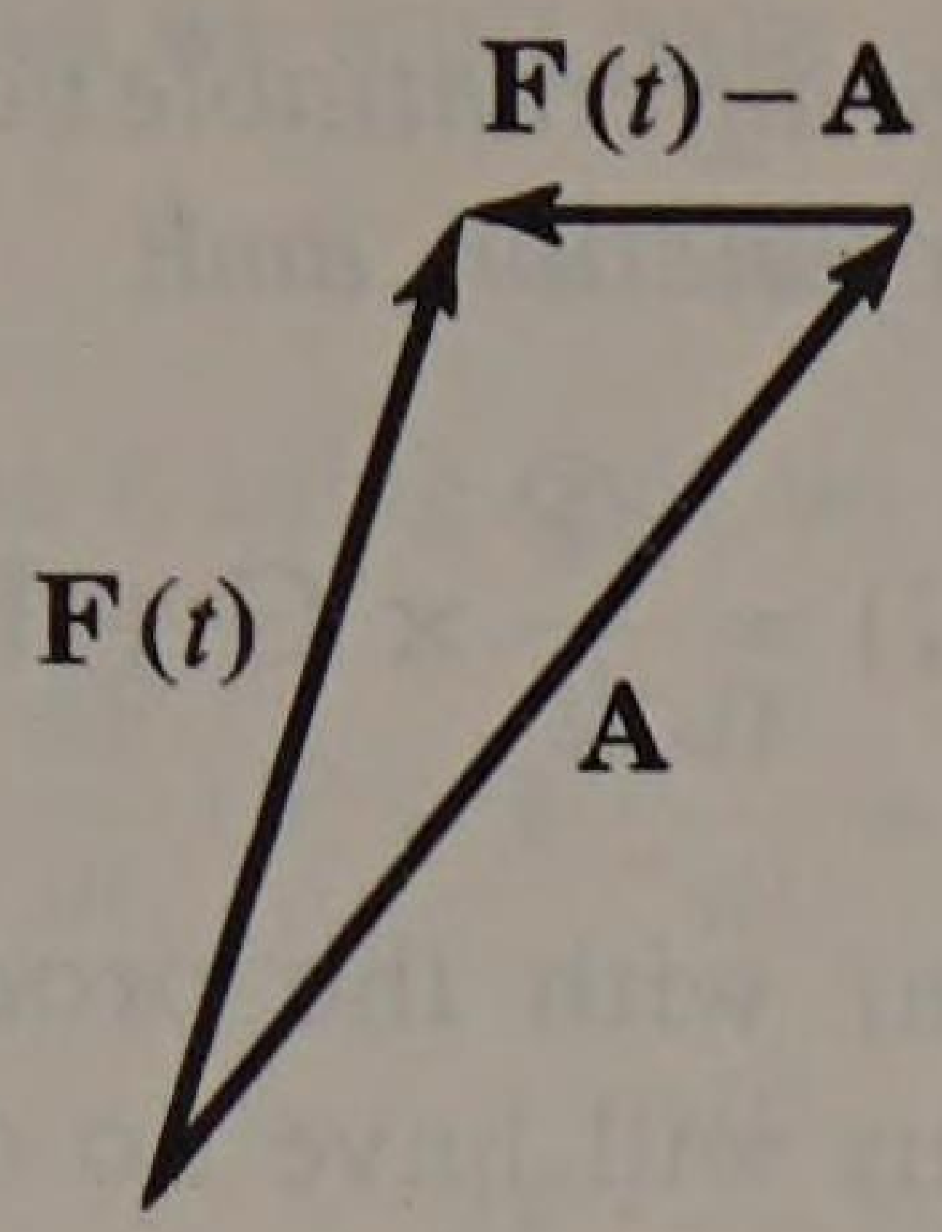

FIGURE 2.1

It is said to be *differentiable* at t_0 if the limit

$$\lim_{\Delta t \to 0} \frac{\mathbf{F}(t_0 + \Delta t) - \mathbf{F}(t_0)}{\Delta t} \tag{2.3}$$

exists; this limit is then called the *derivative of* $\mathbf{F}(t)$ *at* t_0 and is written $\mathbf{F}'(t_0)$ or $\dfrac{d\mathbf{F}}{dt}(t_0)$. The derivative is also a vector function.

If $\mathbf{F}(t)$ is continuous (or differentiable) at *every* point t for which it is defined, we shall simply say $\mathbf{F}(t)$ *is continuous* (or *differentiable*).

The fundamental theorems concerning differentiation of vector-valued functions are similar to those for real-valued functions, except that when differentiating the vector product of two vector functions, one must be careful to preserve the order of factors, since the vector product is not a commutative operation.

THEOREM 2.1 *If* **F** *and* **G** *are differentiable vector functions, then so also is their sum* **F** + **G**, *and the derivative of the function* **F** + **G** *is the sum of the derivatives of* **F** *and* **G** *respectively,*

$$\frac{d}{dt}(\mathbf{F} + \mathbf{G}) = \frac{d\mathbf{F}}{dt} + \frac{d\mathbf{G}}{dt} \tag{2.4}$$

THEOREM 2.2 *If* **F** *is a differentiable vector function, and s is a differentiable scalar function, then the product s***F** *is a differentiable vector function, and*

$$\frac{d}{dt}(s\mathbf{F}) = \frac{ds}{dt}\mathbf{F} + s\frac{d\mathbf{F}}{dt} \tag{2.5}$$

THEOREM 2.3 *If* **F** *and* **G** *are differentiable vector functions, then* **F** · **G** *is a differentiable scalar function, and*

$$\frac{d}{dt}(\mathbf{F} \cdot \mathbf{G}) = \frac{d\mathbf{F}}{dt} \cdot \mathbf{G} + \mathbf{F} \cdot \frac{d\mathbf{G}}{dt} \tag{2.6}$$

THEOREM 2.4 *If* **F** *and* **G** *are differentiable vector functions, then* $\mathbf{F} \times \mathbf{G}$ *is also a differentiable vector function, and*

$$\frac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \frac{d\mathbf{F}}{dt} \times \mathbf{G} + \mathbf{F} \times \frac{d\mathbf{G}}{dt} \tag{2.7}$$

The reader who is familiar with the proofs of the sum and product formulas of elementary calculus will have no difficulty filling in the proofs of these theorems.

Example 2.1 Prove Theorem 2.4.

Solution With the definition of the derivative in mind, we write

$$\frac{\mathbf{F}(t+\Delta t) \times \mathbf{G}(t+\Delta t) - \mathbf{F}(t) \times \mathbf{G}(t)}{\Delta t}$$

$$= \frac{[\mathbf{F}(t+\Delta t) - \mathbf{F}(t)] \times \mathbf{G}(t+\Delta t)}{\Delta t} + \frac{\mathbf{F}(t) \times [\mathbf{G}(t+\Delta t) - \mathbf{G}(t)]}{\Delta t}$$

As $\Delta t \to 0$, the right-hand side approaches a limiting value given by

$$\frac{d\mathbf{F}}{dt} \times \mathbf{G} + \mathbf{F} \times \frac{d\mathbf{G}}{dt}$$

Since the limit of the left-hand side is $\frac{d}{dt}(\mathbf{F} \times \mathbf{G})$, we have proved the theorem.

It follows from (2.4) and (2.5) that if

$$\mathbf{F}(t) = P(t)\mathbf{i} + Q(t)\mathbf{j} + R(t)\mathbf{k}$$

then

$$\mathbf{F}'(t) = P'(t)\mathbf{i} + Q'(t)\mathbf{j} + R'(t)\mathbf{k} \tag{2.8}$$

Thus, vector differentiation is like scalar differentiation, treating **i**, **j**, and **k** as constants.

Example 2.2 Let $\mathbf{F}(t) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Then **F** is a constant vector-valued function, and its derivative with respect to t is identically equal to the zero vector for all t.

Example 2.3 If $\mathbf{F}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\mathbf{k}$, then $\mathbf{F}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \mathbf{k}$

Example 2.4 If $\mathbf{F}(t) = t^3\mathbf{j} - \mathbf{k}$, then $\mathbf{F}'(t) = 3t^2\mathbf{j}$.

Example 2.5 If $\mathbf{F}'(t) = \mathbf{0}$, then $\mathbf{F}(t) = \mathbf{C}$, where the constant **C** is a *vector*.

Example 2.6 Prove that, if $\mathbf{F}(t)$ has constant nonzero magnitude (varies only in direction), then $\mathbf{F}'(t)$ is either the zero vector or it is a nonzero vector perpendicular to $\mathbf{F}(t)$.

Solution If $|\mathbf{F}(t)| = \text{constant}$, then we must have

$$\mathbf{F} \cdot \mathbf{F} = \text{constant}$$

and differentiating with respect to t, using (2.6), we have

$$\frac{d\mathbf{F}}{dt} \cdot \mathbf{F} + \mathbf{F} \cdot \frac{d\mathbf{F}}{dt} = 0$$

$$2\mathbf{F} \cdot \frac{d\mathbf{F}}{dt} = 0$$

Hence the scalar product of **F** with $d\mathbf{F}/dt$ is identically zero. This can happen only if the vectors **F** and $d\mathbf{F}/dt$ are perpendicular, or if one of them is the zero vector. This fact is well worth remembering: *the derivative of a vector of constant length is perpendicular to the vector, or zero.*

Exercises

1. Let $\mathbf{F}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$.
  (a) Find $\mathbf{F}'(t)$.
  (b) Show that $\mathbf{F}'(t)$ is always parallel to the xy plane.
  (c) For what values of t is $\mathbf{F}'(t)$ parallel to the xz plane?
  (d) Does $\mathbf{F}(t)$ have constant magnitude?
  (e) Does $\mathbf{F}'(t)$ have constant magnitude?
  (f) Compute $\mathbf{F}''(t)$.

2. Find $\mathbf{F}'(t)$ in each of the following cases.
  (a) $\mathbf{F}(t) = 3t\mathbf{i} + t^3\mathbf{j}$
  (b) $\mathbf{F}(t) = \sin t\,\mathbf{i} + e^{-t}\mathbf{j} + 3\mathbf{k}$
  (c) $\mathbf{F}(t) = (e^t\mathbf{i} + \mathbf{j} + t^2\mathbf{k}) \times (t^3\mathbf{i} + \mathbf{j} - \mathbf{k})$
  (d) $\mathbf{F}(t) = (\sin t + t^3)(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$
  (e) $\mathbf{F}(t) = 3\mathbf{i} + \mathbf{k}$

3. Find $f'(t)$ in each of the following cases.
  (a) $f(t) = (3t\mathbf{i} + 5t^2\mathbf{j}) \cdot (t\mathbf{i} - \sin t\,\mathbf{j})$
  (b) $f(t) = |2t\mathbf{i} + 2t\mathbf{j} - \mathbf{k}|$
  (c) $f(t) = [(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \times (3t^4\mathbf{i} + t\mathbf{j})] \cdot \mathbf{k}$

4. Show that $\dfrac{d}{dt}\left(\mathbf{R} \times \dfrac{d\mathbf{R}}{dt}\right) = \mathbf{R} \times \dfrac{d^2\mathbf{R}}{dt^2}$.

5. Given the three vectors $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$, $\mathbf{B} = 3\mathbf{i} + 4\mathbf{k}$, and $\mathbf{C} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, evaluate
  (a) $|\mathbf{A}|$
  (b) $\mathbf{A} \cdot \mathbf{B}$
  (c) $\mathbf{B} \times \mathbf{C}$
  (d) $\mathbf{B} \cdot \mathbf{B} \times \mathbf{C}$
  (e) $[\mathbf{A},\mathbf{B},\mathbf{C}]$
  (f) $\mathbf{A}/|\mathbf{B}|$
  (g) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$
  (h) $\dfrac{d}{dt}(\mathbf{A} + \mathbf{B}t)$
  (i) $\dfrac{d}{dt}(\mathbf{B} \times t\mathbf{C})$

2.2 SPACE CURVES, VELOCITIES, AND TANGENTS

In the first chapter, we showed that the parametric equations of a line can be written in vector form

$$\mathbf{R} = \mathbf{R}_0 + t\mathbf{V} \tag{2.9}$$

Here $\mathbf{R}_0$ is the position vector of a fixed point on the line, $\mathbf{V}$ is parallel to the line, and as t assumes values from $-\infty$ to $+\infty$, the tip of the vector $\mathbf{R}$ traces out the line in (x,y,z) space. We can also regard (2.9) as defining $\mathbf{R}$ as a vector function of t (whose derivative is, of course, $\mathbf{V}$).

In this section we shall consider equations of the form $\mathbf{R} = \mathbf{R}(t)$ where the function $\mathbf{R}(t)$ is more complicated than Eq. (2.9). Of course, the equation $\mathbf{R} = \mathbf{R}(t)$ can be written out in terms of its components, giving the system of equations

$$\begin{aligned} x &= x(t) \\ y &= y(t) \\ z &= z(t) \end{aligned} \tag{2.10}$$

where x, y, and z are simply real-valued functions of t.

As t increases from its initial value t_1 to the value t_2, the point (x,y,z) [i.e., the tip of the position vector $\mathbf{R}(t)$] traces out some geometric object in space. In the case of Eq. (2.9), the object is a segment of a straight line. For more complicated (continuous) vector functions, this locus of points will be some more general kind of one-dimensional object which we can call a *space curve* or an *arc*. [We say it's *one*-dimensional because any point on it can be located, via the continuous function $\mathbf{R}(t)$, by specifying the *single* number t.] We use the term "curve" even if the trace of $\mathbf{R}(t)$ is a straight line.

Thus we have associated with every continuous vector function $\mathbf{R}(t)$ a curve in space, which is the set of values assumed by $\mathbf{R}(t)$ as t varies over an interval. Notice that this is quite different from *graphing* x, y, and/or z as functions of t; the *curve* traced by $\mathbf{R}(t)$ is a threadlike collection of points in (x,y,z) space. One cannot read, from the curve alone, the value of t corresponding to a given point. So the curve itself contains much less information than the *function* $\mathbf{R}(t)$. Remember that as far as the curve is concerned, t is a sort of invisible dummy variable, which we have glamorized by awarding it the officious title "parameter."

Remember also that there are any number of different *parametrizations* for a given curve. For example, if $\mathbf{W} = \frac{1}{2}\mathbf{V}$, the function

$$\mathbf{R}_1(t) = \mathbf{R}_0 + t\mathbf{W}$$

traces out exactly the same straight line as (2.9) for $-\infty < t < \infty$, as does the function

$$\mathbf{R}_2(t) = \mathbf{R}_0 + \tan t\,\mathbf{W}$$

for $-\pi/2 < t < \pi/2$. Thus we must keep in mind that many different *functions* may parametrize the same *curve*.

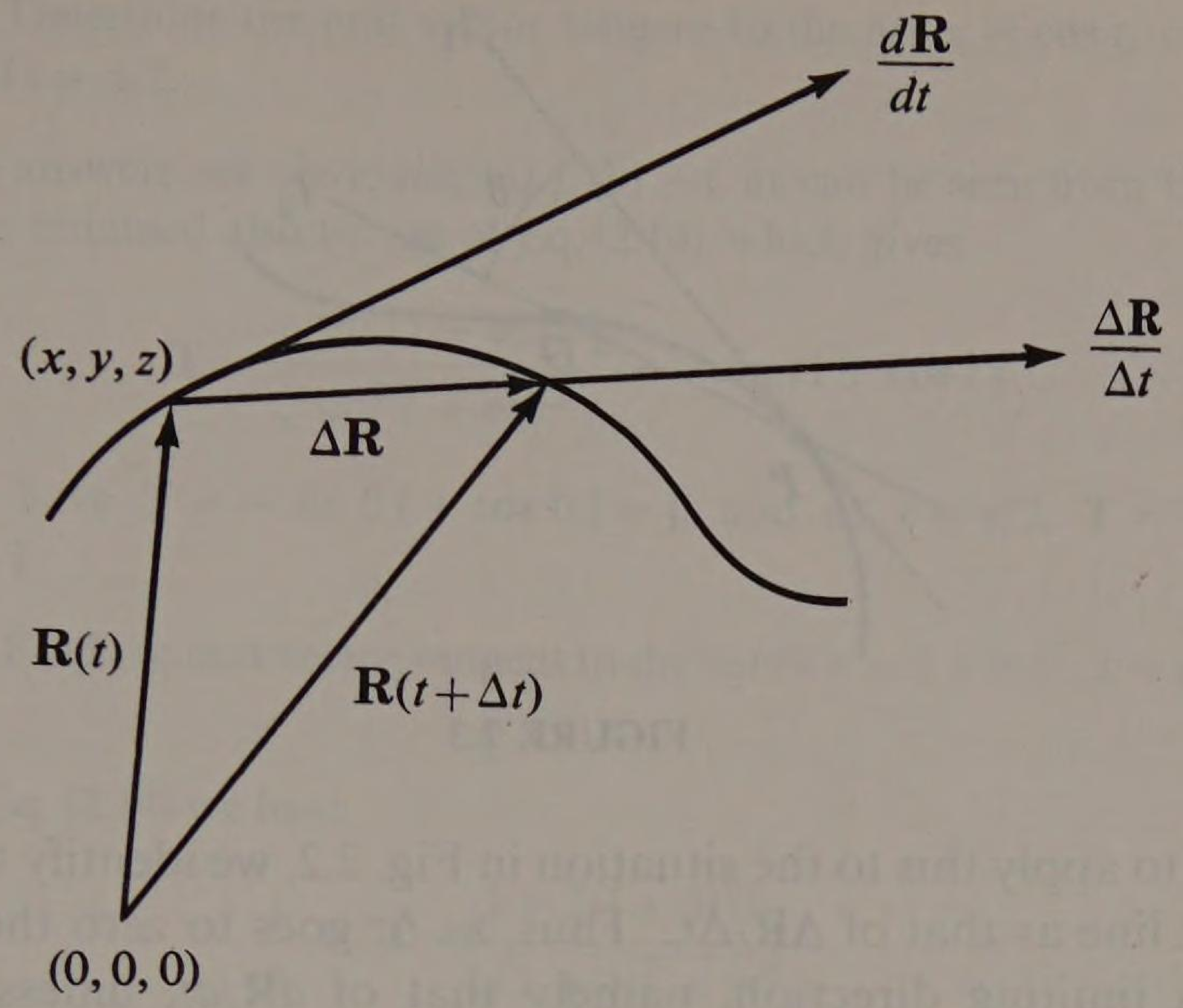

FIGURE 2.2

It is especially useful to think of (x,y,z) as the location of a particle moving through space, with the parameter t representing time. During a time interval of duration Δt, the position vector of the particle changes from the value $\mathbf{R}(t)$ to a new value $\mathbf{R}(t+\Delta t)$. The *displacement* of the particle during this interval of time is

$$\Delta\mathbf{R} = \mathbf{R}(t+\Delta t) - \mathbf{R}(t) = \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j} + \Delta z\,\mathbf{k} \tag{2.11}$$

If the displacement is divided by the scalar Δt, we obtain the *average velocity* of the particle during the time interval,

$$\frac{\Delta\mathbf{R}}{\Delta t} = \frac{\Delta x}{\Delta t}\mathbf{i} + \frac{\Delta y}{\Delta t}\mathbf{j} + \frac{\Delta z}{\Delta t}\mathbf{k} \tag{2.12}$$

(In Fig. 2.2, we take Δt less than unity; hence the vector $\Delta\mathbf{R}/\Delta t$ is greater in magnitude than $\Delta\mathbf{R}$.)

If $\mathbf{R}$ is differentiable, the average velocity $\Delta\mathbf{R}/\Delta t$ tends to a limit as Δt tends to zero. This limit is, by definition, the (instantaneous) *velocity* $\mathbf{v}$:

$$\mathbf{v}(t) = \mathbf{R}'(t) = \frac{d\mathbf{R}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \tag{2.13}$$

The magnitude of $\mathbf{v}$ is called the *speed*; it may be denoted v.

Figure 2.2 seems to indicate that the velocity vector $d\mathbf{R}/dt$ is tangent to the curve. Let us explore this further. Referring to Fig. 2.3, we say informally that the line ℓ_1 is tangent to the curve at P if the angle θ, between ℓ_1 and the secant line ℓ_2 determined by P and Q, goes to zero as Q approaches P along the curve; that is, the direction of the secant line of ℓ_2 approaches that of ℓ_1 as Q approaches P.

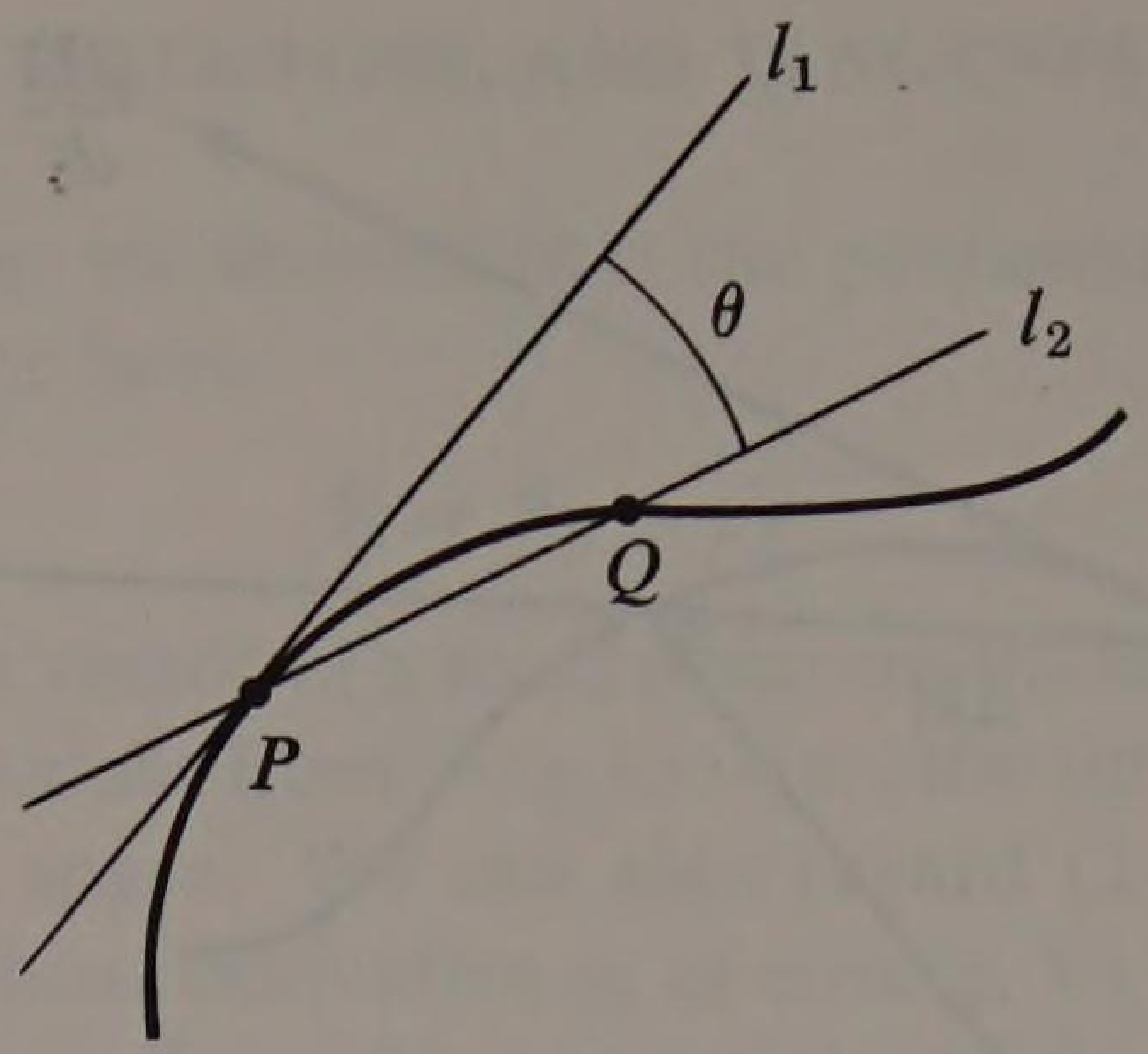

FIGURE 2.3

If we try to apply this to the situation in Fig. 2.2, we identify the direction of the secant line as that of $\Delta\mathbf{R}/\Delta t$. Thus, as Δt goes to zero the secant line must have a limiting direction, namely that of $d\mathbf{R}/dt$, unless, of course, the latter is the zero vector, which has no direction. We have shown the following: if the vector function $\mathbf{R}(t)$ has a nonzero derivative at t_0, then the curve parametrized by $\mathbf{R} = \mathbf{R}(t)$ has a tangent at $\mathbf{R}(t_0)$ whose direction coincides with that of $d\mathbf{R}/dt$. In short, *$d\mathbf{R}/dt$ is tangent to the curve.*

It is conventional to denote by the letter $\mathbf{T}$ a *unit vector* tangent to a curve. Such a $\mathbf{T}$ is defined by the expression

$$\mathbf{T} = \frac{(dx/dt)\mathbf{i} + (dy/dt)\mathbf{j} + (dz/dt)\mathbf{k}}{\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}} \tag{2.14}$$

obtained by dividing the above vector by its own magnitude.

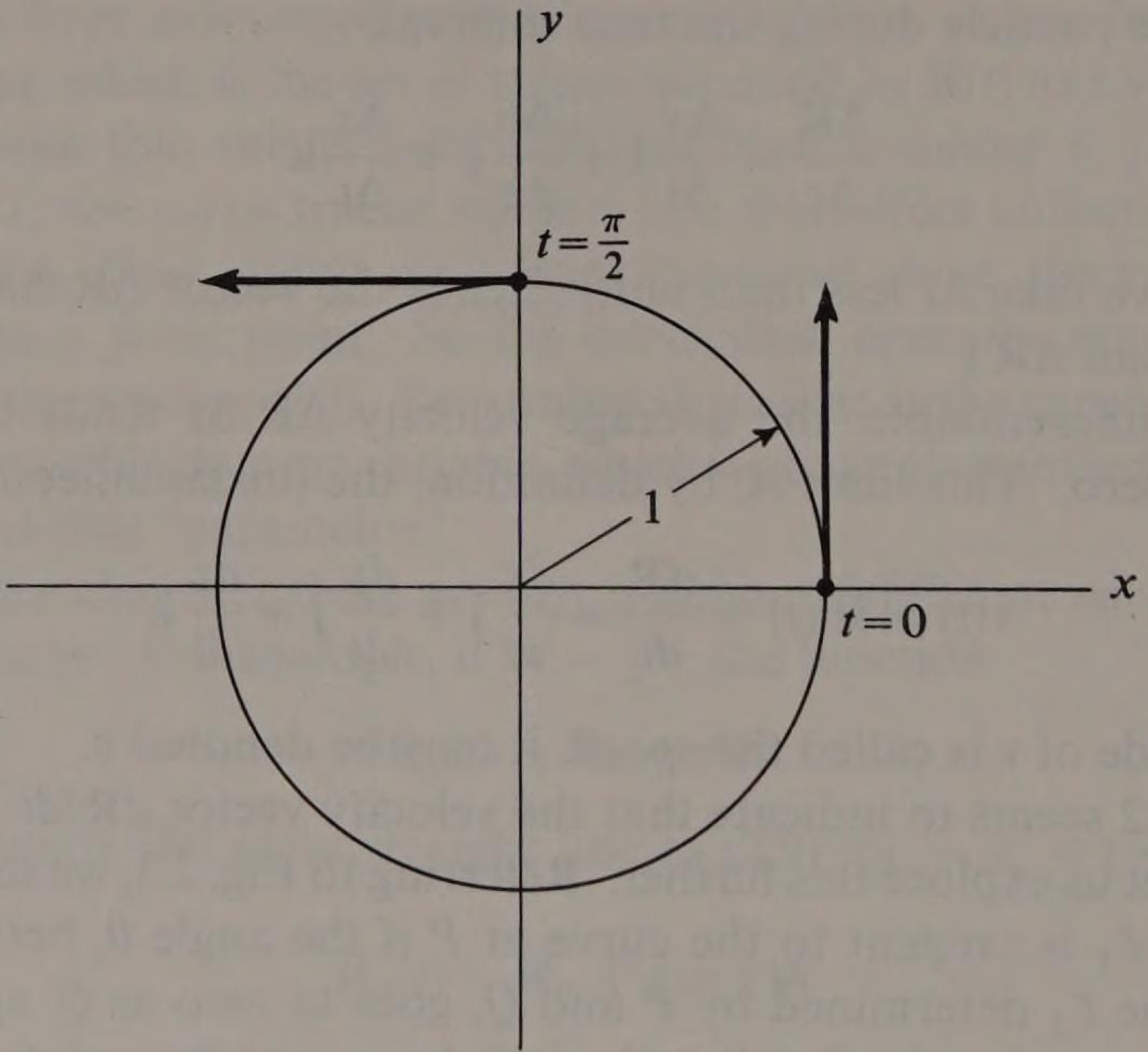

FIGURE 2.4

Example 2.7 Determine the unit vector tangent to the arc $x = \cos t$, $y = \sin t$, $z = 0$, at (a) $t = 0$; (b) $t = \pi/2$.

Solution The answers are obviously (a) $\mathbf{j}$, (b) $-\mathbf{i}$, as can be seen from Fig. 2.4. These answers can be obtained also by use of Eq. (2.14), which gives

$$\mathbf{T} = \frac{-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}}{\sqrt{\sin^2 t + \cos^2 t}} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

At $t = 0$, we have $\mathbf{T} = -\sin 0\,\mathbf{i} + \cos 0\,\mathbf{j} = \mathbf{j}$, and at $t = \pi/2$, $\mathbf{T} = -\sin(\pi/2)\mathbf{i} + \cos(\pi/2)\mathbf{j} = -\mathbf{i}$.

Example 2.8 Find the unit vector tangent to the curve $x = t$, $y = t^2$, $z = t^3$, at the point (2,4,8).

Solution By Eq. (2.14) we have

$$\mathbf{T} = \frac{\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}}{\sqrt{1 + 4t^2 + 9t^4}}$$

When $t = 2$ we have $(x, y, z) = (2,4,8)$ and $\mathbf{T} = (1/\sqrt{161})(\mathbf{i} + 4\mathbf{j} + 12\mathbf{k})$.

We now wish to introduce some nomenclature for describing the curves pictured in Figs. 2.5 to 2.8. The simplest of these is the one in Fig. 2.5, which has a continuously turning tangent at every point and no self-intersections. From the above discussion we can see how to guarantee these properties.

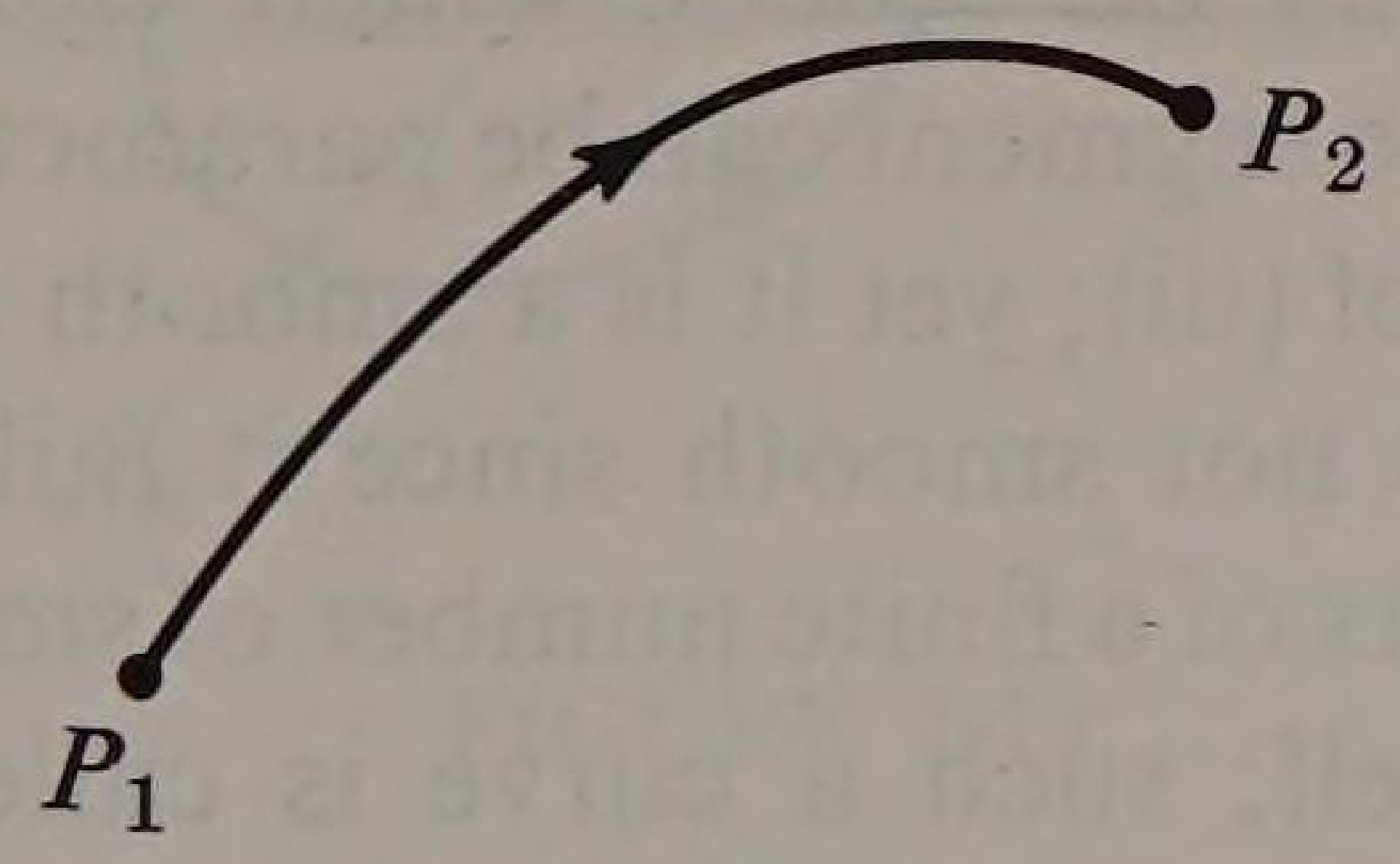

FIGURE 2.5

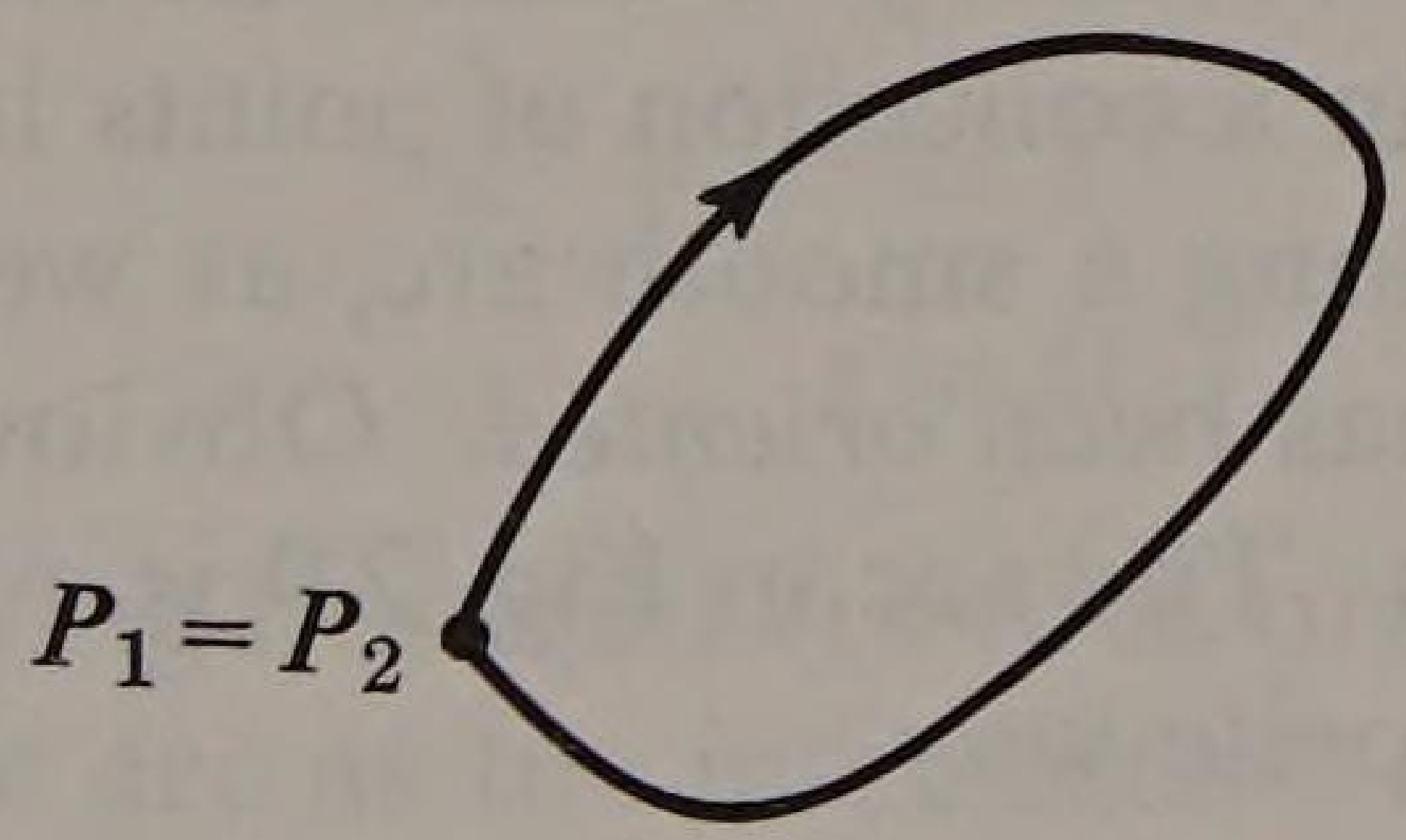

FIGURE 2.6

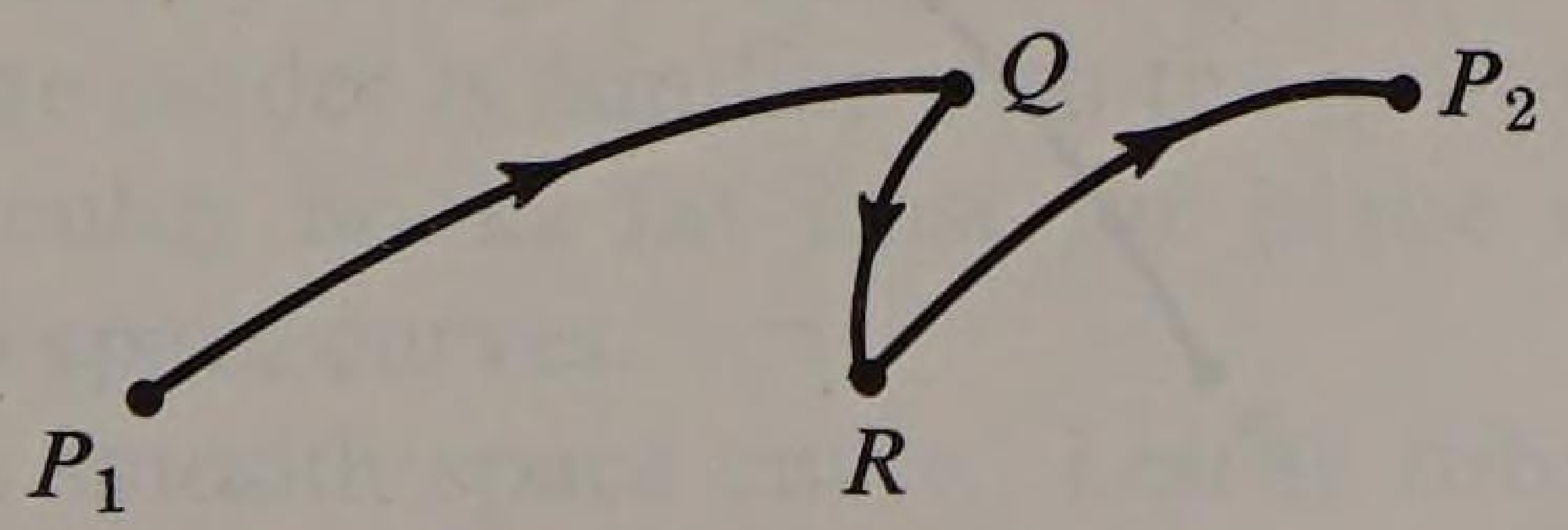

FIGURE 2.7

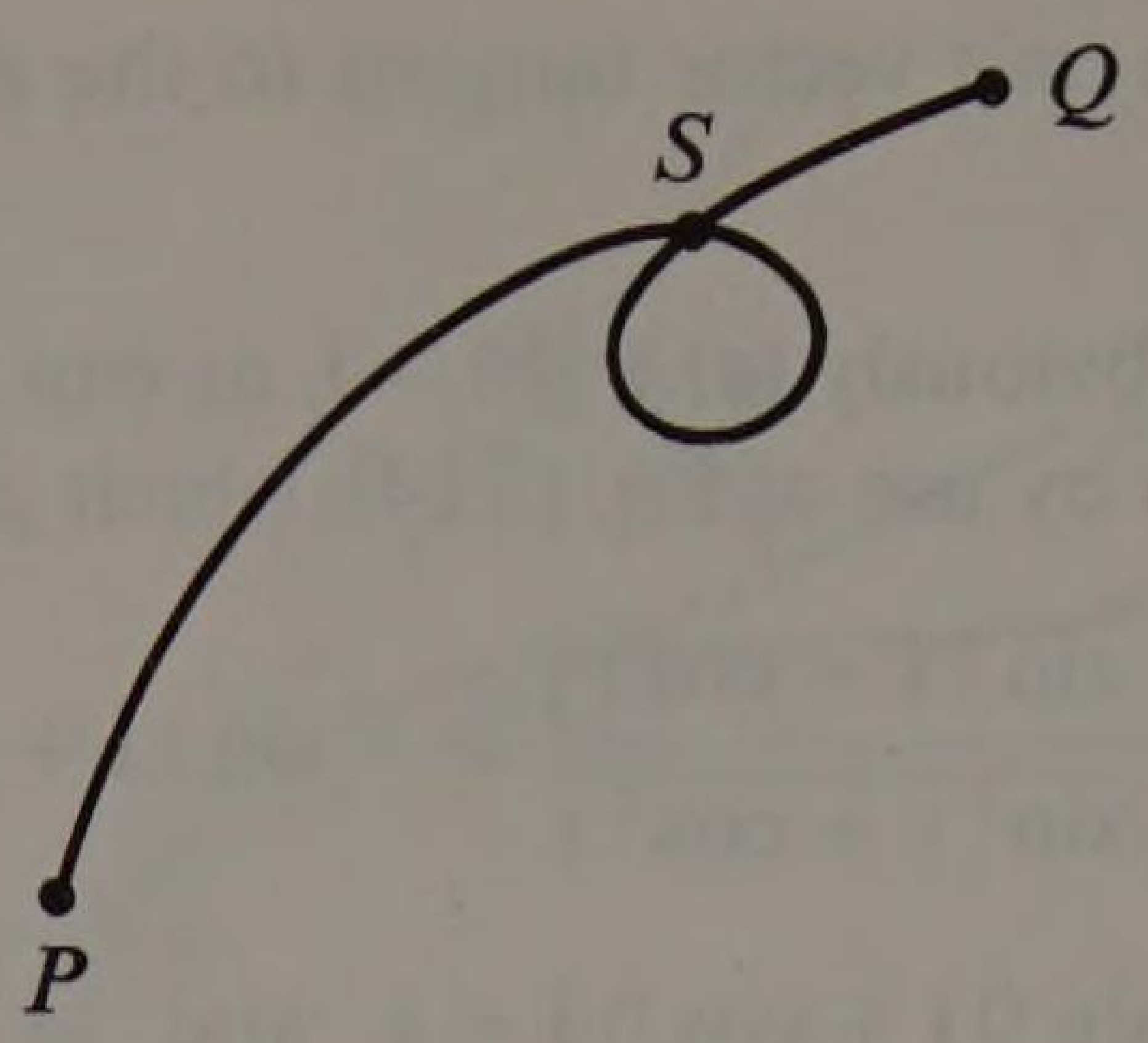

FIGURE 2.8

Accordingly, we say that an arc is *smooth* if it has a parametrization $\mathbf{R} = \mathbf{R}(t)$, $t_1 \le t \le t_2$, satisfying the following conditions:

(*i*) $d\mathbf{R}/dt$ exists and is a continuous function of t, for all values of t in the interval $t_1 \le t \le t_2$,

(*ii*) to distinct values of t in the interval $t_1 < t < t_2$ there correspond distinct points,

(*iii*) there is no value of t in the interval $t_1 \le t \le t_2$ for which $d\mathbf{R}/dt$ is the zero vector.

We allow the possibility that a smooth arc can be *closed*, as in Fig. 2.6, if $\mathbf{R}(t_1) = \mathbf{R}(t_2)$.

Notice that to show an arc is smooth we need only produce *one* such parametrization; there may be others, which violate the three conditions. For example, a straight line segment can be parametrized by $\mathbf{R}(t) = \mathbf{R}_0 + t^3\mathbf{V}$, $-1 \le t \le 1$, in violation of (iii); yet it is a smooth arc.

The arc in Fig. 2.7 is not smooth since it fails to have a tangent at Q and R. However, it consists of a finite number of smooth arcs joined together, and it does not cross itself; such a curve is called *regular.* The curve in Fig. 2.8 is not regular, because of the crossing at S.

In Figs. 2.5, 2.6, and 2.7, we have indicated the direction in which the particle is traversing the curve by a small arrow. Strictly speaking, any curve is nothing more than a collection of points in space. When, however, we indicate a direction along a smooth arc, as we have in these diagrams, then we say that the arc has been *oriented.* Obviously, a smooth arc can be oriented in only two ways. The arc in Fig. 2.9 is a replica of that in Fig. 2.5, but is oriented in the opposite way.

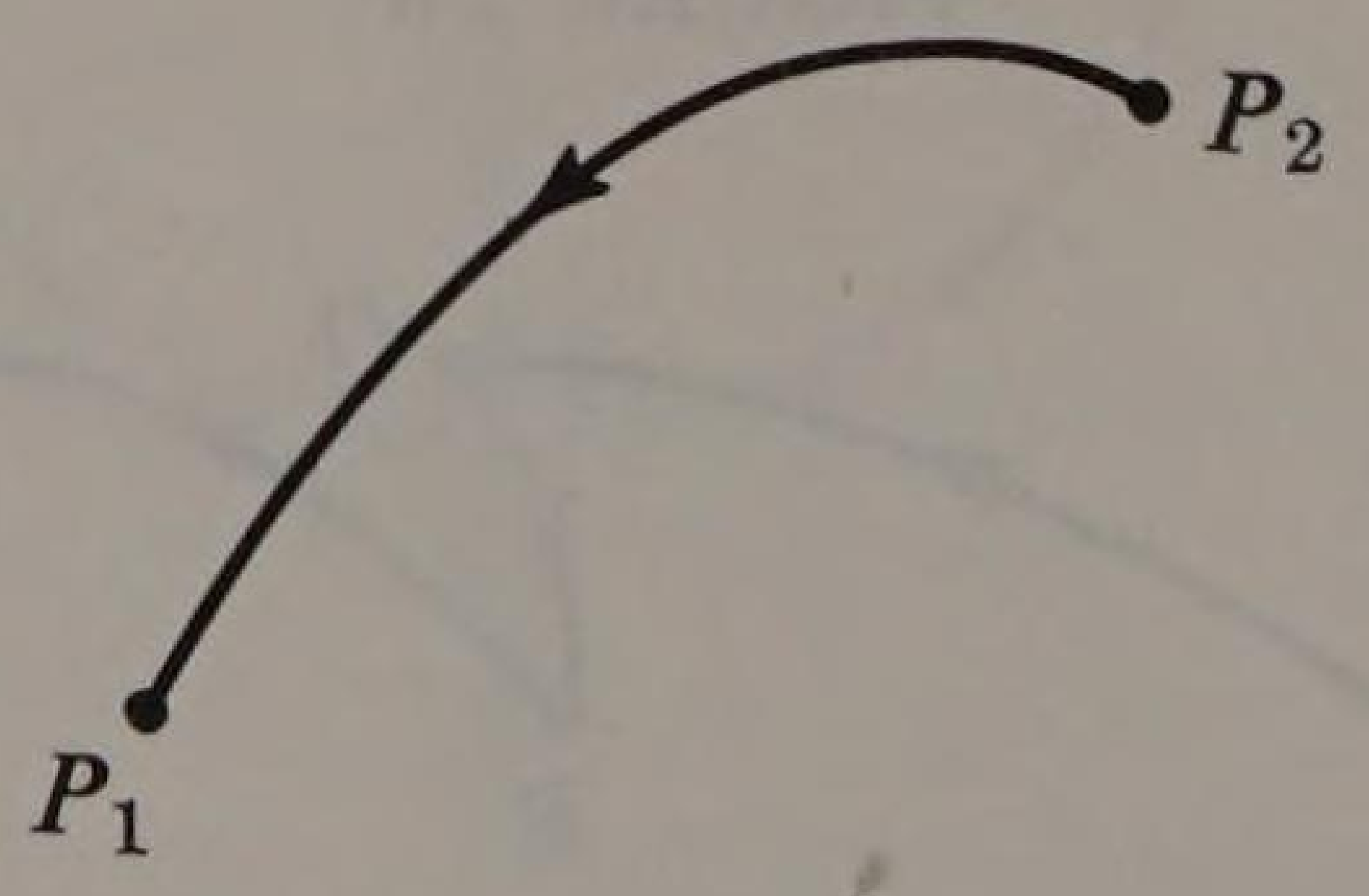

FIGURE 2.9

When an arc is described by equations such as (2.10), in terms of a parameter t, the orientation is usually understood to be determined by that parameter: the direction is the direction of increasing t. For example, the closed arc

$$\begin{aligned} x &= \cos t \\ y &= \sin t \\ z &= 0 \end{aligned} \tag{2.15}$$

is simply a circle of unit radius in the xy plane. As t increases from 0 to 2π, the point moves counterclockwise around the circle, as shown in Fig. 2.10. The same arc with opposite orientation can be given parametrically by

$$\begin{aligned} x &= \cos t \\ y &= -\sin t \\ z &= 0 \end{aligned} \tag{2.16}$$

The Eqs. (2.16) specify the same arc as (2.15), but with opposite orientation, since as t increases from 0 to 2π, the point (x,y,z) traverses the circle in the opposite direction.

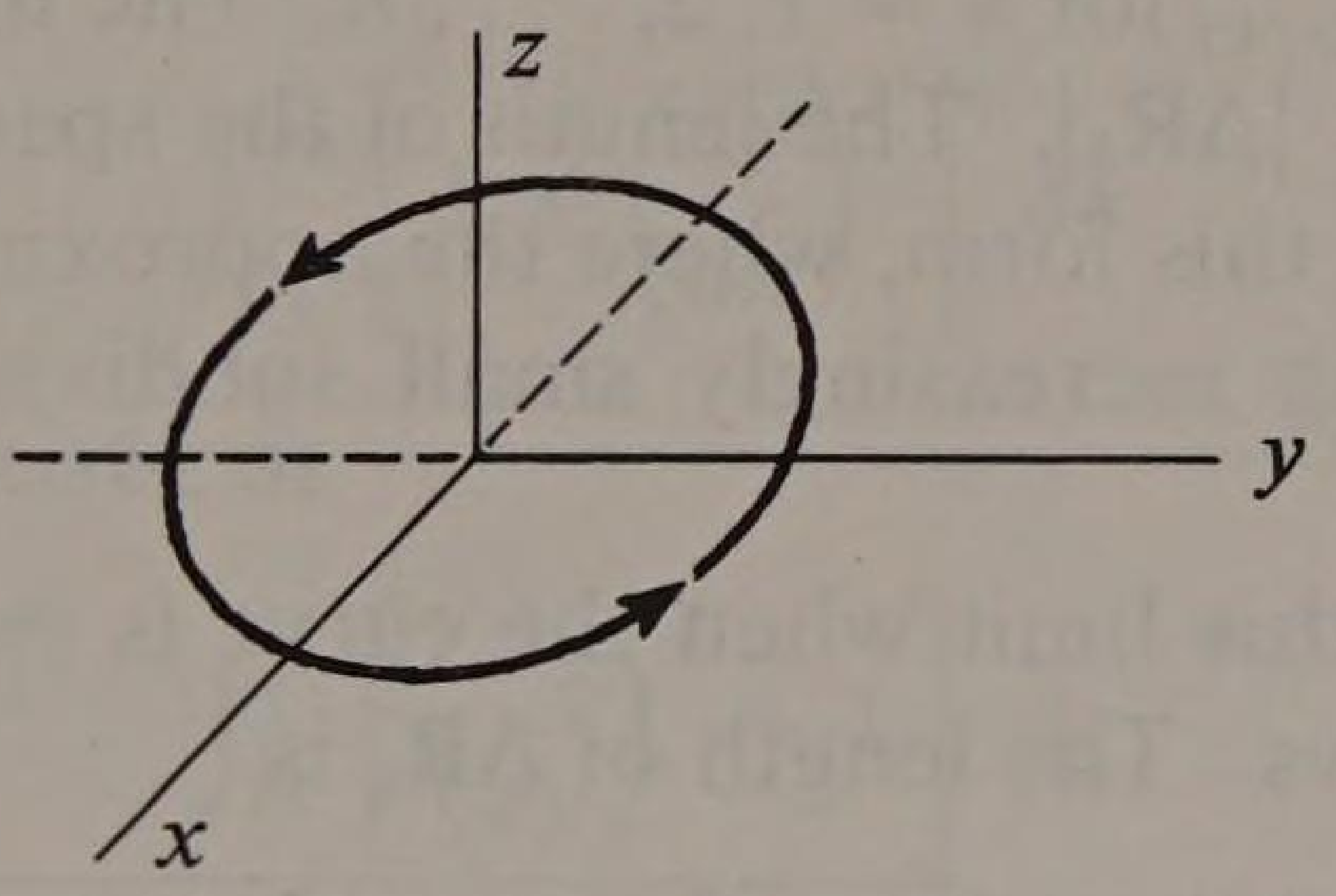

FIGURE 2.10

The same circle can be represented *nonparametrically* (i.e., without a "dummy variable") by the equations

$$\begin{aligned} x^2 + y^2 &= 1 \\ z &= 0 \end{aligned} \tag{2.17}$$

There is no way of knowing from (2.17) which orientation is intended. Note that (2.17) represents the arc as the intersection of two surfaces (a cylinder and a plane). When one specifies an oriented arc as the intersection of two surfaces, by giving two equations, it is necessary to specify the orientation separately, either verbally or by drawing a diagram.

Undoubtedly the reader is familiar with the notion of arc length, which is discussed in calculus books (at least for plane curves). This notion generalizes easily to space curves.

Suppose C is a smooth space curve. Let us subdivide C into smaller arcs, and approximate it by a polygonal path consisting of n straight-line

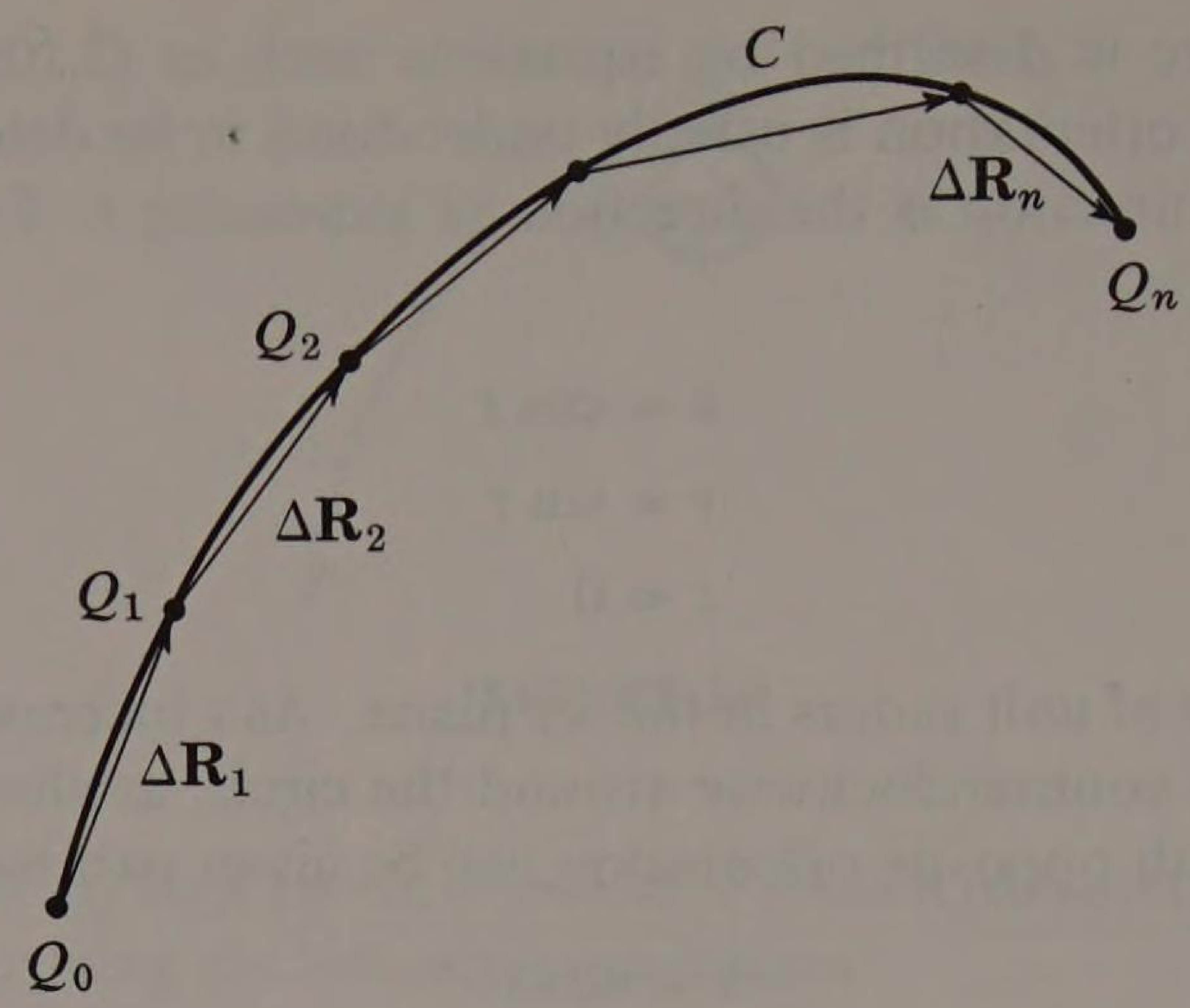

FIGURE 2.11

segments joining the endpoints of the arcs (Fig. 2.11). That is, we select points $Q_0, Q_1, \ldots, Q_n$ along C, in that order, with Q_0 and Q_n the endpoints of C. For each $k = 0, 1, \ldots, n$, let $\mathbf{R}_k$ be the position vector to the point Q_k, and let $\Delta\mathbf{R}_k = \mathbf{R}_k - \mathbf{R}_{k-1}$, for $k = 1, 2, \ldots, n$. The total length of the polygonal path is then $\sum_{k=1}^{n} |\Delta\mathbf{R}_k|$. The length of the space curve C is defined to be the limit of sums of this form, where the approximating polygonal paths are obtained by taking increasingly small subdivisions while n increases without bound.

We can compute this limit when the curve is parametrized by $\mathbf{R}(t)$ for, say, $a \le t \le b$ as follows. The length of $\Delta\mathbf{R}_k$ is

$$|\Delta\mathbf{R}_k| = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2 + (\Delta z_k)^2}$$

Let t_k be the values of t which correspond to the points $\mathbf{R}_k$; i.e., $\mathbf{R}_k = \mathbf{R}(t_k)$. Then, because $d\mathbf{R}/dt$ is continuous, the mean value theorem of calculus ensures us that for some number τ_k between t_{k-1} and t_k,

$$\Delta x_k = x_k - x_{k-1} = (t_k - t_{k-1})\frac{dx}{dt}(\tau_k)$$

Similarly, there are numbers τ_k' and τ_k'' in the same interval such that

$$\Delta y_k = (t_k - t_{k-1})\frac{dy}{dt}(\tau_k')$$

$$\Delta z_k = (t_k - t_{k-1})\frac{dz}{dt}(\tau_k'')$$

So for the length of the polygonal path we have

$$\sum_{k=1}^{n} |\Delta\mathbf{R}_k| = \sum_{k=1}^{n} \left[\left(\frac{dx}{dt}(\tau_k)\right)^2 + \left(\frac{dy}{dt}(\tau_k')\right)^2 + \left(\frac{dz}{dt}(\tau_k'')\right)^2\right]^{\frac{1}{2}} (t_k - t_{k-1})$$

Now if the polygonal subdivision is made finer, the differences $t_k - t_{k-1}$ become smaller and this sum approaches the integral

$$\int_a^b \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{\frac{1}{2}} dt \tag{2.18}$$

as a limit. Recognizing the integrand as $|d\mathbf{R}/dt|$, we see that *the length of the curve C is given by*

$$\int_a^b \left|\frac{d\mathbf{R}}{dt}\right| dt \tag{2.19}$$

The arc length of a regular curve is defined to be the sum of the lengths of the various smooth curves that constitute it.

Sometimes it is possible to write two of the variables, say y and z, in terms of the other, say x. In that case dy and dz may be expressed in terms of x and dx, and the integral is taken with respect to x, the limits of integration being the values of x corresponding to t_1 and t_2.

If the arc P_1P_2 lies entirely in the xy plane, which is the simplest case treated in calculus books, then z is identically equal to zero and so $dz/dt = 0$ and, by eliminating the parameter t, Eq. (2.18) may be written in the familiar alternative form

$$\int_{x_1}^{x_2} \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx \tag{2.20}$$

provided the integral exists, or

$$\int_{y_1}^{y_2} \left[\left(\frac{dx}{dy}\right)^2 + 1\right]^{\frac{1}{2}} dy \tag{2.21}$$

provided this integral exists. It is possible that these integrals may not exist. For example, if the arc P_1P_2 contains a segment that is parallel to the y axis, then dy/dx will not exist along this segment (that is, dy/dx is "infinite") and (2.20) will not make sense.

Example 2.9 Find the arc length between (0,0,1) and (1,0,1) of the curve

$$y = \sin 2\pi x \qquad z = \cos 2\pi x$$

(This is a helix winding about the x axis.)

Solution

$$dx^2 + dy^2 + dz^2 = dx^2 + 4\pi^2\cos^2 2\pi x\, dx^2 + 4\pi^2\sin^2 2\pi x\, dx^2$$

Hence the integral is

$$\int_0^1 (1 + 4\pi^2)^{\frac{1}{2}}\, dx = (1 + 4\pi^2)^{\frac{1}{2}}$$

The expression (2.18) is sometimes written

$$\int_C [(dx)^2 + (dy)^2 + (dz)^2]^{\frac{1}{2}} \qquad \text{or} \qquad \int_C |d\mathbf{R}| \tag{2.22}$$

where it is understood that dx, dy, and dz are expressed in terms of the parameter t and the differential dt (so that t is the variable over which the integration is performed). The form (2.22) emphasizes that the arc length is a property of the curve alone, and does not depend on the particular parametrization.

Returning to (2.19) we see that the arc length measured along the curve from some arbitrary initial position $\mathbf{R}(t_1)$ to a *variable* position $\mathbf{R}(t)$ is given by

$$s = s(t) = \int_{t_1}^{t} \left| \frac{d\mathbf{R}}{dt} \right| dt \qquad (t \geq t_1)$$

This suggests the possibility of using s itself as the parameter. In principle, at least, we may invert the above equation to get t in terms of s; substituting into the function $\mathbf{R}(t)$ gives $\mathbf{R}$ as a function of s.

In practice, the direct computation of $\mathbf{R}(s)$ is prohibitively difficult except for some standard, contrived, examples, to wit:

Example 2.10 Reparametrize the curves

$$(i) \quad \mathbf{R}(t) = \frac{t^2}{2}\mathbf{i} + \frac{t^3}{3}\mathbf{k} \qquad 0 \leq t \leq 2$$

$$(ii) \quad \mathbf{R}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} \qquad 0 \leq t \leq 2\pi$$

in terms of arc length.

Solution (*i*) Choosing $t_1 = 0$, we have

$$s = \int_0^t \left| \frac{d\mathbf{R}}{dt} \right| dt = \int_0^t (t^2 + t^4)^{\frac{1}{2}}\, dt = \frac{(t^2+1)^{\frac{3}{2}} - 1}{3}$$

Inverting this produces

$$t = [(3s+1)^{\frac{2}{3}} - 1]^{\frac{1}{2}}$$

and the new parametrization is

$$\mathbf{R}(s) = \frac{(3s+1)^{\frac{2}{3}} - 1}{2}\mathbf{i} + \frac{[(3s+1)^{\frac{2}{3}} - 1]^{\frac{3}{2}}}{3}\mathbf{k}$$

(*ii*) Again with $t_1 = 0$, we find

$$s = \int_0^t \left| \frac{d\mathbf{R}}{dt} \right| dt = \int_0^t (4\sin^2 t + 4\cos^2 t)^{\frac{1}{2}}\, dt = 2t$$

Hence $t = s/2$ and the arc-length parametrization reads

$$\mathbf{R}(s) = 2\cos\frac{s}{2}\mathbf{i} + 2\sin\frac{s}{2}\mathbf{j}$$

The arc length parametrization possesses some advantages. By the fundamental theorem of calculus, we then have

$$\frac{ds}{dt} = \left| \frac{d\mathbf{R}}{dt} \right| \qquad (= |\mathbf{v}|) \tag{2.23}$$

(This identifies the *speed* with the rate of change of arc length, a reassuring fact.) In coordinate form this becomes

$$\frac{ds}{dt} = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{\frac{1}{2}}$$

Because our assumptions guarantee that $ds/dt \neq 0$, it follows from the chain rule that

$$\frac{d\mathbf{R}}{ds} = \frac{d\mathbf{R}}{dt}\bigg/\frac{ds}{dt}$$

Since $d\mathbf{R}/dt$ is tangent to the curve, this shows $d\mathbf{R}/ds$ is also. (This reflects the obvious fact that the tangent *direction* is independent of the parametrization used to describe the curve.) Moreover, $d\mathbf{R}/ds$ is a *unit* tangent vector; so by Eq. (2.14)

$$\mathbf{T} = \frac{d\mathbf{R}}{ds}$$

In Example 2.10 these vectors are

$$(i)\quad \frac{d\mathbf{R}}{ds} = (3s+1)^{-\frac{1}{3}}\mathbf{i} + [(3s+1)^{\frac{2}{3}} - 1]^{\frac{1}{2}}(3s+1)^{-\frac{1}{3}}\mathbf{k}$$

$$(ii)\quad \frac{d\mathbf{R}}{ds} = -\sin\frac{s}{2}\,\mathbf{i} + \cos\frac{s}{2}\,\mathbf{j}$$

and the reader can verify that both are unit vectors.

Exercises

1. Suppose that P_1P_2 is a smooth arc in the xy plane. Is it necessarily true that dy/dx exists at every point on this arc?

2. Study the consequences of dropping condition (*iii*) in the definition of a smooth arc. (*Hint*: Sketch the arc $\mathbf{R} = t^2\mathbf{i} + t^3\mathbf{j}$.)

3. By using identities concerning hyperbolic functions, eliminate the parameter t from the equations

$$x = \cosh t \qquad y = \sinh t \qquad z = 0$$

4. As t varies from -1 to 1, the point (x,y,z) where

$$x = t \qquad y = |t| \qquad z = 0$$

traces a regular curve. At what point on this curve is there no tangent?

5. Observe that

$$x = t \qquad y = \sin 2\pi t \qquad z = \cos 2\pi t$$

is a parametrization of the helix in Example 2.9. Compute the arc length between the same two endpoints using formula (2.19). What is the unit tangent vector at (0,0,1)?

6. If **T** denotes the unit tangent to the curve

$$x = t \qquad y = 2t + 5 \qquad z = 3t$$

show that $d\mathbf{T}/dt = 0$. Interpret this.

7. Find the arc length of the curve described in Exercise 6, between (0,5,0) and (1,7,3), (a) by using (2.19), and (b) by using a little common sense.

8. (a) Determine the arc length of the curve

$$x = e^t \cos t \qquad y = e^t \sin t \qquad z = 0$$

between $t = 0$ and $t = 1$.
(b) Reparametrize the curve in terms of arc length.

9. For the curve

$$x = \sin t - t \cos t$$
$$y = \cos t + t \sin t$$
$$z = t^2$$

find (a) the arc length between (0,1,0) and $(-2\pi,1,4\pi^2)$, (b) $\mathbf{T}(t)$, (c) $\mathbf{T}(\pi)$.

10. Find the unit vector tangent to the oriented closed curve

$$x = a \cos t \qquad y = b \sin t \qquad z = 0$$

at $t = \frac{3}{2}\pi$.

11. Show that the graph of any continuously differentiable function $y = f(x)$ is a smooth curve. (*Hint*: Check the parametrization $x = t$, $y = f(t)$, $z = 0$.)

2.3 ACCELERATION AND CURVATURE

The *acceleration* of a particle is defined to be the time rate of change of its velocity. Since velocity is a vector quantity, this acceleration may be associated with a change in either the magnitude or the direction of the velocity, or both.

Suppose first that the direction of the velocity is constant. Then the motion of the particle takes place along a straight line and the magnitude of the acceleration is the rate of change of speed,

$$|\mathbf{a}| = \frac{d}{dt}|\mathbf{v}| = \frac{d^2s}{dt^2}$$

where s is arc length along the trajectory [recall Eq. (2.23)]. The acceleration is directed along the straight line. On the other hand, if the particle moves at constant speed around a circle of radius ρ, it is well known that it undergoes a "centripetal" acceleration of magnitude

$$|\mathbf{a}| = \frac{|\mathbf{v}|^2}{\rho} = \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2$$

directed towards the center of the circle. This is due solely to the change of direction.

One of the aims of this section is to show that for motion along a *general* curve with **v** changing direction and magnitude the acceleration vector can be expressed as the sum of two orthogonal vectors, one giving the rate of change of speed and the other giving the instantaneous centripetal acceleration corresponding to a related circular trajectory.

If motion along a curve is to be related to motion on a circle, we clearly need to select the circle which "best" approximates the curve at a given point. In Fig. 2.12 we indicate the circle approximating the curve at P_1. Two properties that the circle must have are clear: it should pass through the point P_1, and its tangent must coincide with the tangent to the curve at P_1. It remains for us to decide what radius ρ the circle should have in order that it fit the curve as well as possible.

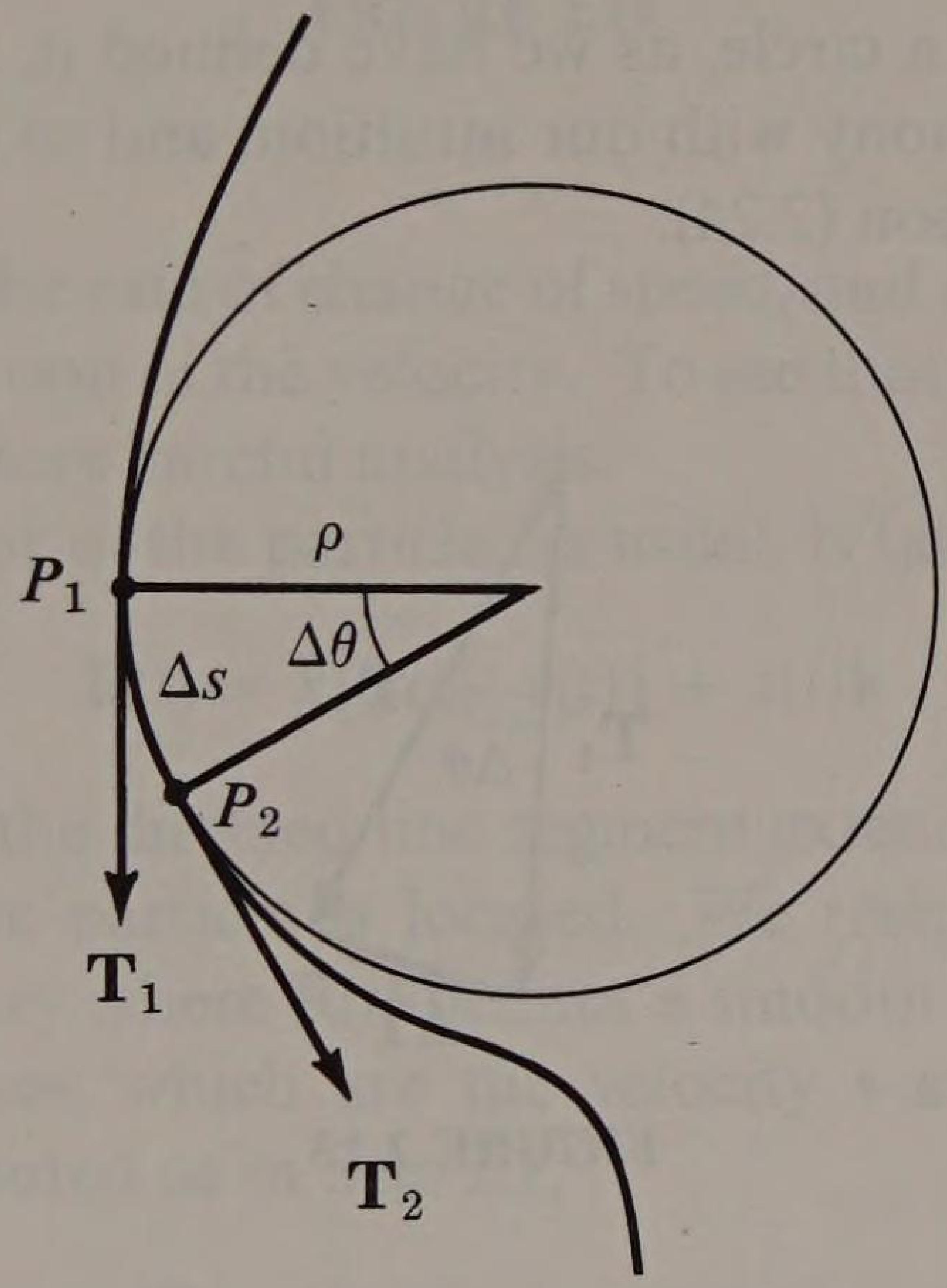

FIGURE 2.12

Observe that circles with small radii are more sharply curved than circles with large radii. Thus, by choosing ρ appropriately, we ought to be able to select a circle with the same *curvature* as the given curve, at P_1. But how do we measure this curvature? Intuitively, curvature arises as a result of the tangent direction changing as we move along the curve; a straight line has no curvature and an arc is more sharply curved if the tangent turns faster along the length of the curve. Let us therefore *define* the curvature k as the rate at which the *unit* tangent vector turns, with respect to arc length along the curve:

$$k = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\right| \Big/ \frac{ds}{dt} \tag{2.24}$$

What does this give for the curvature of a circle of radius ρ? In Fig. 2.12 the arc length between P_1 and P_2 on the circle is $\Delta s = \rho\,\Delta\theta$. The unit tangent vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ also make an angle $\Delta\theta$, and the change in the unit tangent as we proceed from P_1 and P_2 is

$$\Delta\mathbf{T} = \mathbf{T}_2 - \mathbf{T}_1$$

For small $\Delta\theta$, the magnitude of $\Delta\mathbf{T}$ is approximately $\Delta\theta$, as we see from Fig. 2.13 (keep in mind that the magnitudes of $\mathbf{T}_1$ and $\mathbf{T}_2$ are unity). Thus

$$\left|\frac{\Delta\mathbf{T}}{\Delta s}\right| \approx \frac{\Delta\theta}{\Delta s} = \frac{1}{\rho}$$

This approximation becomes exact as Δs approaches zero, so we can write

$$\left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{\rho}$$

Thus the curvature of a circle, as we have defined it, is the reciprocal of its radius. This is in harmony with our intuition, and so we shall feel confident in adopting the definition (2.24).

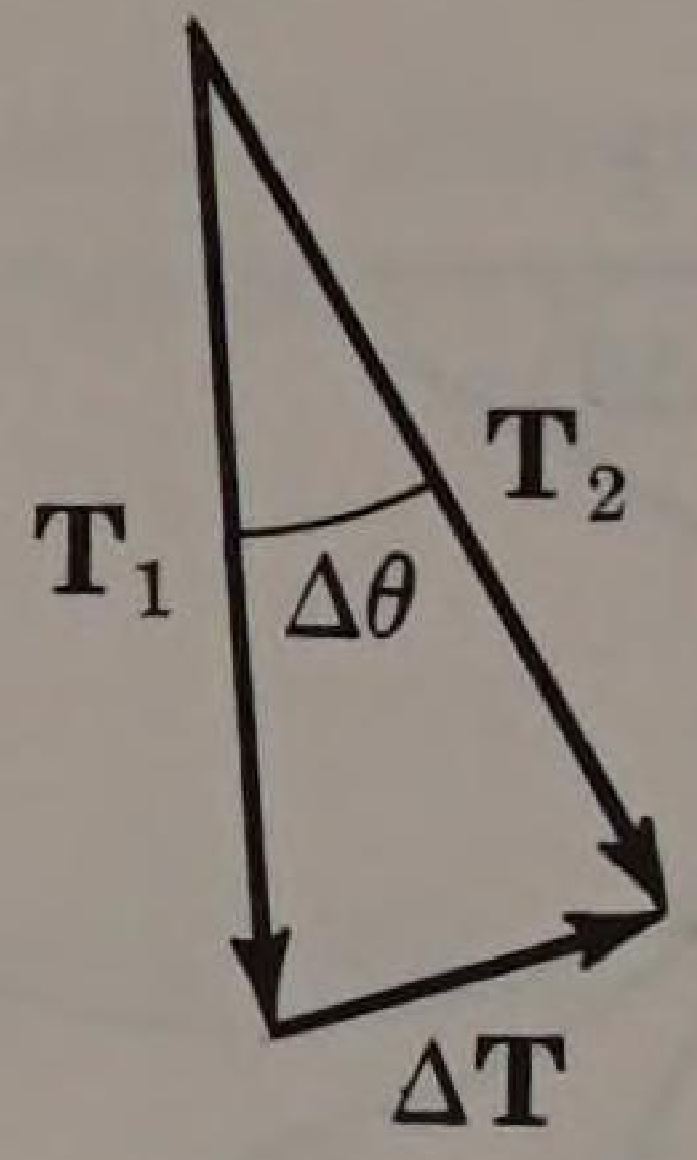

FIGURE 2.13

Consequently, the radius ρ of our approximating circle (called the "osculating circle"), is given by

$$\rho = \frac{1}{k} = 1 \bigg/ \left|\frac{d\mathbf{T}}{ds}\right|$$

Now we let $\mathbf{N}$ denote a *unit* vector pointing towards the center of the approximating circle, and as usual let $\mathbf{T}$ denote the unit tangent vector. The directions of both $\mathbf{T}$ and $\mathbf{N}$ may vary at different points along the curve, but they are always at right angles with each other, as shown in Fig. 2.14.

If our earlier considerations about circular motion can be generalized to motion along curve, then we are led to anticipate that the acceleration $\mathbf{a}$ can be expressed as a sum of two components,

$$\mathbf{a} = a_t\mathbf{T} + a_n\mathbf{N} \tag{2.25}$$

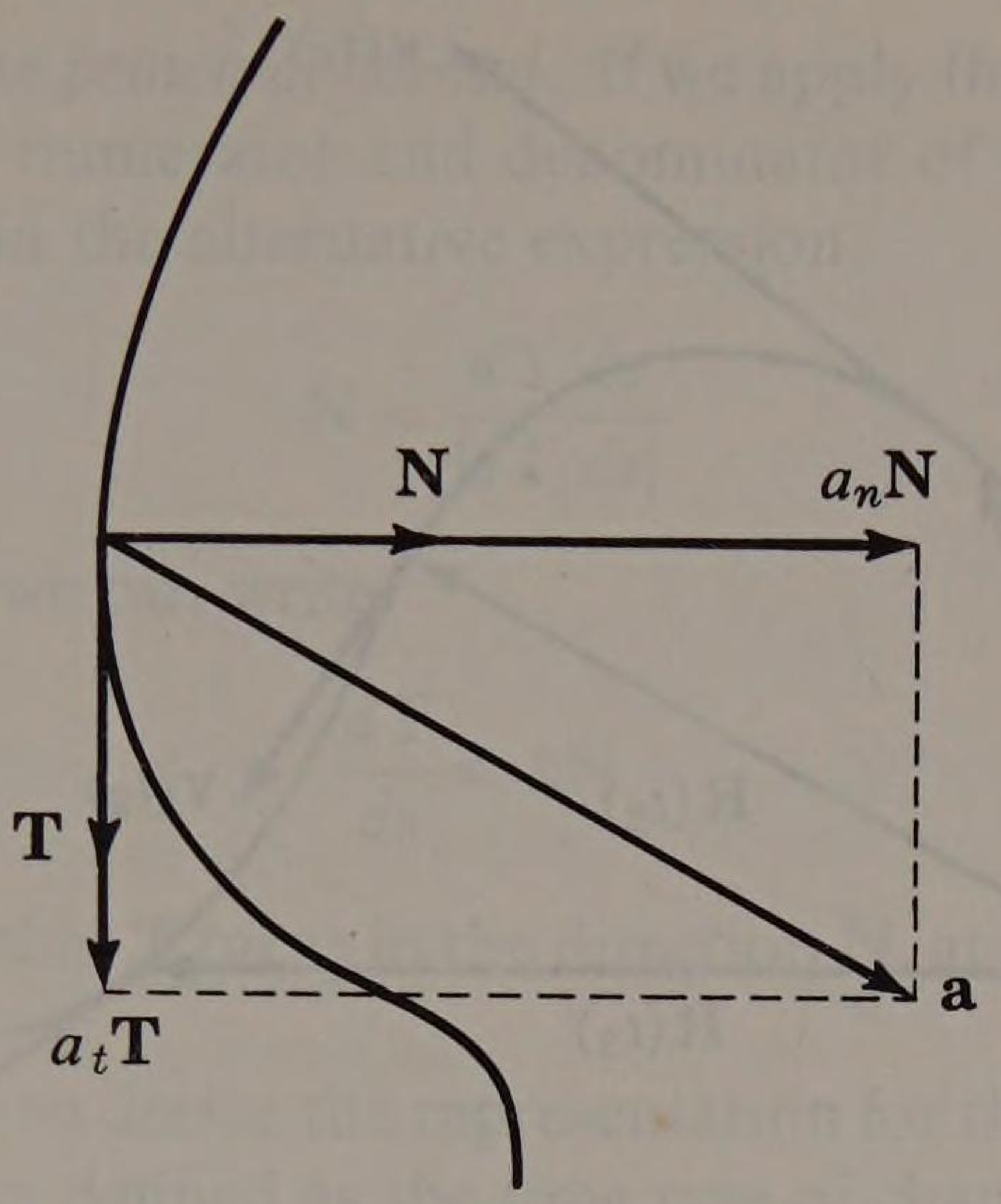

FIGURE 2.14

where $a_t = d^2s/dt^2$ is the rate of change of speed, and $a_n = |\mathbf{v}|^2/\rho$ results from the change in the direction of the velocity. To see that this is, in fact, the case, we start over with a more careful analysis.

The position vector of the particle, as usual, is taken to be

$$\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

which we visualize as the directed line segment extending from the origin to the point at which the particle is located. We restrict our attention to a portion of the trajectory where $\mathbf{R}(t)$ defines a smooth arc and is twice differentiable. Its derivatives, which are the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$, respectively, are computed as in Sec. 2.1,

$$\mathbf{v} = \frac{d\mathbf{R}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \tag{2.26}$$

$$\mathbf{a} = \frac{d^2\mathbf{R}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \tag{2.27}$$

It is convenient to visualize $\mathbf{v}(t)$ as a directed line segment with its tail at the point where the particle is located. As t varies, the corresponding vector $\mathbf{v}(t)$ may vary either in direction or magnitude, or both (Fig. 2.15). The speed of the particle is the magnitude of the velocity ds/dt, where the arc length s is measured along the curve from some arbitrary initial point:

$$v = |\mathbf{v}(t)| = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{\frac{1}{2}} = \frac{ds}{dt} \tag{2.28}$$

The unit tangent vector $\mathbf{T}$ may be obtained by dividing the velocity

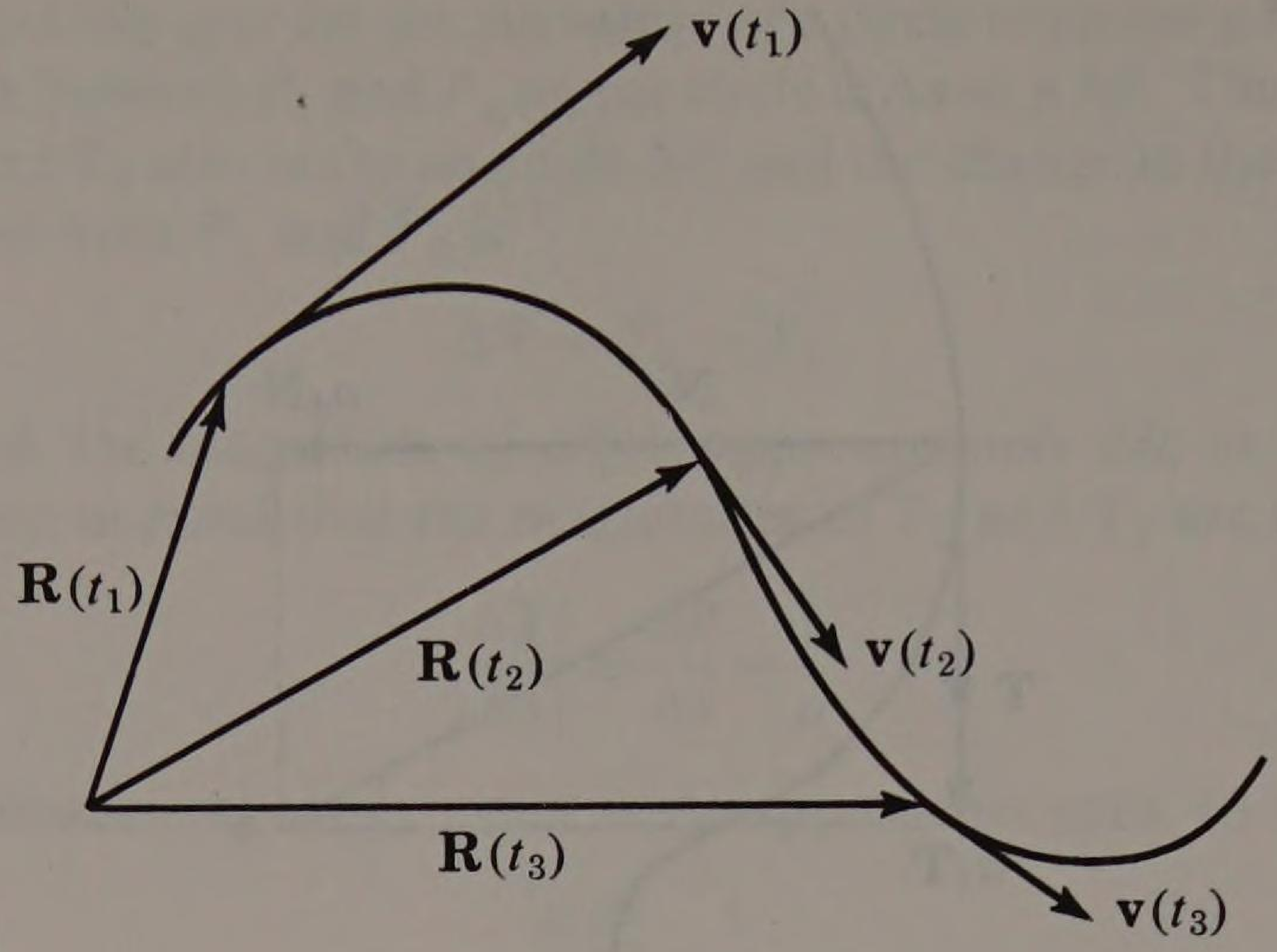

FIGURE 2.15

$\mathbf{v}(t)$ by the speed $|\mathbf{v}(t)|$, since our assumptions guarantee that $|\mathbf{v}(t)|$ is never zero.

$$\mathbf{T} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} \tag{2.29}$$

We note that $\mathbf{T}$ is also given by the expression

$$\mathbf{T} = \frac{d\mathbf{R}}{ds} \tag{2.30}$$

The *curvature* k of the curve at any point is defined to be the *magnitude* of the vector $d\mathbf{T}/ds$ at that point [recall Eq. (2.24)]:

$$k = \left|\frac{d\mathbf{T}}{ds}\right| \tag{2.31}$$

If $k \neq 0$, the *radius of curvature* ρ is defined to be the reciprocal of the curvature,

$$\rho = \frac{1}{k} \tag{2.32}$$

The motivation for this definition of ρ was given above. By introducing k we will be able to avoid using the term "infinite radius of curvature." Thus, the curvature of a straight line is $k = 0$.

Since $\mathbf{T}$ has constant *magnitude*, the derivative of $\mathbf{T}$ with respect to t is either the zero vector or it is a nonzero vector perpendicular to $\mathbf{T}$. This was proved in Example 2.6; moreover, it is clear geometrically from Fig. 2.13, where we see that $\Delta\mathbf{T}$ is approximately perpendicular to $\mathbf{T}$ if $\Delta\mathbf{T}$ is small.

If $d\mathbf{T}/dt$ is not the zero vector, we define the unit vector $\mathbf{N}$ to be $d\mathbf{T}/dt$ divided by its own magnitude,

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \tag{2.33}$$

This vector is called the *principal normal.* If we apply the chain rule $d\mathbf{T}/dt = (d\mathbf{T}/ds)(ds/dt)$ to both numerator and denominator of this fraction, we can cancel ds/dt and obtain the alternative expression

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$$

and since $k = |d\mathbf{T}/ds|$ we can write

$$\frac{d\mathbf{T}}{ds} = k\mathbf{N}$$

In words, we may say that **T** turns in the direction **N**, at a rate k (with respect to arc length).

Now we are ready to derive the representation for the acceleration of the particle. This has been defined as the time rate of change of the velocity,

$$\mathbf{a}(t) = \mathbf{v}'(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \tag{2.34}$$

Since $|\mathbf{v}(t)| = ds/dt$, we can write

$$\mathbf{v}(t) = \frac{ds}{dt}\mathbf{T} \tag{2.35}$$

So by the product rule for derivatives (Sec. 2.1),

$$\begin{aligned}\mathbf{a}(t) = \mathbf{v}'(t) &= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\\ &= \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 k\mathbf{N}\end{aligned}$$

In other words, we have

$$\mathbf{a} = a_t\mathbf{T} + a_n\mathbf{N} \tag{2.36}$$

where $a_t = d^2s/dt^2$ and $a_n = kv^2$. This is exactly what we anticipated in Eq. (2.25).

We note that at any point on the curve where $k = 0$, the normal vector **N** is not defined. This does not matter, since we have $a_n = 0$ in that case and hence have no need for **N** in (2.36). In case $k \neq 0$, we can write $a_n = v^2/\rho$, the way we did in the previous heuristic discussion.

Since **T** and **N** are mutually perpendicular vectors at any point where they are defined, we have, by the pythagorean theorem,

$$a^2 = a_t^{\,2} + a_n^{\,2} \tag{2.37}$$

To compute a, we need only find $d^2\mathbf{R}/dt^2$ by differentiation, and calculate the magnitude of this vector. To compute a_t we need only find $\mathbf{v} = d\mathbf{R}/dt$, calculate its magnitude $|d\mathbf{R}/dt| = ds/dt$, and differentiate this with respect to t. Having computed a and a_t, it is then easy to obtain a_n by using (2.37). In some problems, this is more convenient than using the expression kv^2.

Example 2.11 The position of a particle moving around the circle $x^2 + y^2 = r^2$ in the xy plane, with angular velocity ω, is

$$x = r \cos \omega t \qquad y = r \sin \omega t \qquad z = 0$$

Find the normal and tangential components of acceleration of the particle, and determine the curvature of the circle.

Solution We have

$$\mathbf{R} = r \cos \omega t\,\mathbf{i} + r \sin \omega t\,\mathbf{j}$$

$$\frac{d\mathbf{R}}{dt} = -r\omega \sin \omega t\,\mathbf{i} + r\omega \cos \omega t\,\mathbf{j}$$

$$\frac{d^2\mathbf{R}}{dt^2} = -r\omega^2 \cos \omega t\,\mathbf{i} - r\omega^2 \sin \omega t\,\mathbf{j}$$

The magnitudes of these vectors are

$$v = \frac{ds}{dt} = \left|\frac{d\mathbf{R}}{dt}\right| = (r^2\omega^2 \sin^2 \omega t + r^2\omega^2 \cos^2 \omega t)^{\frac{1}{2}} = \omega r$$

$$a = \left|\frac{d^2\mathbf{R}}{dt^2}\right| = \omega^2 r$$

Since ds/dt is a constant, $a_t = d^2s/dt^2 = 0$, and $a = a_n$. Therefore $kv^2 = \omega^2 r$; and since $v = \omega r$, we have $k = \omega^2 r/\omega^2 r^2 = 1/r$. This verifies that the curvature of a circle is the reciprocal of its radius. The answers are: $a_n = \omega^2 r$, $a_t = 0$, $k = 1/r$.

Example 2.12 The coordinates of a particle at time t are

$$x = \sin t - t \cos t$$
$$y = \cos t + t \sin t$$
$$z = t^2$$

Find the speed, the normal and tangential components of acceleration, and the curvature of the path, in terms of t.

Solution

$$\mathbf{R} = (\sin t - t \cos t)\mathbf{i} + (\cos t + t \sin t)\mathbf{j} + t^2\mathbf{k}$$

$$\frac{d\mathbf{R}}{dt} = (t \sin t)\mathbf{i} + (t \cos t)\mathbf{j} + 2t\mathbf{k}$$

$$\frac{d^2\mathbf{R}}{dt^2} = (t \cos t + \sin t)\mathbf{i} + (-t \sin t + \cos t)\mathbf{j} + 2\mathbf{k}$$

The speed is $ds/dt = |d\mathbf{R}/dt| = (t^2 \sin^2 t + t^2 \cos^2 t + 4t^2)^{\frac{1}{2}} = \sqrt{5}t$. The tangential component of acceleration is $a_t = d^2s/dt^2 = \sqrt{5}$.

From (2.37),

$$\begin{aligned} a_n &= [a^2 - a_t{}^2]^{\frac{1}{2}} \\ &= [(t \cos t + \sin t)^2 + (-t \sin t + \cos t)^2 + 2^2 - 5]^{\frac{1}{2}} = t \end{aligned}$$

Since $a_n = kv^2$ we have $k = a_n/v^2 = t/5t^2 = 1/5t$.

One can derive a fairly simple expression for the curvature k by taking the vector cross product of

$$\mathbf{R}'(t) = |\mathbf{v}|\mathbf{T} \qquad \text{and} \qquad \mathbf{R}''(t) = \frac{d^2s}{dt^2}\mathbf{T} + k|\mathbf{v}|^2\mathbf{N}$$

which, since $\mathbf{T} \times \mathbf{T} = \mathbf{0}$, gives

$$\mathbf{R}' \times \mathbf{R}'' = k|\mathbf{v}|^3(\mathbf{T} \times \mathbf{N})$$

Since $\mathbf{T}$ and $\mathbf{N}$ are mutually perpendicular unit vectors, their cross product $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is a unit vector; this vector is called the *binormal.* We have $\mathbf{R}' \times \mathbf{R}'' = k|\mathbf{v}|^3\mathbf{B}$ and

$$|\mathbf{R}' \times \mathbf{R}''| = k|\mathbf{v}|^3$$

and hence

$$k = \frac{|\mathbf{R}' \times \mathbf{R}''|}{(\mathbf{R}' \cdot \mathbf{R}')^{\frac{3}{2}}} = \frac{|\mathbf{R}' \times \mathbf{R}''|}{|\mathbf{R}'|^3} \tag{2.38}$$

However, in most cases it is easier to use Eq. (2.24).

Once again we point out that although many of these formulas involve derivatives with respect to arc length s, one never needs to actually compute the reparametrization $\mathbf{R}(s)$, because of the chain rule.

OPTIONAL READING: THE FRENET FORMULAS

Because of its importance in geometry, it may be well to say more about the vector $\mathbf{B}$, which is a unit vector mutually perpendicular to both $\mathbf{T}$ and $\mathbf{N}$. The vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$, in that order, form a right-handed system. It is useful to think of these three vectors as attached to a particle moving along the curve: as the particle moves, its associated triad of mutually perpendicular unit vectors moves and rotates (see Fig. 2.16). For a *plane curve*, $\mathbf{T}$ and $\mathbf{N}$ lie in the plane of the curve, so that $\mathbf{B}$ is a constant unit vector always perpendicular to the plane.

Let us try to describe how the triad rotates as a particle proceeds along a space curve. As we have seen, the vector $\mathbf{T}$ turns towards the vector $\mathbf{N}$ at a rate k, measured with respect to arc length.

$$\frac{d\mathbf{T}}{ds} = k\mathbf{N} \tag{2.39}$$

But since $\mathbf{N}$ is always perpendicular to $\mathbf{T}$, these vectors will turn *together* like a rigid body. $\mathbf{N}$ must therefore turn towards the direction $-\mathbf{T}$ at the same rate, k. In addition, it is also possible for $\mathbf{N}$ to *rotate about* $\mathbf{T}$ *as an axis*; this would happen if the instantaneous plane of the curve were to "tilt". In such a case, $d\mathbf{N}/ds$ would have a component perpendicular to

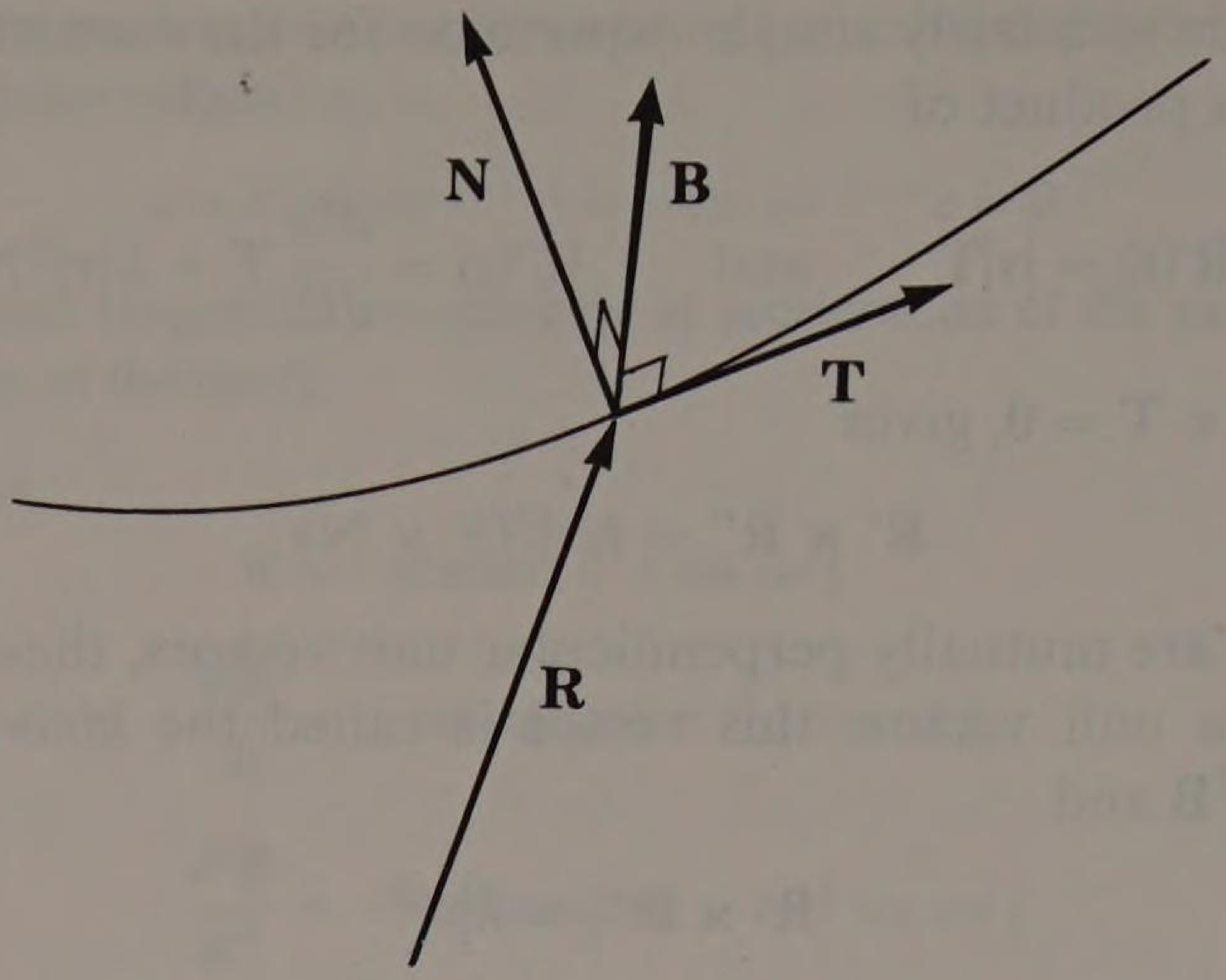

FIGURE 2.16

both **T** and **N**, i.e., along **B**. Thus we would have

$$\frac{d\mathbf{N}}{ds} = -k\mathbf{T} + \tau\mathbf{B} \tag{2.40}$$

where τ measures the rate at which the curve *twists*; accordingly, it is known as the *torsion*.

The torsion can be visualized by observing the cross section of a piece of solder wire bent into the shape of the curve, as in Fig. 2.17. The torsion, or twisting, of the wire will effect a *rotation* of the cross-sectional pattern. One can shape the wire into any plane curve without introducing torsion, but if the curve is nonplanar, the wire must twist.

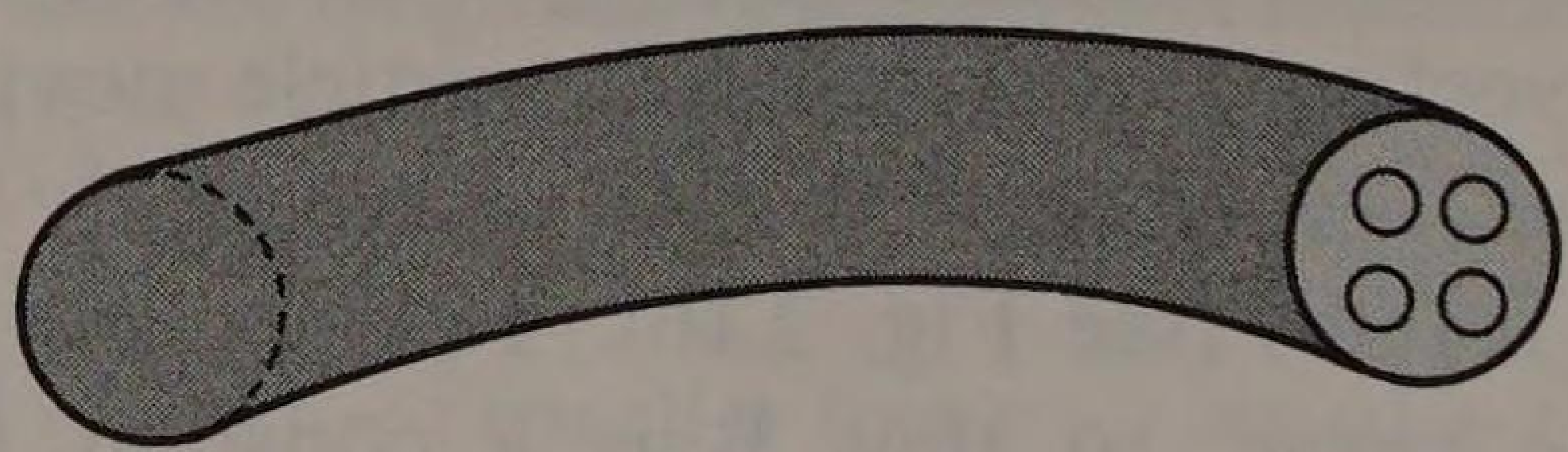

FIGURE 2.17

Once again, the fact that **N** turns towards **B** at a rate τ, and the fact that **N** and **B** are rigidly fixed at right angles, imply that **B** turns towards $-\mathbf{N}$ at the same rate. At first glance it seems conceivable that **B** might also rotate about **N** as an axis, thus turning in the direction of **T**. However, if this happened, **T** would be forced by rigidity into turning in the direction $-\mathbf{B}$; but **T** turns only in the direction **N**, *by definition*! Thus we have

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \tag{2.41}$$

Equations (2.39), (2.40), and (2.41) are called the *Frenet formulas.* They are important in differential geometry, where it is shown that any two curves with identical corresponding values of curvature and torsion are congruent (as usual, subject to certain restrictions).

Exercises

In the first four problems below, the coordinates of a moving particle are given as a function of the time t. Find (a) the speed, (b) the tangential and normal components of acceleration, (c) the unit tangent vector **T**, and (d) the curvature of the curve, as functions of time.

1. $x = e^t \cos t, \quad y = e^t \sin t, \quad z = 0$

2. $x = 3t \cos t, \quad y = 3t \sin t, \quad z = 4t$

3. $x = e^t \cos t, \quad y = e^t \sin t, \quad z = e^t$

4. $x = 5 \sin 4t, \quad y = 5 \cos 4t, \quad z = 10t$

5. The position vector of a moving particle is

$$\mathbf{R} = \cos t(\mathbf{i} - \mathbf{j}) + \sin t(\mathbf{i} + \mathbf{j}) + \tfrac{1}{2}t\mathbf{k}$$

(a) Determine the velocity and the speed of the particle.
(b) Determine the acceleration of the particle.
(c) Find a unit tangent to the path of the particle, in the direction of motion.
(d) Show that the curve traversed by the particle has constant curvature k, and find its value.

6. Find the curvature of the space curve

$$x = 3t^2 - t^3 \qquad y = 3t^2 \qquad z = 3t + t^3$$

7. Find the curvature and torsion for the helix

$$x = t \qquad y = \sin t \qquad z = \cos t$$

8. The position vector of a particle is given by

$$\mathbf{R}(t) = \sqrt{2} \cos 3t\,\mathbf{i} + \sqrt{2} \cos 3t\,\mathbf{j} + 2 \sin 3t\,\mathbf{k}$$

Find its speed, the curvature and torsion of its path, and describe the path geometrically.

9. If **F** is a function of t possessing derivatives of all orders, find the derivative of

$$\mathbf{F} \times \frac{d\mathbf{F}}{dt} \cdot \frac{d^2\mathbf{F}}{dt^2}$$

10. By inspection, write down the values of each of the following:

(a) $\dfrac{d\mathbf{R}}{ds} \cdot \mathbf{T}$ (d) $\mathbf{T} \cdot \mathbf{N}$ (g) $[\mathbf{T},\mathbf{N},\mathbf{B}]$

(b) $\dfrac{d}{ds}(\mathbf{T} \cdot \mathbf{T})$ (e) $\dfrac{d\mathbf{R}}{dt} \cdot \mathbf{T}$ (h) $\left|\dfrac{d^2\mathbf{R}}{ds^2}\right|$

(c) $\dfrac{d^2\mathbf{R}}{dt^2} \cdot \mathbf{T}$ (f) $\dfrac{d\mathbf{N}}{ds} \cdot \mathbf{B}$ (i) $\dfrac{d\mathbf{B}}{ds}$

11. The *Darboux* vector is defined to be $\boldsymbol{\omega} = \tau\mathbf{T} + k\mathbf{B}$. Show that the equation

$$\frac{d\mathbf{U}}{ds} = \boldsymbol{\omega} \times \mathbf{U}$$

is satisfied for $\mathbf{U} = \mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$. Notice the resemblance of this equation to the equation in Sec. 1.12 describing angular velocity.

12. A rigorous derivation of the Frenet formulas proceeds as follows:

(a) Regard Eq. (2.39) as the defining equation for k and $\mathbf{N}$.

(b) Show that $\dfrac{d\mathbf{N}}{ds} + k\mathbf{T}$ is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$. (Here it is helpful to differentiate the relation $\mathbf{T} \cdot \mathbf{N} = 0$.) Thus (2.40) can be regarded as the defining equation for τ.

(c) Prove (2.41) from (2.39) and (2.40) by differentiating the relation $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

Carry out the details of this program.

13. True or false:

(a) If $\mathbf{R}$ is the position vector of a particle, t denotes time, and s denotes arc length, $d^2\mathbf{R}/dt^2$ is a scalar multiple of $d^2\mathbf{R}/ds^2$.

(b) A moving particle achieves its maximum speed at the instant $t = 3$. (Before and after that instant, its speed is less than its speed at $t = 3$.) It follows from this that its acceleration is zero at the instant $t = 3$.

(c) The acceleration of a particle moving along a curve with binormal $\mathbf{B}$ is always perpendicular to $\mathbf{B}$. [More precisely, $\mathbf{a}(t)$ and $\mathbf{B}(t)$ are orthogonal for each fixed value of t.]

2.4 PLANAR MOTION IN POLAR COORDINATES

In this section we consider the motion of a particle in the xy plane in which the position of the particle is given in polar coordinates, r and θ. We remind the reader that r and θ provide alternative descriptions of points in the plane, and they are sometimes more convenient when circular symmetries are present. They are depicted in Fig. 2.18 and are related to (x, y) coordinates through the equations

$$x = r\cos\theta \qquad r = (x^2 + y^2)^{\frac{1}{2}}$$

$$y = r\sin\theta \qquad \theta = \sin^{-1}\frac{y}{(x^2 + y^2)^{\frac{1}{2}}} = \cos^{-1}\frac{x}{(x^2 + y^2)^{\frac{1}{2}}}$$

The extra equations for θ are necessary to avoid quadrant ambiguities; customarily one takes $-\pi < \theta \leq \pi$.

We assume that the particle's trajectory is specified by giving r and θ as functions of the time t, and that these functions possess second derivatives.

In order to work directly with polar coordinates it is convenient to introduce unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$, which point respectively along the position vector and at right angles to it (in the direction of increasing θ), as shown in Fig. 2.18.

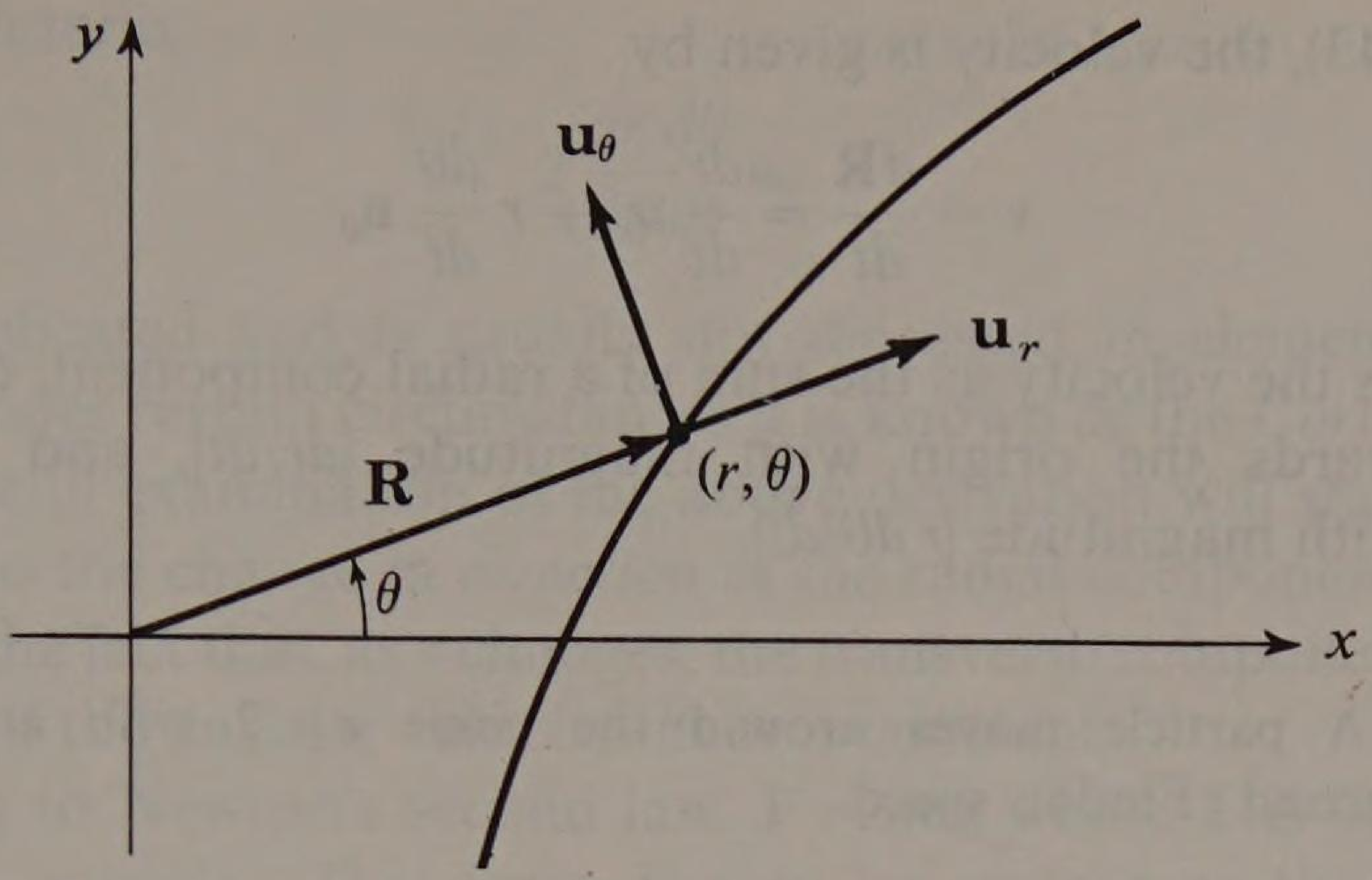

FIGURE 2.18

It is easy to see that $\mathbf{u}_r$ and $\mathbf{u}_\theta$ can be written in terms of **i** and **j** as follows:

$$\begin{aligned}\mathbf{u}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\\ \mathbf{u}_\theta &= -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\end{aligned} \tag{2.42}$$

Note that $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are functions of θ and are defined at every point in space except the origin. Unlike **i** and **j**, $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are not constants. For example, along the positive x axis, $\mathbf{u}_r = \mathbf{i}$, but along the positive y axis, $\mathbf{u}_r = \mathbf{j}$. It follows that we must be careful in differentiating vectors written in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$.

Directly from Eq. (2.42) we see that

$$\begin{aligned}\frac{d\mathbf{u}_r}{d\theta} &= \mathbf{u}_\theta\\ \frac{d\mathbf{u}_\theta}{d\theta} &= -\mathbf{u}_r\end{aligned} \tag{2.43}$$

(Notice that these important formulas reinforce the observations we made in the previous section about the derivatives of unit vectors rigidly attached to each other.)

The position vector of a particle located at a point (r,θ) is

$$\mathbf{R} = r\mathbf{u}_r \tag{2.44}$$

We obtain the velocity by differentiating Eq. (2.44) and using the chain rule,

$$\begin{aligned}\frac{d\mathbf{R}}{dt} &= \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt}\\ &= \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt}\end{aligned}$$

Hence, by (2.43), the velocity is given by

$$\mathbf{v} = \frac{d\mathbf{R}}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta \tag{2.45}$$

This expresses the velocity as the sum of a radial component, directed away from or towards the origin with magnitude $|dr/dt|$, and a transverse component with magnitude $|r\,d\theta/dt|$.

Example 2.13 A particle moves around the circle $r = 2$ with angular velocity $d\theta/dt = 5$ rad/second. Find its speed.

Solution Since r is a constant, $dr/dt = 0$. Hence

$$\mathbf{v} = r\left(\frac{d\theta}{dt}\right)\mathbf{u}_\theta = 10\mathbf{u}_\theta$$

Therefore $|\mathbf{v}| = 10$.

Example 2.14 A circular disk rotates with constant angular velocity 3 rad/sec. A fly walks from the center of the disk outward to the rim at a rate of 2 cm/sec (relative to the disk). Find the speed of the fly 4 seconds after he starts at the center.

Solution Since $dr/dt = 2$, we have $r = r_0 + 2t$. Since the fly starts at the center, $r_0 = 0$. Hence by Eq. (2.45)

$$\mathbf{v} = 2\mathbf{u}_r + 3r\mathbf{u}_\theta$$

At time $t = 4$, $r = 2t = 8$, so $\mathbf{v} = 2\mathbf{u}_r + 24\mathbf{u}_\theta$. The speed is then $(2^2 + 24^2)^{\frac{1}{2}} = (580)^{\frac{1}{2}}$ cm/sec.

Returning to (2.45), we differentiate again to obtain the acceleration:

$$\begin{aligned}
\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\frac{d\mathbf{u}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\frac{d\theta}{dt}\frac{d\mathbf{u}_\theta}{dt} \\
&= \frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\frac{d\theta}{dt}\frac{d\theta}{dt}\frac{d\mathbf{u}_\theta}{d\theta} \\
&= \frac{d^2r}{dt^2}\mathbf{u}_r + 2\frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta - r\left(\frac{d\theta}{dt}\right)^2\mathbf{u}_r
\end{aligned}$$

Combining terms,

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{u}_\theta \tag{2.46}$$

The first term in Eq. (2.46), $(d^2r/dt^2)\mathbf{u}_r$, gives the acceleration for pure radial motion; and the third term, $r(d^2\theta/dt^2)\mathbf{u}_\theta$, measures the effect of angular acceleration. In the special case that r is a constant, we have motion in a circle with center at the origin; then $\mathbf{u}_\theta$ and $\mathbf{u}_r$ are, respectively, the vectors $\mathbf{T}$ and $-\mathbf{N}$ of the preceding section. In this special case the second term is the centripetal acceleration term.

The fourth term,

$$2\frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta$$

is more complicated and is usually not discussed in elementary physics textbooks. Under certain circumstances it is known as the *Coriolis acceleration.* As a careful examination of the above derivation will show, this term is due partly to the change in *direction* of the radial component of velocity, and partly to the fact that, as r changes, the transverse component of velocity changes, even if the angular velocity $d\theta/dt$ is constant.

According to Newton's second law, $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ is the total force acting on the particle. This force $\mathbf{F}$ may be written as the sum of two components,

$$\mathbf{F} = F_r\mathbf{u}_r + F_\theta\mathbf{u}_\theta$$

The motion of the particle is then governed by the two differential equations

$$F_r = m\frac{d^2r}{dt^2} - mr\left(\frac{d\theta}{dt}\right)^2 \tag{2.47}$$

$$F_\theta = mr\frac{d^2\theta}{dt^2} + 2m\frac{dr}{dt}\frac{d\theta}{dt} \tag{2.48}$$

If both sides of Eq. (2.48) are multiplied by r, (2.48) can be written in the form

$$rF_\theta = \frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right) \tag{2.49}$$

which in some cases may be interpreted as stating that the torque applied to the particle equals the time rate of change of its angular momentum.

If $F_\theta = 0$, (2.49) may be integrated to yield $mr^2\,d\theta/dt = C$. In other words, if the force is always directed radially toward or away from the origin (a "central force field"), then the angular momentum of the particle will be constant. This immediately implies Kepler's second law of planetary motion, that the radius vector in a central force field sweeps over area at a constant rate, since the rate at which the vector $\mathbf{R}$ sweeps out area is

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$$

Exercises

1. Find $d^3\mathbf{R}/dt^3$ in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$.
2. A particle moves in a plane with constant angular velocity ω about the origin, but r varies so that the rate of increase of its acceleration is parallel to the position vector $\mathbf{R}$. Show that $d^2r/dt^2 = r\omega^2/3$.

3. Find $\mathbf{v}$ and $\mathbf{a}$ if a particle moves so that
$$r = b(1 - \cos\theta)$$
and
$$\frac{d\theta}{dt} = 4$$

4. Find $\mathbf{v}$ and $\mathbf{a}$ if
$$r = b(1 + \sin t)$$
$$\theta = e^{-t} - 1$$

5. The force $\mathbf{F}$ exerted by a magnetic field $\mathbf{B}$ on a particle carrying a charge q is given by $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$, where $\mathbf{v}$ is the velocity of the particle. Draw a diagram showing the relative directions of $\mathbf{v}$, $\mathbf{B}$, and $\mathbf{F}$, in some special cases. Under what circumstances will the field exert no force on the particle?

6. A particle of mass m and charge q moves in a constant magnetic field $\mathbf{B}$ directed parallel to the z axis. If the resulting trajectory is a circle of radius r in the xy plane, express q/m in terms of v, r, and $\mathbf{B}$.

7. Which terms in Eq. (2.46) will be nonzero, in each of the following cases?
   (a) A particle moves around a circle with center at the origin with constant nonzero angular velocity.
   (b) A particle moves around a circle with center at the origin with constant nonzero angular acceleration.
   (c) A particle moves along a straight line not passing through the origin, with constant speed.
   (d) A person is walking from the center of a merry-go-round towards its outer edge (discuss various possibilities).

8. A particle moves along a straight line, not passing through the origin. Is $r(d\theta/dt)^2$ nonzero?

9. A particle moves with constant radial speed 2 cm/sec away from the center of a platform rotating with uniform angular velocity of 30 rev/min.
   (a) What is its radial acceleration?
   (b) What is its Coriolis acceleration?

10. Find the magnitude of the Coriolis acceleration of a particle moving in the xy plane with position given by
$$x = 3t\cos 4\pi t$$
$$y = 3t\sin 4\pi t$$

11. A particle of mass m moves in a force field $\mathbf{F}$. Assume that the sum of its kinetic and potential energy is $E = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} - \int_0^t \mathbf{F}\cdot\mathbf{v}\,dt$. Using Eqs. (2.45), (2.47), and (2.48), express E in polar coordinates and show that $dE/dt = 0$, and hence that E is constant.

2.5 OPTIONAL READING: TENSOR NOTATION

As shown in Sec. 2.1, differentiation of a vector function proceeds componentwise. That is, the ith component of $d\mathbf{F}/dt$ is the derivative of the ith component of $\mathbf{F}$:
$$\left(\frac{d\mathbf{F}}{dt}\right)_i = \frac{dF_i}{dt}$$

This happy circumstance makes the tensor notation for the rules in Theorems 2.1 through 2.4, and their proofs, quite apparent. Thus for the cross product we have

$$\frac{d}{dt}\varepsilon_{ijk}F_jG_k = \varepsilon_{ijk}\frac{dF_j}{dt}G_k + \varepsilon_{ijk}F_j\frac{dG_k}{dt}$$

by the rules of ordinary calculus. Interpreted in vector notation, this says

$$\frac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \frac{d\mathbf{F}}{dt} \times \mathbf{G} + \mathbf{F} \times \frac{d\mathbf{G}}{dt}$$

which is Eq. (2.7).

The other theorems are equally straightforward.

Exercise

Derive the rule for the derivative of the dot product.

Supplementary Problems

1. Let C be the curve given by the equation

$$\mathbf{R}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \log \sec t\,\mathbf{k} \qquad 0 \le t < \frac{\pi}{2}$$

find:

(a) the element of arc length, ds, along C, in terms of t,
(b) the unit tangent $\mathbf{T}$,
(c) the unit normal $\mathbf{N}$, and
(d) the curvature k.

2. If C is the curve given parametrically by

$$\mathbf{R}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}$$

find:

(a) the normal $\mathbf{N}$ and the binormal $\mathbf{B}$ for this curve at $t = 0$, and
(b) the equation of the plane passing through the point $\mathbf{R}(0)$ and parallel to both vectors $\mathbf{N}$ and $\mathbf{B}$ of part (a).

3. A particle moves so that its coordinates at time t are given by

$$x(t) = e^{-t}\cos t \qquad y(t) = e^{-t}\sin t \qquad z(t) = e^{-t}$$

Find its velocity, speed, and acceleration, and the curvature of its path at time t.

4. A particle moves so that its position $\mathbf{R}$ at time t is given by

$$\mathbf{R}(t) = \log(t^2 + 1)\mathbf{i} + (t - 2\,\text{Arctan}\,t)\mathbf{j} + 2\sqrt{2}t\mathbf{k}$$

(a) Show that this particle moves with constant speed $v = 3$.
(b) Find the curvature of the path of this particle.

5. A point moves along a curve so that its position $\mathbf{R}$ is given by

$$\mathbf{R} = t^2\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$$

find:

(a) its speed v,

(b) the unit tangent $\mathbf{T}$ to its path, and

(c) the vector $k\mathbf{N}$.

6. (a) Find the unit tangent $\mathbf{T}$, the principal normal $\mathbf{N}$, and the binormal $\mathbf{B}$ for the curve

$$x = \cos^3 t \qquad y = \sin^3 t \qquad z = 2\sin^2 t \qquad 0 < t \le \frac{\pi}{2}$$

(b) Find the curvature and the torsion for the preceding curve.

7. A particle moves so that its position (r,θ) in polar coordinates is given by

$$r = 2(1 + \sin\theta) \qquad \theta = e^{-t}$$

find its velocity $\mathbf{v}$ in terms of the vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$.

8. An experiment is being designed in which a particle of mass 1 is to exhibit the following planar motion in polar coordinates:

$$\left.\begin{aligned} r(t) &= 1 + t \\ \theta(t) &= \frac{\pi}{1+t} \end{aligned}\right\} \quad \text{for } t \ge 0$$

Determine:

(a) the position and velocity of this particle at the time $t = 1$, illustrating your answer in a diagram, and

(b) the radial and transverse forces $F_r(t)$ and $F_\theta(t)$ needed on the particle to attain the desired motion.

(c) If the forces acting on this particle are removed at $t = 1$, find its position at $t = 5$.

9. A charged particle moves along the curve

$$r = \frac{1}{1 + 2\cos\theta} \qquad \text{with} \qquad \frac{d\theta}{dt} = \frac{1}{r^2}$$

(a) By differentiating the equation $\mathbf{R} = r\mathbf{u}_r$, show that

$$\frac{d\mathbf{R}}{dt} = 2\sin\theta\,\mathbf{u}_r + \frac{1}{r}\,\mathbf{u}_\theta$$

(b) Find $\dfrac{d^2\mathbf{R}}{dt^2}$ and simplify.

10. A particle, starting at $t = 0$ from the point $r = 2$, $\theta = 0$ in polar coordinates, moves so that

$$r = 2 + \sin t \qquad v = \sqrt{2}\cos t$$

Find a formula for the angle θ in terms of t, and determine the position of this particle at time $t = \pi/2$. (Assume that $\theta \ge 0$ for all t.)

11. A disc rotates back and forth with angular velocity $\cos t$ radians/second. An insect starting 1 cm from the center of the disc at time $t = 0$ crawls outward at a rate of $2t$ cm/sec. Find the position, velocity, and speed of the insect after 2π seconds.

12. The *evolute* of a curve $\mathbf{R}(t)$ is the locus of the centers of curvature of the curve. Using the parametric formulas, show that the tangent to the evolute is normal to the original curve.

13. If the curve $\mathbf{R}(t)$ lies on a sphere $|\mathbf{R}(t)| = constant$, prove that

$$\mathbf{R} = -\rho\mathbf{N} - \frac{1}{\tau}\frac{d\rho}{ds}\mathbf{B}$$

using the terminology of Sec. 2.3. (*Hint*: Keep differentiating $\mathbf{R} \cdot \mathbf{R} = constant$, using the Frenet formulas.)

CHAPTER 3

Scalar and Vector Fields

3.1 SCALAR FIELDS; ISOTIMIC SURFACES; GRADIENTS

If to each point (x,y,z) of a region in space there is made to correspond a number $f(x,y,z)$, we say that f is a *scalar field.* In other words, a scalar field is simply a scalar-valued function of three variables.

For the sake of fixing ideas, the following scalar fields are given as examples that will be referred to repeatedly:

1. $f(x,y,z) = x + 2y - 3z$
2. $f(x,y,z) = x^2 + y^2 + z^2$
3. $f(x,y,z) = x^2 + y^2$
4. $f(x,y,z) = \dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2$
5. $f(x,y,z) = \sqrt{x^2 + y^2} - z$
6. $f(x,y,z) = \dfrac{1}{x^2 + y^2}$

The fields in examples 1 through 5 are defined at every point in space. The field in example 6 is defined at every point (x,y,z) except where $x^2 + y^2 = 0$, that is, everywhere except on the z axis.

If f is a scalar field, any surface defined by $f(x,y,z) = C$, where C is a constant, is called an *isotimic surface* (from the Greek *isotimos*, meaning *of equal value*). Sometimes, in physics, more specialized terms are used. For instance, if f denotes either electric or gravitational field potential, such surfaces are called *equipotential surfaces.* If f denotes temperature, they are called *isothermal surfaces.* If f denotes pressure, they are called *isobaric surfaces.*

In the above examples, the isotimic surfaces are:

1. All planes perpendicular to the vector $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.
2. All spheres with center at the origin.
3. All right circular cylinders with the z axis as axis of symmetry.
4. A family of ellipsoids.
5. A family of cones.
6. The same as in example 3.

It is impossible for distinct isotimic surfaces of the same scalar field to intersect, since only one number $f(x,y,z)$ is associated with any one point (x,y,z).

Here are some physical examples of scalar fields: the mass density of the atmosphere, the temperature at each point in an insulated wall, the water pressure at each point in the ocean, the gravitational potential of points in astronomical space, the electrostatic potential of the region between two condenser plates. Such scalar fields as density and pressure are only approximate idealizations of a complicated physical situation, since they take no account of the atomic properties of matter.

Let us consider the behavior of a scalar field in the neighborhood of a point (x_0,y_0,z_0) within its region of definition. Let us imagine a line segment passing through (x_0,y_0,z_0) parallel to a given vector $\mathbf{u}$. Let s denote the displacement measured along the line segment in the direction of $\mathbf{u}$ (Fig. 3.1) with $s = 0$ corresponding to (x_0,y_0,z_0). To each value of the parameter s there corresponds a point (x,y,z) on the line segment, and hence a corresponding scalar $f(x,y,z)$. The derivative df/ds at $s = 0$, if this derivative exists, is called the *directional derivative* of f at (x_0,y_0,z_0), in the direction of the vector $\mathbf{u}$.

In other words, the directional derivative of f is simply the rate of change of f, per unit distance, in some prescribed direction. The directional derivative df/ds will generally depend on the location of the point (x_0,y_0,z_0) and also on the direction prescribed.

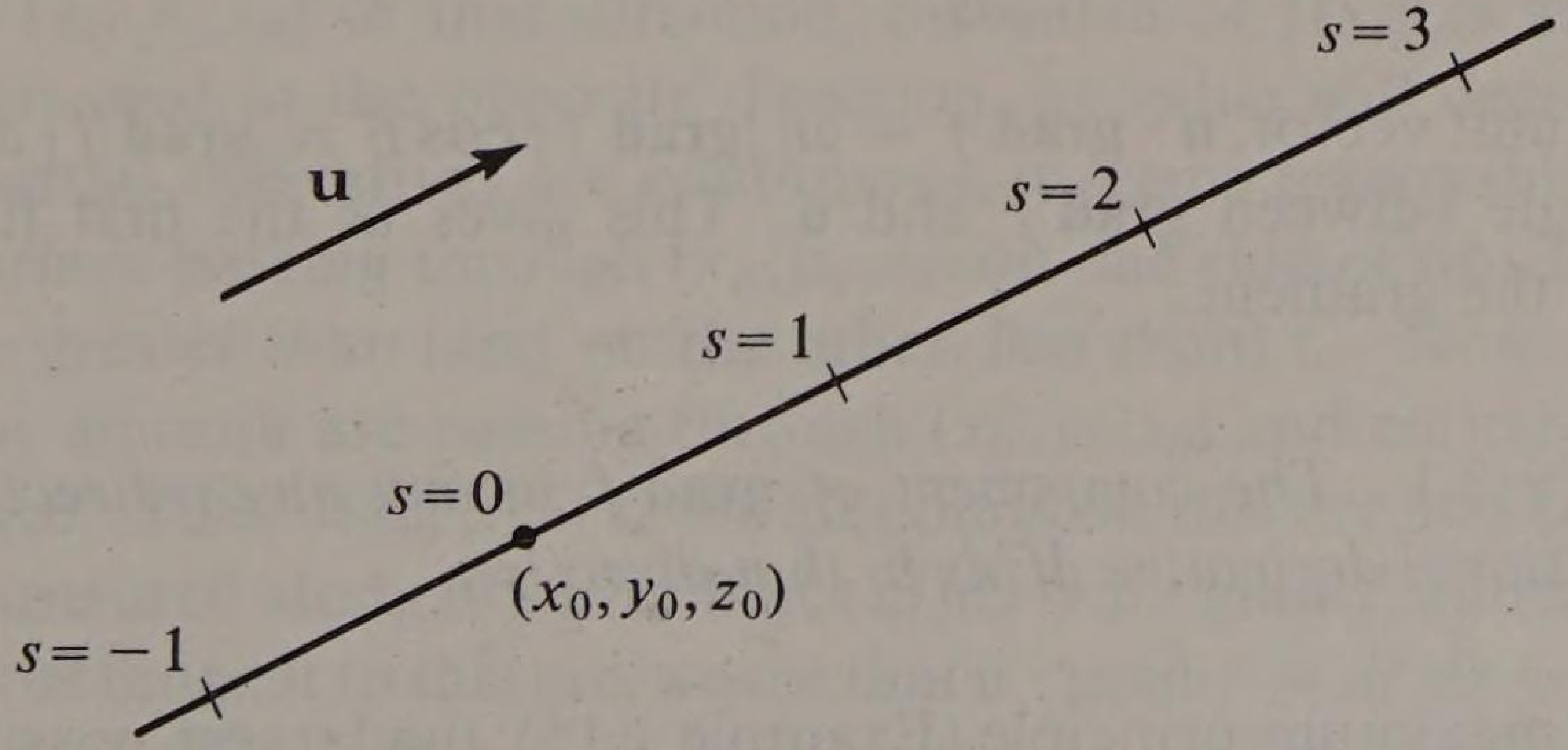

FIGURE 3.1

The directional derivative of a scalar field f in a direction parallel to the x axis, with s measured as increasing in the positive x direction, is conventionally denoted $\partial f/\partial x$, and is called the partial derivative of f with respect to x. Similarly, the directional derivative of f in the positive y direction is called $\partial f/\partial y$, and that in the positive z direction, $\partial f/\partial z$. We assume that the reader has had some experience with partial derivatives.

The directional derivative of a scalar field f in a direction that is not parallel to any of the coordinate axes is conventionally denoted df/ds, but of course this symbol is ambiguous; it would not make sense to ask "what is df/ds" without specifying the direction in which s is to be measured.

A convenient way of specifying the desired direction is by prescribing a vector $\mathbf{u}$ pointing in that direction. Although the magnitude of $\mathbf{u}$ is immaterial, it is conventional to take $\mathbf{u}$ to be a unit vector. We have already seen (Sec. 2.3) that a unit vector in a desired direction can be obtained by computing $d\mathbf{R}/ds$ in that direction, where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. That is,

$$\mathbf{u} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k} \tag{3.1}$$

is a unit vector pointing in the direction in which s is measured. Here we are thinking of x, y, and z as functions of the parameter s, for points (x,y,z) on the line segment; s is, of course, arc length along the segment.

If the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ exist and are continuous throughout a region, then it is well known (see Appendix B for a proof) that the following chain rule is valid:

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \tag{3.2}$$

If we define the *gradient* of f to be the vector

$$\mathbf{grad}\ f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{3.3}$$

we see that the right side of (3.2) is the dot product of $\mathbf{u}$ with $\mathbf{grad}\ f$,

$$\frac{df}{ds} = \mathbf{u} \cdot \mathbf{grad}\ f \tag{3.2$'$}$$

Since $\mathbf{u}$ is a unit vector, $\mathbf{u} \cdot \mathbf{grad}\ f = |\mathbf{u}|\,|\mathbf{grad}\ f| \cos\theta = |\mathbf{grad}\ f| \cos\theta$, where θ is the angle between $\mathbf{grad}\ f$ and $\mathbf{u}$. This gives us the first fundamental property of the gradient:

PROPERTY 3.1 *The component of* $\mathbf{grad}\ f$ *in any given direction gives the directional derivative* df/ds *in that direction.*

By the maximum principle (Example 1.15), the largest possible value of $\mathbf{u} \cdot \mathbf{grad}\ f$, for unit vectors $\mathbf{u}$, is obtained when $\mathbf{u}$ is in the same direction as

grad f (assuming **grad** $f \neq \mathbf{0}$). Since $\mathbf{u} \cdot \textbf{grad}\, f = df/ds$, it follows that the maximum value of df/ds is obtained in the direction of **grad** f. This is the second fundamental property of the gradient:

> PROPERTY 3.2 **grad** f *points in the direction of the maximum rate of increase of the function* f.

If **u** points in the direction of **grad** f, then

$$\mathbf{u} \cdot \textbf{grad}\, f = |\mathbf{u}|\, |\textbf{grad}\, f| \cos\theta = |\textbf{grad}\, f|$$

which gives the third fundamental property of the gradient:

> PROPERTY 3.3 *The magnitude of* **grad** f *equals the maximum rate of increase of* f *per unit distance.*

Experience has shown that the wording of these fundamental properties makes them rather easy to memorize [and they *should* be memorized, together with the definition (3.3)].

The fourth fundamental property of the gradient of a function makes it possible to use the gradient concept in solving geometrical problems:

> PROPERTY 3.4 *Through any point* (x_0,y_0,z_0) *where* **grad** $f \neq \mathbf{0}$, *there passes an isotimic surface* $f(x,y,z) = C$; **grad** f *is normal (i.e., perpendicular) to this surface at the point* (x_0,y_0,z_0).

This property holds only when $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ exist and are continuous in a neighborhood of the point in question. The constant C is, of course, equal to $f(x_0,y_0,z_0)$. If **grad** $f = \mathbf{0}$ the locus of points satisfying $f(x,y,z) = C$ might not form a surface. (Consider, for example, this locus if f is a *constant* function.)

We omit a detailed proof of this fourth property, but the following discussion may make it seem reasonable. Let C denote the value of f at (x_0,y_0,z_0). Since **grad** $f \neq \mathbf{0}$, it follows from the preceding fundamental properties that df/ds will be positive in some direction. If, then, we proceed away from (x_0,y_0,z_0) in that direction, the value of $f(x,y,z)$ will increase, and if we proceed in the opposite direction, its value will decrease. Since f and its partial derivatives are continuous, it seems reasonable that there will be a surface passing through (x_0,y_0,z_0) on one side of which the values of f will be greater than (and on the other, less than) C. Now suppose we consider any smooth arc passing through (x_0,y_0,z_0) and entirely contained in this surface. Then $f(x,y,z) = C$ for all points on this arc and so $df/ds = 0$, where s is measured along this arc. Since $df/ds = \mathbf{u} \cdot \textbf{grad}\, f$, and in this case **u** is a unit vector tangent to this arc, we see that $\mathbf{u} \cdot \textbf{grad}\, f = df/ds = 0$, implying that **grad** f is perpendicular to **u**. This reasoning applies to any smooth arc in the surface passing through (x_0,y_0,z_0). Hence **grad** f is perpendicular

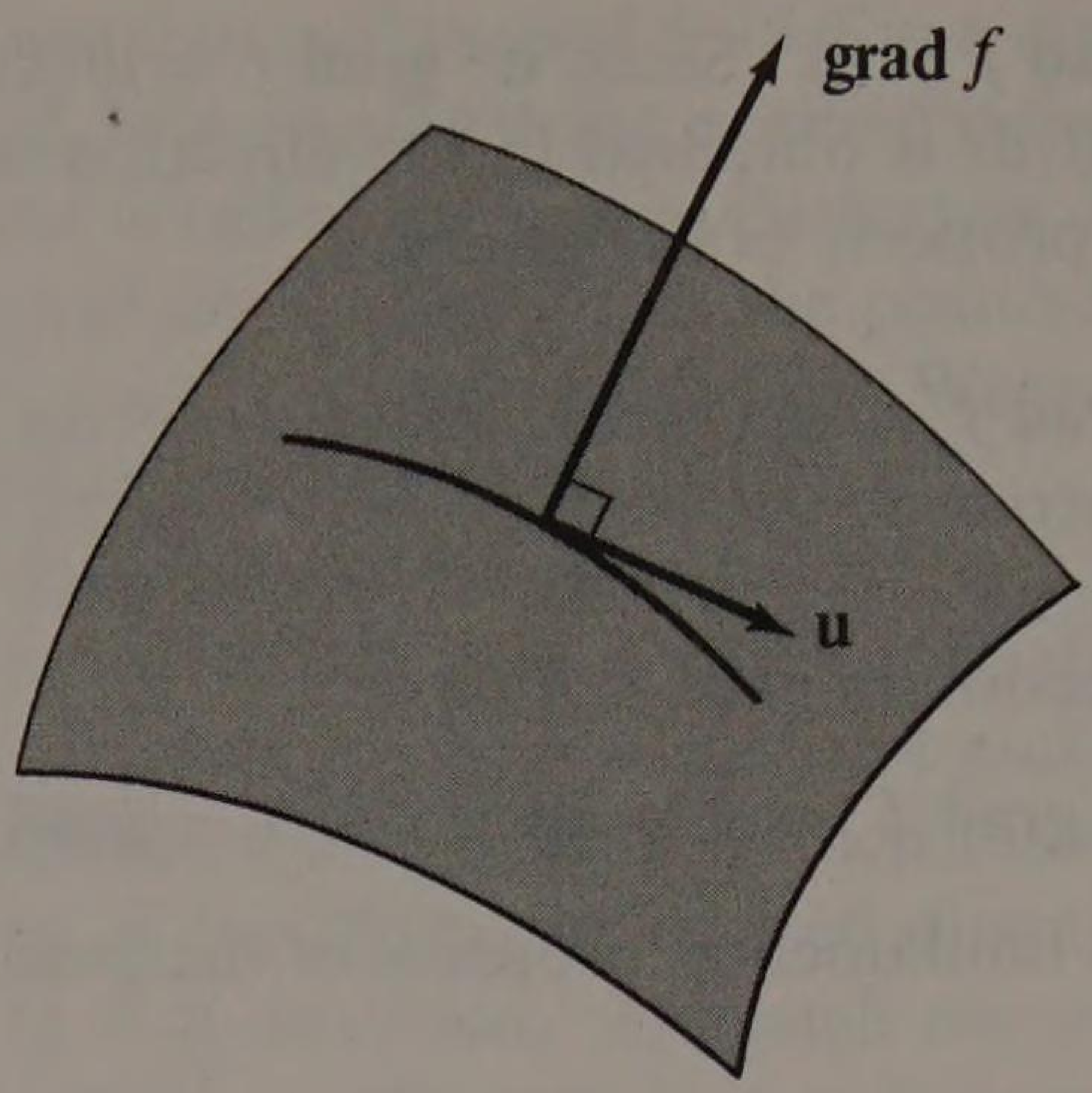

FIGURE 3.2

to every such arc, at that point, which can be the case only if **grad** f is perpendicular to the surface (Fig. 3.2).

We now return to the six examples given previously. In each case the gradient is easily computed using the definition (3.3):

1. **grad** $f = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$
2. **grad** $f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$
3. **grad** $f = 2x\mathbf{i} + 2y\mathbf{j}$
4. **grad** $f = \dfrac{x}{2}\mathbf{i} + \dfrac{2y}{9}\mathbf{j} + 2z\mathbf{k}$
5. **grad** $f = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}} - \mathbf{k}$
6. **grad** $f = -\dfrac{2x}{(x^2 + y^2)^2}\mathbf{i} - \dfrac{2y}{(x^2 + y^2)^2}\mathbf{j}$

1. (This is the only one of the six examples for which **grad** f is a constant.) We already know from Sec. 1.10 that $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ is perpendicular to any plane of the form $x + 2y - 3z = C$. We see that **grad** $f = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. Thus we have verified the fourth fundamental property, in this special case.
2. In this case the isotimic surfaces are spheres centered at the origin, so the normals to these surfaces must be vectors pointing directly away from the origin. Sure enough, we have **grad** $f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = 2\mathbf{R}$, and we know that the vector $2\mathbf{R}$ always points directly away from the origin. To see the significance of the number 2 here, let r denote the distance from the origin to the point (x,y,z). Then we can, in this example, write the function in terms of r: it is simply r^2. Moreover, if we move away from any point in the direction of maximum increase of r^2, which obviously means moving directly away from the origin, then the element

of arc length is simply dr. In this direction, the derivative df/ds is df/dr, and $(d/dr)(r^2) = 2r$. Also, $|2\mathbf{R}| = 2r$, so we have verified the third fundamental property in this special case.

3. The reader familiar with cylindrical coordinates can do the same thing here as we just did with example 2. Let $\rho = (x^2 + y^2)^{\frac{1}{2}}$, the distance from the point (x,y,z) to the z axis. The function f in this example is simply ρ^2, and obviously increases most rapidly in a direction perpendicular to the z axis. Its derivative in this direction is 2ρ, which is also the magnitude $|\mathbf{grad}\ f| = (4x^2 + 4y^2)^{\frac{1}{2}}$. The direction is clearly normal to the isotimic surfaces, since the latter are right circular cylinders centered on the z axis. The second, third, and fourth fundamental properties are extremely transparent in this case, as they were in example 2.
5. (We skip example 4.) All we care to note here is the elementary geometrical significance of the $-\mathbf{k}$ term in $\mathbf{grad}\ f$. The isotimic surfaces of this function are conical; each has an apex on the z axis and spreads outward with increasing z. Thus, we see easily that the normal to one such surface will not point directly away from the z axis, as it does in example 3, but will have an additional, constant component in the negative z direction.

The following are some sample problems that illustrate the use of the fundamental properties of the gradient of a scalar field.

Example 3.1 Find df/ds in the direction of the vector $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, at the point $(1,1,2)$, if $f(x,y,z) = x^2 + y^2 - z$.

Solution $\mathbf{grad}\ f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ at $(1,1,2)$. A unit vector in the desired direction is $\mathbf{u} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$ (obtained by dividing $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ by its own length). Property 3.1 then gives $df/ds = \mathbf{u} \cdot \mathbf{grad}\ f = \frac{4}{3} + \frac{4}{3} + \frac{1}{3} = 3$. This means that the value of the function f is increasing 3 units per unit distance, if we proceed from $(1,1,2)$ in the direction stated.

Example 3.2 The temperature of points in space is given by $f(x,y,z) = x^2 + y^2 - z$. A mosquito located at $(1,1,2)$ desires to fly in such a direction that he will get cool as soon as possible. In what direction should he move?

Solution As we saw in Example 3.1, $\mathbf{grad}\ f = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ at $(1,1,2)$. The mosquito should move in the direction $-\mathbf{grad}\ f$, since $\mathbf{grad}\ f$ is in the direction of increasing temperature.

Example 3.3 A mosquito is flying at a speed of 5 units of distance per second, in the direction of the vector $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. The temperature is given by $f(x,y,z) = x^2 + y^2 - z$. What is his rate of increase of temperature, per unit time, at the instant he passes through the point $(1,1,2)$?

Solution As shown in Example 3.1 above, df/ds in this direction is 3 units per unit distance. The rate of increase of temperature per unit time is thus $df/dt = (df/ds)(ds/dt) = (3)(5) = 15$ degrees per second.

Example 3.4 What is the maximum possible df/ds, if $f(x,y,z) = x^2 + y^2 - z$, at the point (1,4,2)?

Solution $|\mathbf{grad}\, f| = |2\mathbf{i} + 8\mathbf{j} - \mathbf{k}| = \sqrt{69}$. The answer is approximately 8.31 units per unit distance.

Example 3.5 Find a unit vector normal to the surface $x^2 + y^2 - z = 6$ at the point (2,3,7).

Solution This is an isotimic surface for the function $f(x,y,z) = x^2 + y^2 - z$. At (2,3,7), we have $\mathbf{grad}\, f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} = 4\mathbf{i} + 6\mathbf{j} - \mathbf{k}$. The length of this vector is $\sqrt{53}$. Thus, an answer is $(\sqrt{53}/53)(4\mathbf{i} + 6\mathbf{j} - \mathbf{k})$. (The negative of this vector is also a correct answer.)

The reader may have observed that the number "6", the constant on the right-hand side of the equation defining the isotimic surface in Example 3.5, appears to have no effect on the normal, **grad** f. This is not quite true. Granted, the formula for **grad** f ignores the 6, but when it is evaluated at (x,y,z) the numbers x, y, and z must satisfy $x^2 + y^2 - z = 6$. Clearly, $2^2 + 3^2 - 7 = 6$.

Exercises

1. Compute **grad** f if
  (a) $f = \sin x + e^{xy} + z$
  (b) $f = 1/|\mathbf{R}|$
  (c) $f = \mathbf{R} \cdot \mathbf{i} \times \mathbf{j}$

2. If $f(x,y,z) = x^2 + y^2$, what is the locus of points in space for which **grad** f is parallel to the y axis?

3. What can you say about a function whose gradient is everywhere parallel to the y axis?

4. Find all functions $f(x,y,z)$ such that $\mathbf{grad}\, f = 2x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$.

5. Describe **grad** f in words, without actually doing any calculating, given that $f(x,y,z)$ is the distance between (x,y,z) and the z axis.

6. Find the derivative of $f(x,y,z) = x + xyz$ at the point $(1,-2,2)$ in the direction of (a) $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, (b) $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

7. Find the directional derivative df/ds at $(1,3,-2)$ in the direction of $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ if
  (a) $f(x,y,z) = yz + xy + xz$
  (b) $f(x,y,z) = x^2 + 2y^2 + 3z^2$
  (c) $f(x,y,z) = xy + x^3y^3$
  (d) $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$

8. Given $f(x,y,z) = x^2 + y^2 + z^2$, find the maximum value of df/ds at the point (3,0,4),
  (a) by using the gradient of f;
  (b) by interpreting f geometrically.

9. Find the magnitude of the greatest rate of change of $f(x,y,z) = (x^2 + z^2)^3$ at $(1,3,-2)$. Interpret geometrically.

10. Find a vector normal to the surface $x^2 + yz = 5$ at $(2,1,1)$.

11. Find an equation of the plane tangent to the sphere $x^2 + y^2 + z^2 = 21$ at $(2,4,-1)$.

12. Find a vector normal to the cylinder $x^2 + z^2 = 8$ at (2,0,2,),
(a) by inspection (draw a diagram);
(b) by finding the gradient of the function $f(x,y,z) = x^2 + z^2$ at $(2,0,2)$.

13. Find an equation of the plane tangent to the surface $z^2 - xy = 14$ at $(2,1,4)$.

14. Find equations of the line normal to the sphere $x^2 + y^2 + z^2 = 2$ at $(1,1,0)$,
(a) by inspection (draw a diagram);
(b) by computing the gradient of $f(x,y,z) = x^2 + y^2 + z^2$ at $(1,1,0)$, and using this to find the normal.

15. Find a unit vector normal to the plane $3x - y + 2z = 3$,
(a) by the methods of Sec. 1.10;
(b) by the methods of the preceding section.

16. Find an equation of the plane tangent to the surface $z = x^2 + y^2$ at $(2,3,13)$. [*Hint*: Consider the function $f(x,y,z) = x^2 + y^2 - z$.]

17. Find a unit vector tangent to the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the sphere $x^2 + y^2 + z^2 = 9$ at the point $(\sqrt{2},\sqrt{2},\sqrt{5})$,
(a) by drawing a diagram, obtaining the answer by inspection;
(b) by finding the vector product of the normals to the two surfaces at that point;
(c) by writing the equations of the curve in parametric form. (*Hint*: Let $x = 2 \sin t$ and $y = 2 \cos t$.)

18. Determine the angle between the normals of the intersecting spheres $x^2 + y^2 + z^2 = 16$ and $(x-1)^2 + y^2 + z^2 = 16$, at the point $(1/2, 3/2, 3\sqrt{6}/2)$.

19. At what angle does the line $2x = y = 2z$ intersect the ellipsoid $2x^2 + y^2 + 2z^2 = 8$?

3.2 VECTOR FIELDS AND FLOW LINES

A *vector field* **F** is a rule associating with each point (x,y,z) in a region a vector $\mathbf{F}(x,y,z)$. In other words, a vector field is a vector-valued function of three variables.

Some vector fields are not defined for all points in space. For example, the vector field

$$\mathbf{F}(x,y,z) = \frac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$$

is not defined along the z axis, since $x^2 + y^2 = 0$ for points on the z axis.

In visualizing a vector field, we imagine that from each point in the region there extends a vector. Both direction and magnitude may vary with position (Fig. 3.3).

Any vector field may be written in terms of its components:

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$$

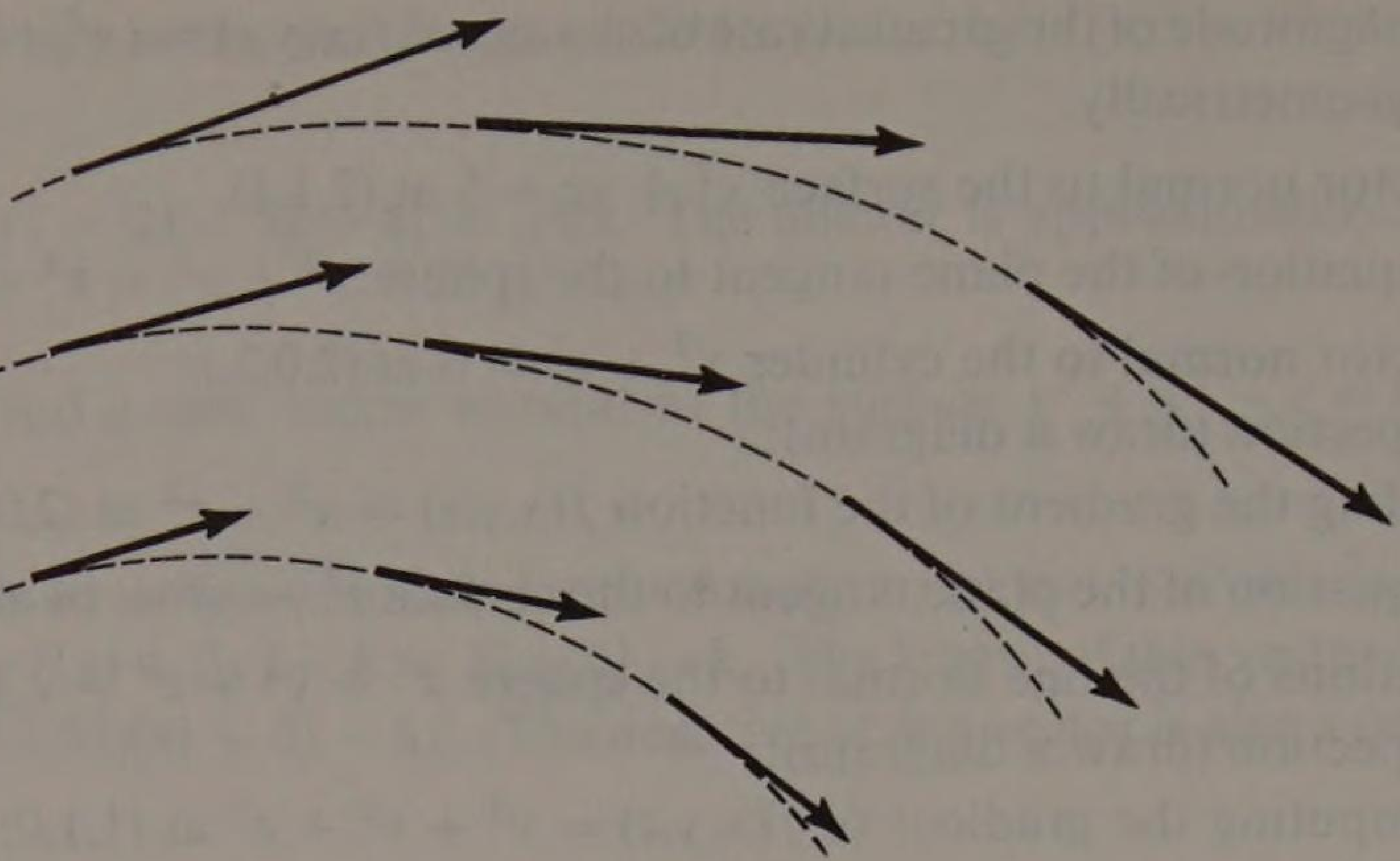

FIGURE 3.3

Example 3.6 If $f(x,y,z)$ is a *scalar* field, **grad** f is a *vector* field.

Example 3.7 Each of the "vectors" $\mathbf{u}_r$ and $\mathbf{u}_\theta$ (Sec. 2.4) is a vector field defined in the plane.

Example 3.8 In hydrodynamics, one associates with each point of a region the velocity of the fluid passing that point. In this manner one obtains, at any instant of time, a vector field describing the instantaneous velocity of the fluid at every point.

Example 3.9 In theoretical physics, there is associated with each point in space an electric intensity vector, representing the force that would be exerted, per unit charge, on a charged particle, if it were located at that point. This electric field, at any instant of time, constitutes a vector field. (Magnetic fields and gravitational fields also provide examples of vector fields defined in space.)

Let us consider a vector field $\mathbf{F}$ that is defined and nonzero at every point of a region in space. Any curve passing through this region is called a *flow line* of $\mathbf{F}$ provided that, at every point on the curve, $\mathbf{F}$ is tangent to the curve. (Flow lines are also called *stream lines or characteristic curves* of $\mathbf{F}$. If $\mathbf{F}$ is a force field, the flow lines are commonly called *lines of force.*) In Fig. 3.3, three flow lines are indicated as dotted curves.

This may be looked at in another way. The vector field $\mathbf{F}$ determines, at each point in the region, a direction. If a particle moves in such a manner that the direction of its velocity at any point coincides with the direction of the vector field $\mathbf{F}$ at that point, the space curve traced out is a flow line.

If the vector field $\mathbf{F}(x,y,z)$ describes the velocity at each point in a hydrodynamic system, the flow lines are the paths which are traversed by the component particles of the fluid, assuming that $\mathbf{F}$ is not a function of time. (The situation is more complicated for time-varying flows.)

Note that, if $g(x,y,z)$ is a scalar field that is not zero at any point, the flow lines of the vector field $g(x,y,z)\mathbf{F}(x,y,z)$ will be the same as those of $\mathbf{F}(x,y,z)$, since only the *direction* of $\mathbf{F}$ at any point is relevant in determining the flow lines.

Since the direction of a flow line is uniquely determined by the field **F**, it is impossible to have two different directions at the same point, and therefore it is impossible for two flow lines to cross. If the magnitude of **F** is zero at some point in space, then no direction is defined at that point and no flow line passes through that point. Now let's see how to calculate flow lines.

If **R** is the position vector to an arbitrary point of a flow line, and if s represents arc length measured along the flow line, then the unit vector tangent to the curve at that point is given by

$$\mathbf{T} = \frac{d\mathbf{R}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k} \tag{3.4}$$

The requirement that **T** have the same direction as **F** can be written

$$\mathbf{T} = \beta\mathbf{F} \tag{3.5}$$

where β is a scalar-valued function of x, y, and z. This can be written in terms of components,

$$\beta F_1 = \frac{dx}{ds} \qquad \beta F_2 = \frac{dy}{ds} \qquad \beta F_3 = \frac{dz}{ds} \tag{3.6}$$

If F_1, F_2, and F_3 are all nonzero, we may eliminate β and write (3.6) in differential form,

$$\frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3} \tag{3.7}$$

If one of these functions (say F_3) is identically zero in a region, then we obtain directly from (3.6) that the curve lies in a plane (say, z = constant) parallel to one of the coordinate planes.

Example 3.10 If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$, then $F_1 = x$, $F_2 = y$, and $F_3 = 1$, giving $dx/x = dy/y = dz$. Solving the differential equations $dx/x = dz$ and $dy/y = dz$, we obtain $x = C_1e^z$, $y = C_2e^z$. Thus the equations of the flow line passing through the point (3,4,7) are $x = 3e^{z-7}$, $y = 4e^{z-7}$. The equations of the flow line passing through the origin are $x = 0$, $y = 0$—i.e., the z axis.

Example 3.11 If $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, then $F_1 = x$, $F_2 = y$, and $F_3 = 0$. In this case (3.6) becomes $\beta x = dx/ds$, $\beta y = dy/ds$, and $0 = dz/ds$. Eliminating β from the first two equations we obtain $dx/x = dy/y$, and, solving, we obtain $y = Cx$. From the third equation we obtain z = constant. The field is zero when both x and y equal zero, and so the flow lines are not defined along the z axis. The flow lines are straight half-lines parallel to the xy plane, extending outward from the z axis.

Example 3.12 If $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, then $-\beta y = dx/ds$, $\beta x = dy/ds$, and $0 = dz/ds$. Thus $-dx/y = dy/x$, and hence $x^2 + y^2$ = constant. Also, we have z = constant. The flow lines are circles surrounding the z axis and are parallel to the xy plane. As in Example 3.11, no flow lines pass through points on the z axis.

Flow lines may be infinite in extent, as in Examples 3.10 and 3.11, or they may close upon themselves, as in Example 3.12.

Exercises

1. A vector field **F** is defined in the xy plane by $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. Draw a diagram similar to Fig. 3.3, showing the values of **F** at the points $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$, $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$, and a scattering of other points. Indicate flow lines.
2. Let $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + \mathbf{k}$.
   (a) Find the general equation of a flow line.
   (b) Find the flow line through the point $(1,1,2)$.
3. Without doing any calculating at all, describe the flow lines of the vector field $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. [*Hint*: If a particle located at (x,y,z) has velocity **R**, in what direction is it moving relative to the origin?]
4. The flow lines of the gradient of a scalar field cross the isotimic surfaces orthogonally. Explain.

3.3 DIVERGENCE

The concept of *gradient*, as we have presented it, applies only to scalar fields. We now consider the more complicated problem of describing the rate of change of a *vector* field. There are two fundamental measures of the rate of change of a vector field: the *divergence* and the *curl*.

Roughly speaking, the divergence of a vector field is a scalar field that tells us, at each point, the extent to which the field diverges from that point. The curl of a vector field is a vector field that gives us, at each point, an indication of how the field swirls in the vicinity of that point. However, to describe divergence and curl in such a brief manner is not only useless but a bit dangerous, since (if taken literally) both of these preceding sentences are not only vague but technically incorrect. As we shall see, it is possible for a field to have a positive divergence without appearing to "diverge" at all, and it is possible for a field to have a nontrivial curl and yet have flow lines that do not bend at all.

In this section we consider only the divergence. We begin by presenting a heuristic discussion which will serve to motivate the formal definition.

As usual, the vector field will be denoted by

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$$

Let us, for the moment, interpret $\mathbf{F}(x,y,z)$ as the velocity of a fluid particle located at (x,y,z), as in Example 3.8 of the previous section; **F** is thus the velocity field of the fluid. Now consider a small planar patch of surface

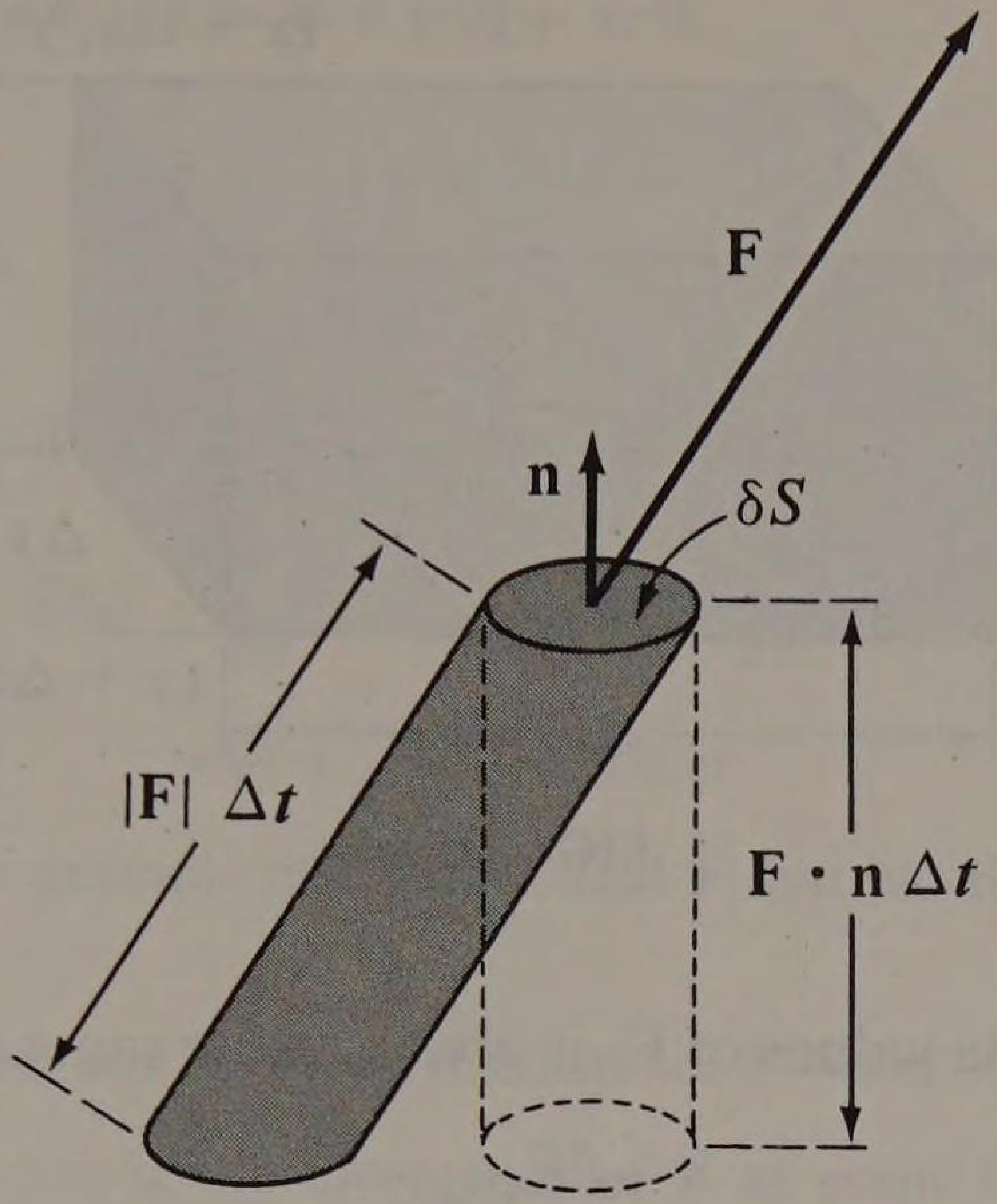

FIGURE 3.4

inside the fluid. Let δS be the area of the patch, and let **n** be a unit vector normal to the patch. We would like to find an expression for the amount of the fluid that flows through this area per unit time.

As shown in Fig. 3.4, if **F** is the velocity of the fluid at some point on the patch, then the body of fluid that will pass through the patch in the time Δt is, approximately, the fluid in the tube with base δS and central axis $\mathbf{F}\,\Delta t$. The approximation becomes exact as δS and Δt become small. If we assume the density of the fluid is unity (i.e., the fluid is incompressible), then the amount of fluid in this tube is given by its volume. Its base area is δS, and its height is $\mathbf{F}\,\Delta t\cdot\mathbf{n}$. Multiplying these and dividing by Δt, we see that *the amount of fluid crossing the area* δS (in the direction **n**) *per unit time is approximately* $\mathbf{F}\cdot\mathbf{n}\,\delta S$. This is called the *flux of the vector field* **F** *through the area* δS.

To define the divergence of the field **F**, we consider a small rectangular parallelepiped having corners at (x,y,z), $(x+\Delta x, y, z)$, $(x, y+\Delta y, x)$, $(x, y, z+\Delta z)$, etc. (Fig. 3.5). We shall compute the total flux of the field **F** through the six sides of this box in the outwards direction (i.e., on each side we choose **n** to be the outward normal). We then divide this flux by the volume of the box and take the limit as the dimensions of the box go to zero. *This limit is called the divergence of* **F** *at the point* (x,y,z).

The computation of this limit proceeds as follows. On face number I in Fig. 3.5 the outward normal is $-\mathbf{i}$. Thus, according to the above analysis, the flux out of this face is approximately $-F_1(x,y,z)\,\Delta y\,\Delta z$. The flux out of face number II, whose outward normal is **i**, is $F_1(x+\Delta x, y, z)\,\Delta y\,\Delta z$. The total flux out of faces I and II is thus

$$[F_1(x+\Delta x, y, z) - F_1(x, y, z)]\,\Delta y\,\Delta z$$

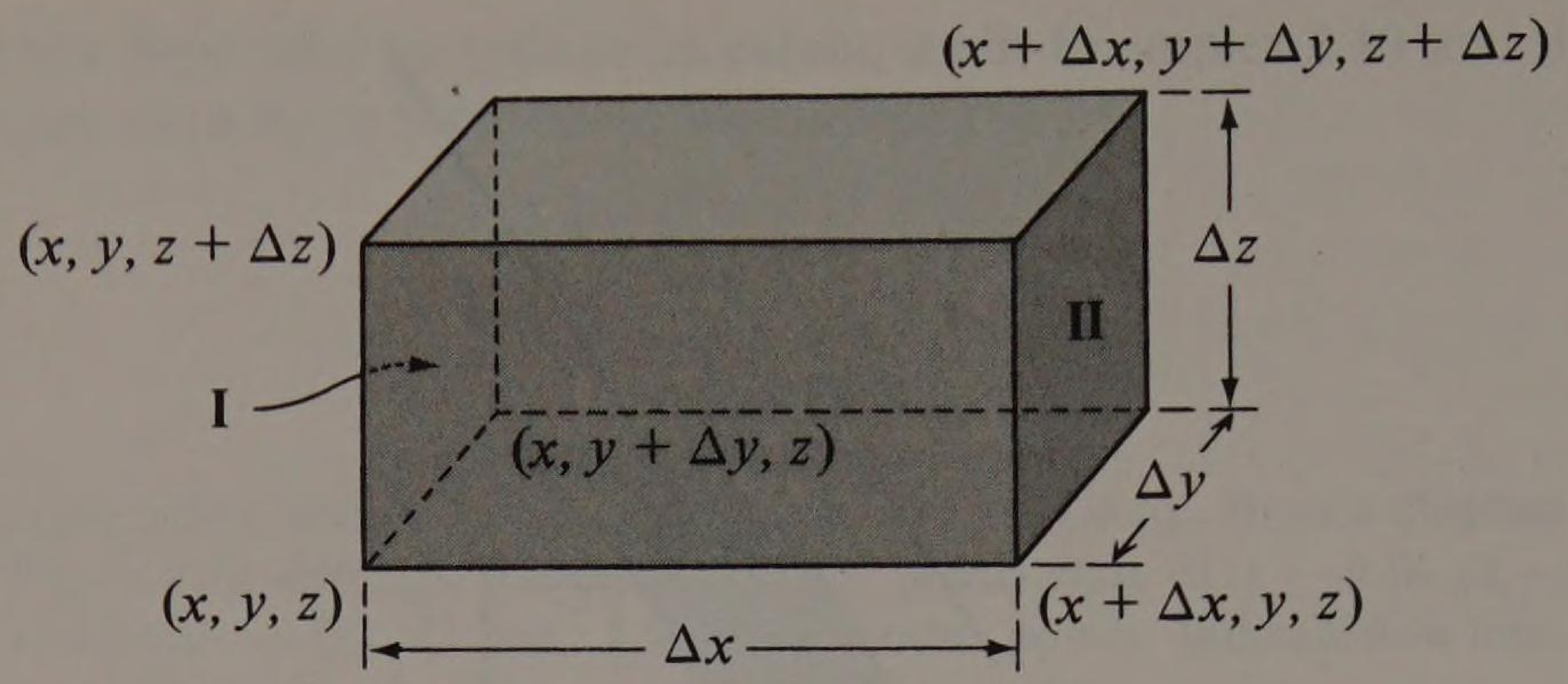

FIGURE 3.5

The difference in these values of F_1 is given, to the same order of accuracy, by

$$\frac{\partial F_1}{\partial x}\Delta x$$

Thus the contribution to the net outward flux from faces I and II is

$$\frac{\partial F_1}{\partial x}\Delta x\,\Delta y\,\Delta z$$

Similarly, the two faces in the y direction contribute

$$\frac{\partial F_2}{\partial y}\Delta y\,\Delta x\,\Delta z$$

and, adding the contribution of the two remaining faces we see that the net outward flux is approximately

$$\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)\Delta x\,\Delta y\,\Delta z$$

After we divide by the volume $\Delta x\,\Delta y\,\Delta z$, our approximations become accurate as we take the limit and we are led to the following statement, which we take as our formal *definition* of divergence:

The divergence of a vector field

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \tag{3.8}$$

is a scalar field, denoted div **F**, *defined by*

$$\operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z} \tag{3.9}$$

It is easy to compute the divergence of a vector field, as we demonstrate with examples. Keep in mind that div **F** *is defined* by Eq. (3.9), and that our heuristic discussion gives us the interpretation of div **F** as net outflux per unit volume.

Example 3.13 Find div **F**, if $\mathbf{F} = x\mathbf{i} + y^2z\mathbf{j} + xz^3\mathbf{k}$.

Solution

$$\begin{aligned}\text{div } \mathbf{F} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y^2z) + \frac{\partial}{\partial z}(xz^3)\\ &= 1 + 2yz + 3xz^2\end{aligned}$$

Example 3.14 Find div **F**, if $\mathbf{F} = xe^y\mathbf{i} + e^{xy}\mathbf{j} + \sin yz\,\mathbf{k}$.

Solution

$$\begin{aligned}\text{div } \mathbf{F} &= \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial y}(e^{xy}) + \frac{\partial}{\partial z}(\sin yz)\\ &= e^y + xe^{xy} + y\cos yz\end{aligned}$$

Example 3.15 Give an example of a vector field **F** that has divergence equal to 3 at every point in space.

Solution Many solutions can be given, for instance $\mathbf{F} = 3x\mathbf{i}$ or $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Example 3.16 In Fig. 3.6, is the divergence of **F** at point P positive or negative? Assume no variation of **F** in the z direction and that F_3 is identically zero.

Solution We see from the diagram that F_1 is approximately constant, so $\partial F_1/\partial x = 0$. Below P, F_2 is negative, and above P, F_2 is positive, so $\partial F_2/\partial y$ is positive. Since $F_3 = 0$, we have $\partial F_3/\partial z = 0$. It follows that div **F** is positive at point P.

Heuristically, we can see that the flux through the x faces of a parallelepiped at P will cancel, while there is definitely flux *out* of both y faces. Since there is no flux in the z direction, the divergence is positive.

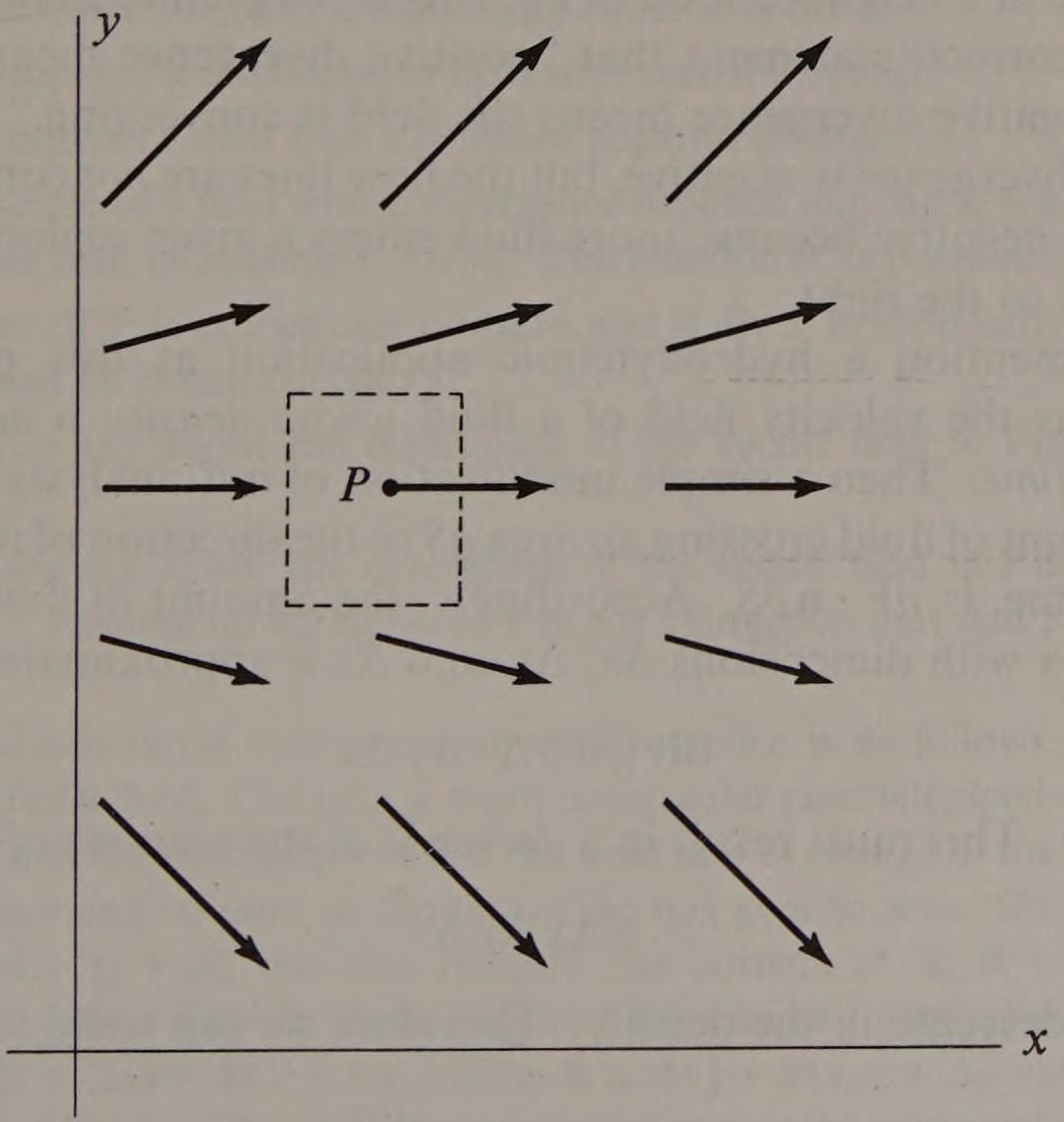

FIGURE 3.6

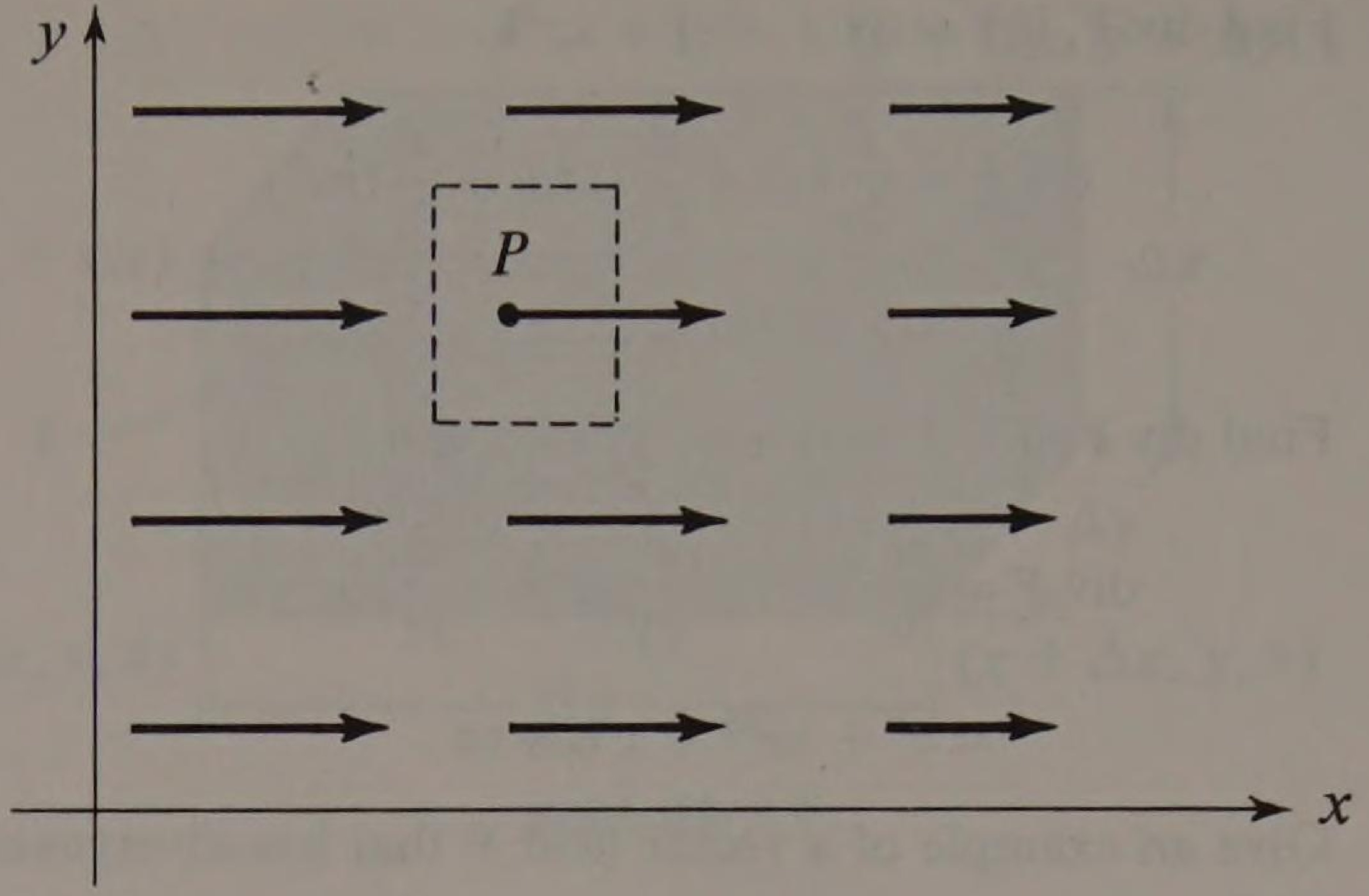

FIGURE 3.7

Example 3.17 In Fig. 3.7, is the divergence of **F** at point P positive or negative? Assume no variation of **F** in the z direction and that F_3 is identically zero.

Solution We see from the diagram that F_1 is decreasing with increasing x, hence $\partial F_1/\partial x$ is negative. F_2 and F_3 are zero at every point. It follows that the divergence of **F** is negative at every point.

Again heuristically, there is no flux in the y or z direction, and the flux in the x direction *decreases* as we move to the right. So the net flux through the sides of a box at P is inward, and the divergence must be negative.

In Fig. 3.6, where the divergence is positive, the lines of flux do, in a sense, diverge in a neighborhood of P. This is the picture that motivates the common (incorrect) statement that "positive divergence means the field is diverging, negative divergence means the field is converging." Note that in Fig. 3.7, the divergence is negative, but the flow lines are *not* converging. The divergence is negative because more fluid enters a given region from the left than leaves it to the right.

Let us mention a hydrodynamic application at this point. Again, interpret **F** as the velocity field of a fluid *whose density* ρ *may depend on position and time.* Then a simple modification of our analysis of flux shows that the amount of fluid crossing an area δS in the direction of its unit normal **n**, per unit time, is $\rho\mathbf{F}\cdot\mathbf{n}\,\delta S$. Accordingly, the amount of fluid flowing out of a small box with dimensions Δx, Δy, and Δz is approximately

$$\operatorname{div}(\rho\mathbf{F})\,\Delta x\,\Delta y\,\Delta z$$

per unit time. This must result in a *decrease* of the amount of fluid,

$$\rho\,\Delta x\,\Delta y\,\Delta z$$

and hence a decrease in the density. Therefore we can write

$$\operatorname{div}(\rho\mathbf{F}) = -\frac{\partial\rho}{\partial t}$$

This is called the *equation of continuity* in fluid mechanics; it expresses the law of conservation of mass.

The heuristic reasoning employed in this section is, of course, subject to criticism, as are most arguments involving "infinitesimals." Its rigorous justification rests on a result known, appropriately enough, as the *divergence theorem*, and we will study it in the next chapter. For the present we are satisfied with having a formal, precise definition of div **F** in Eq. (3.9), and an intuitive picture of what it represents.

Exercises

1. Find div **F**, given that $\mathbf{F} = e^{xy}\mathbf{i} + \sin xy\,\mathbf{j} + \cos^2 zx\,\mathbf{k}$.

2. Find div **F**, given that $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

3. Find div **F**, given that $\mathbf{F} = \mathbf{grad}\ \phi$ where $\phi = 3x^2y^3z$.

4. Find the divergence of the field

$$\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

Is the divergence of this field defined at every point in space?

5. Show in detail that $\operatorname{div}(\phi\mathbf{F}) = \phi \operatorname{div}\mathbf{F} + \mathbf{F}\cdot\mathbf{grad}\ \phi$.

6. Construct an example of a scalar field ϕ and a vector field **F**, neither of which is constant, for which $\operatorname{div}(\phi\mathbf{F})$ is identically equal to $\phi \operatorname{div}\mathbf{F}$.

7. Give an example of a nonconstant field with zero divergence.

8. Give an example of a field with a constant negative divergence.

9. Give an example of a field whose divergence depends only on x, is always positive, and increases with increasing x. (*Hint*: The function e^x is positive for every x.)

10. True or false: If **F** is everywhere nonzero and if div **F** is identically zero, the flow lines of **F** must be closed curves.

11. What can you say about the divergence of the vector field in Fig. 3.8 at points P, Q, and R?

12. What can you say about the divergence of the vector field in Fig. 3.9 at points P, Q, and R? Assume no variation of **F** in the z direction and that F_3 is identically zero.

13. Another hydrodynamic interpretation of divergence is as follows: Let **F** be the velocity field of a fluid. Consider a small rectangular parallelepiped of fluid located at (x, y, z). Then the divergence of **F** is the time rate of change of the volume of this body fluid, per unit volume, as the size of the box goes to zero. Show this. [*Hint*: With $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the box initially has corners at $\mathbf{R}$, $\mathbf{R} + \Delta x\mathbf{i}$, $\mathbf{R} + \Delta y\mathbf{j}$, $\mathbf{R} + \Delta z\mathbf{k}$, etc. After time Δt these corners have moved to the new positions $\mathbf{R} + \mathbf{F}(x, y, z)\,\Delta t$, $\mathbf{R} + \Delta x\mathbf{i} + \mathbf{F}(x + \Delta x, y, z)\,\Delta t$, $\mathbf{R} + \Delta y\mathbf{j} + \mathbf{F}(x, y + \Delta y, z)\,\Delta t$, $\mathbf{R} + \Delta z\mathbf{k} + \mathbf{F}(x, y, z + \Delta z)\,\Delta t$, etc. Calculate the new volume using the triple scalar product, and compute the limit described above.]

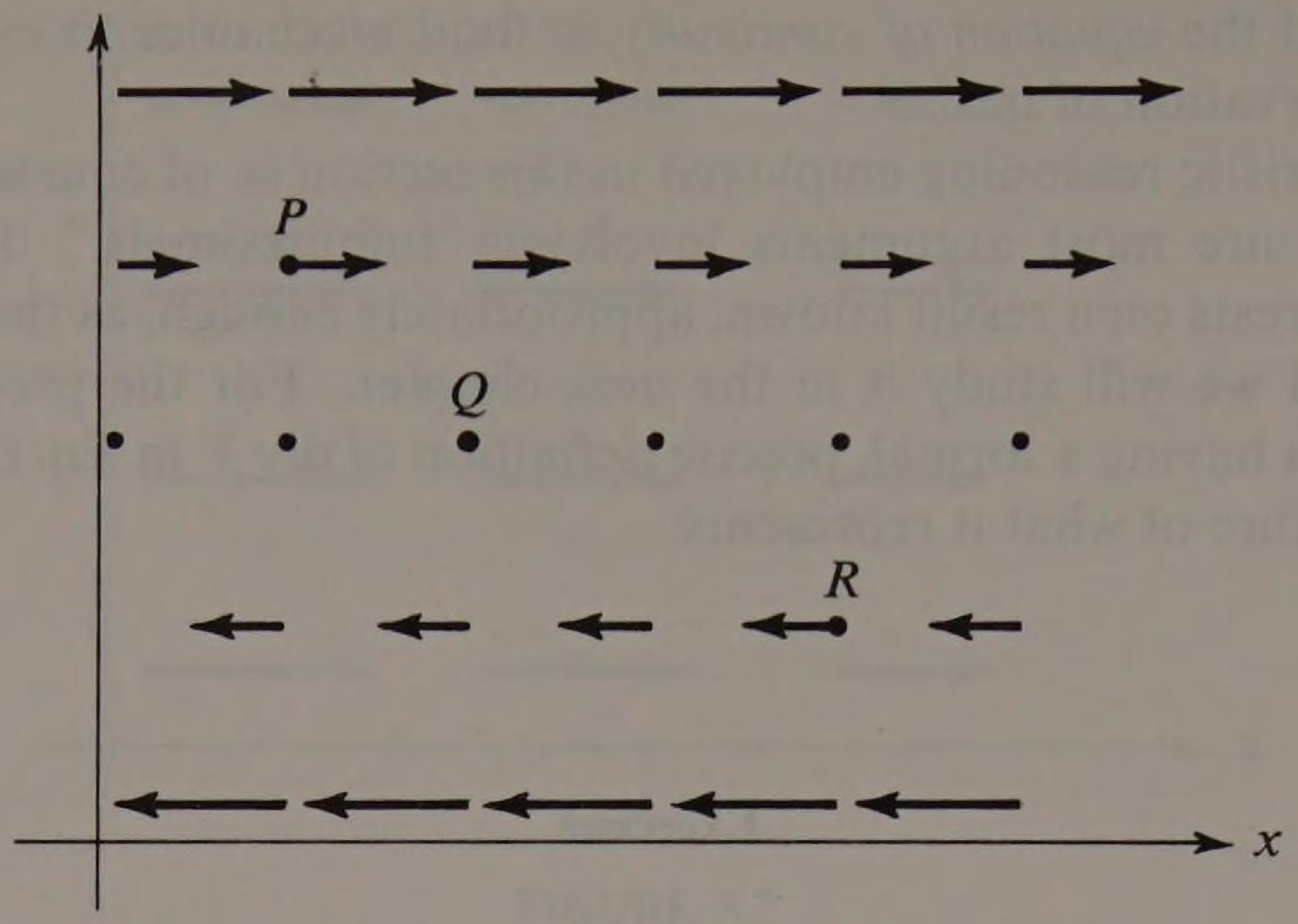

FIGURE 3.8

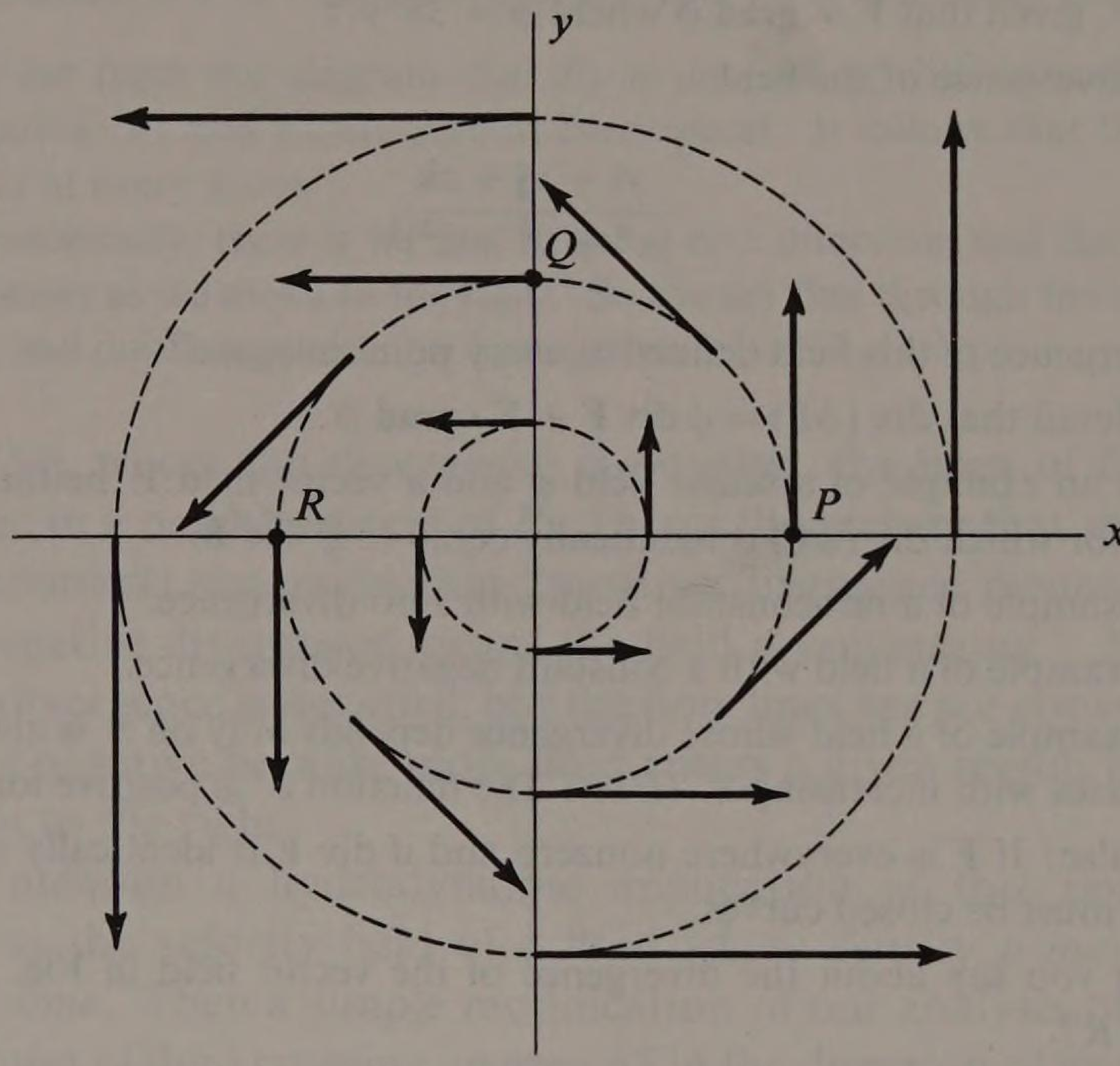

FIGURE 3.9

3.4 CURL

As in the previous section, we shall preface our formal definition of the curl of a vector field with some heuristic considerations. Once again, it is convenient to imagine that **F** represents the velocity field of an incompressible liquid. Now let us imagine we have a little paddle wheel, like that shown in Fig. 3.10, that is free to rotate about its axis AA'.

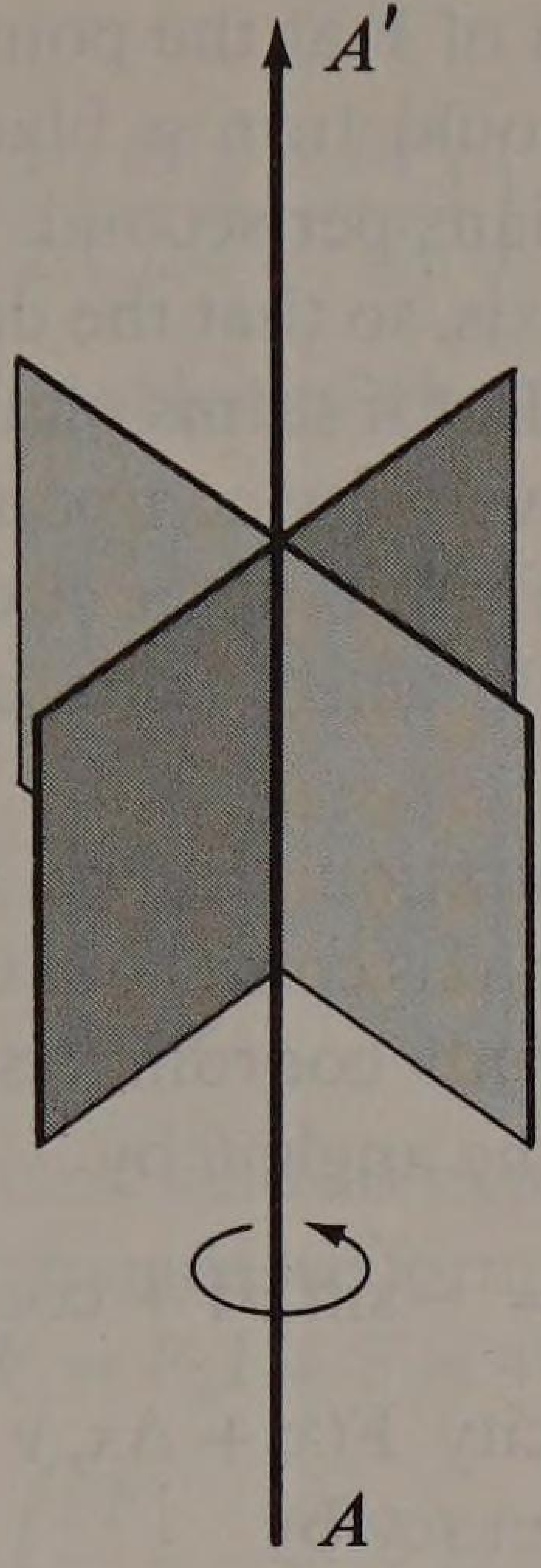

FIGURE 3.10

Imagine that we immerse this paddle wheel in the liquid. Because of the flow of the liquid, it will tend to rotate with some angular velocity. This angular velocity will vary, depending on where we locate the paddle wheel and on the positioning of its axis. For definiteness we shall try to compute the angular velocity with the paddle wheel lined up along the z axis.

The mechanism which rotates the wheel is provided by the tendency of the fluid to *swirl* around the z axis; this motion is due to the counterclockwise components of the velocity near the axis. If we impose a polar coordinate system centered around the paddle wheel axis, as in Fig. 3.11, then the

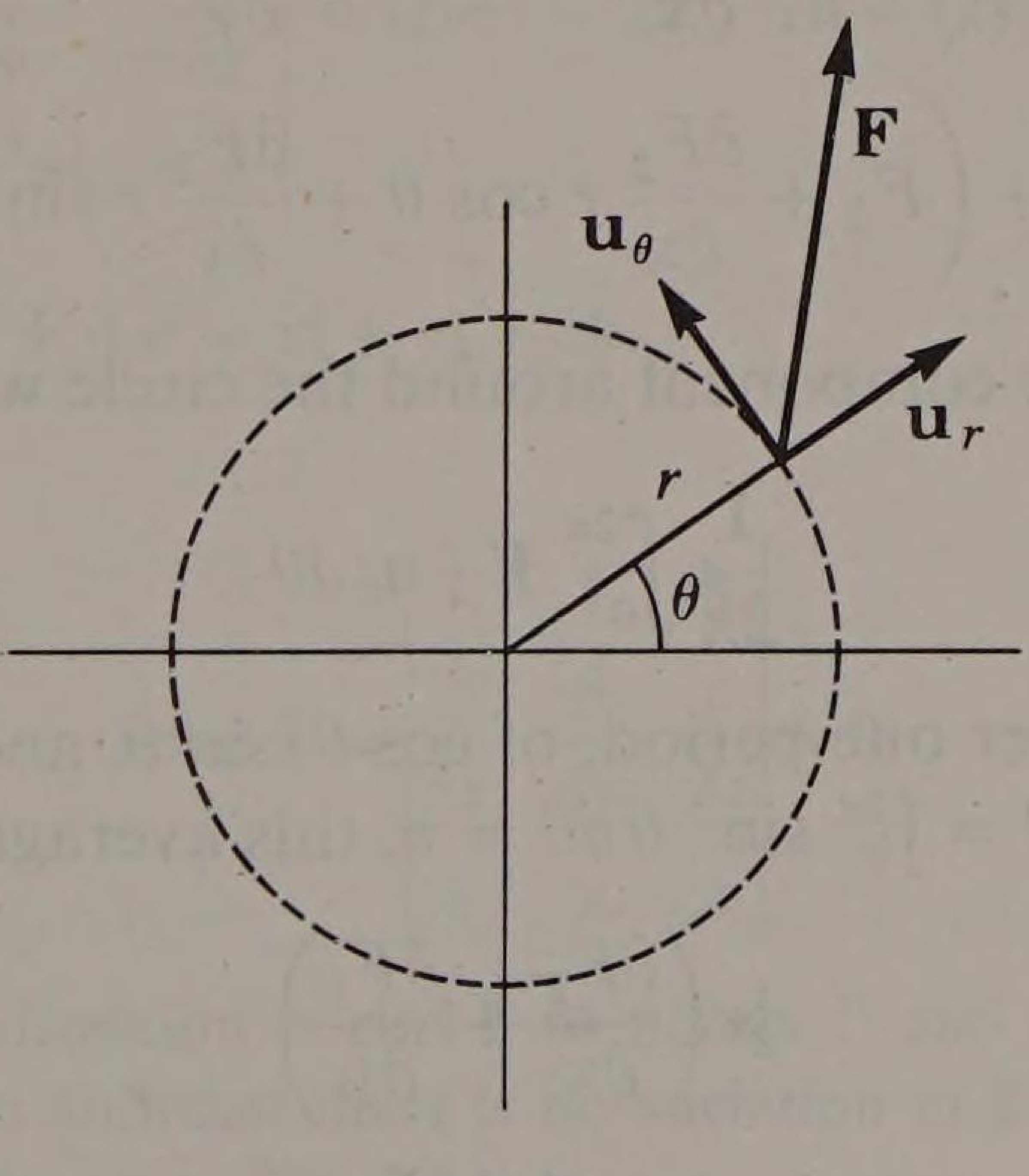

FIGURE 3.11

counter clockwise component of **F** at the point (r,θ) is given by $\mathbf{F}\cdot\mathbf{u}_\theta$. This component of the velocity would turn a blade of the paddle wheel at an *angular* rate of $\mathbf{F}\cdot\mathbf{u}_\theta/r$, in radians per second. Of course, this rate will differ from point to point near the axis, so that the different blades of the wheel are "pushed" at different speeds. But it seems plausible to expect that if we took the *average* counterclockwise velocity component over a small circle around the axis, and then divide by the radius of the circle, the quotient would give the angular velocity of the paddle wheel (whose blades we regard as rigidly fixed to each other).

Let us perform this computation. In Fig. 3.11 (x,y,z) are the coordinates of the center of the circle; the z axis comes out of the page towards the reader. At the point on the circle with coordinates $(x+\Delta x, y+\Delta y, z)$, the unit vector $\mathbf{u}_\theta$ is given in terms of the angle θ by

$$\mathbf{u}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

The components of the velocity $\mathbf{F}(x+\Delta x, y+\Delta y, z)$ at this point can be expressed to first order of accuracy by

$$F_1(x+\Delta x, y+\Delta y, z) = F_1(x,y,z) + \frac{\partial F_1}{\partial x}\Delta x + \frac{\partial F_1}{\partial y}\Delta y$$

$$F_2(x+\Delta x, y+\Delta y, z) = F_2(x,y,z) + \frac{\partial F_2}{\partial x}\Delta x + \frac{\partial F_2}{\partial y}\Delta y$$

F_3 does not concern us, since we are only interested in the counterclockwise component $\mathbf{F}\cdot\mathbf{u}_\theta$. Expressing Δx and Δy in terms of r and θ,

$$\Delta x = r\cos\theta$$
$$\Delta y = r\sin\theta$$

we have, for the counterclockwise component of velocity at (r,θ) on the circle,

$$\mathbf{F}\cdot\mathbf{u}_\theta = -\left(F_1 + \frac{\partial F_1}{\partial x}r\cos\theta + \frac{\partial F_1}{\partial y}r\sin\theta\right)\sin\theta$$
$$+\left(F_2 + \frac{\partial F_2}{\partial x}r\cos\theta + \frac{\partial F_2}{\partial y}r\sin\theta\right)\cos\theta$$

The average clockwise component around the circle will be

$$\frac{1}{2\pi}\int_0^{2\pi}\mathbf{F}\cdot\mathbf{u}_\theta\,d\theta$$

Since the integrals, over one period, of $\cos\theta$, $\sin\theta$, and $\sin\theta\cos\theta$ are zero, and since $\int_0^{2\pi}\cos^2\theta\,d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta = \pi$, this average is seen to be

$$\tfrac{1}{2}r\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

and dividing by r, we conclude that *the angular velocity of the fluid about the* *z* axis is

$$\frac{1}{2}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

The computation of the angular velocity about the x axis yields

$$\frac{1}{2}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)$$

and, for the y axis,

$$\frac{1}{2}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)$$

We want the curl of a vector field to express its tendency to swirl; so, dropping the factor $\frac{1}{2}$ for convenience we formulate the following definition: *The curl of a vector field* $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ *is the vector field*

$$\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathbf{k} \tag{3.10}$$

Rather than memorize (3.10), the student is advised to write the curl in the form of a symbolic determinant:

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{3.11}$$

Example 3.18 Find **curl F**, if $\mathbf{F} = xyz\mathbf{i} + x^2y^2z^2\mathbf{j} + y^2z^3\mathbf{k}$.

Solution

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & x^2y^2z^2 & y^2z^3 \end{vmatrix} = (2yz^3 - 2x^2y^2z)\mathbf{i} + (xy)\mathbf{j} + (2xy^2z^2 - xz)\mathbf{k}$$

Example 3.19 Find **curl F** if $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Solution

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$$

Example 3.20 In what direction is **curl F** at points P and Q in Fig. 3.12? Assume that F_3 is identically zero and that there is no variation in **F** in the z direction. (This field is the same as that shown in Fig. 3.9.)

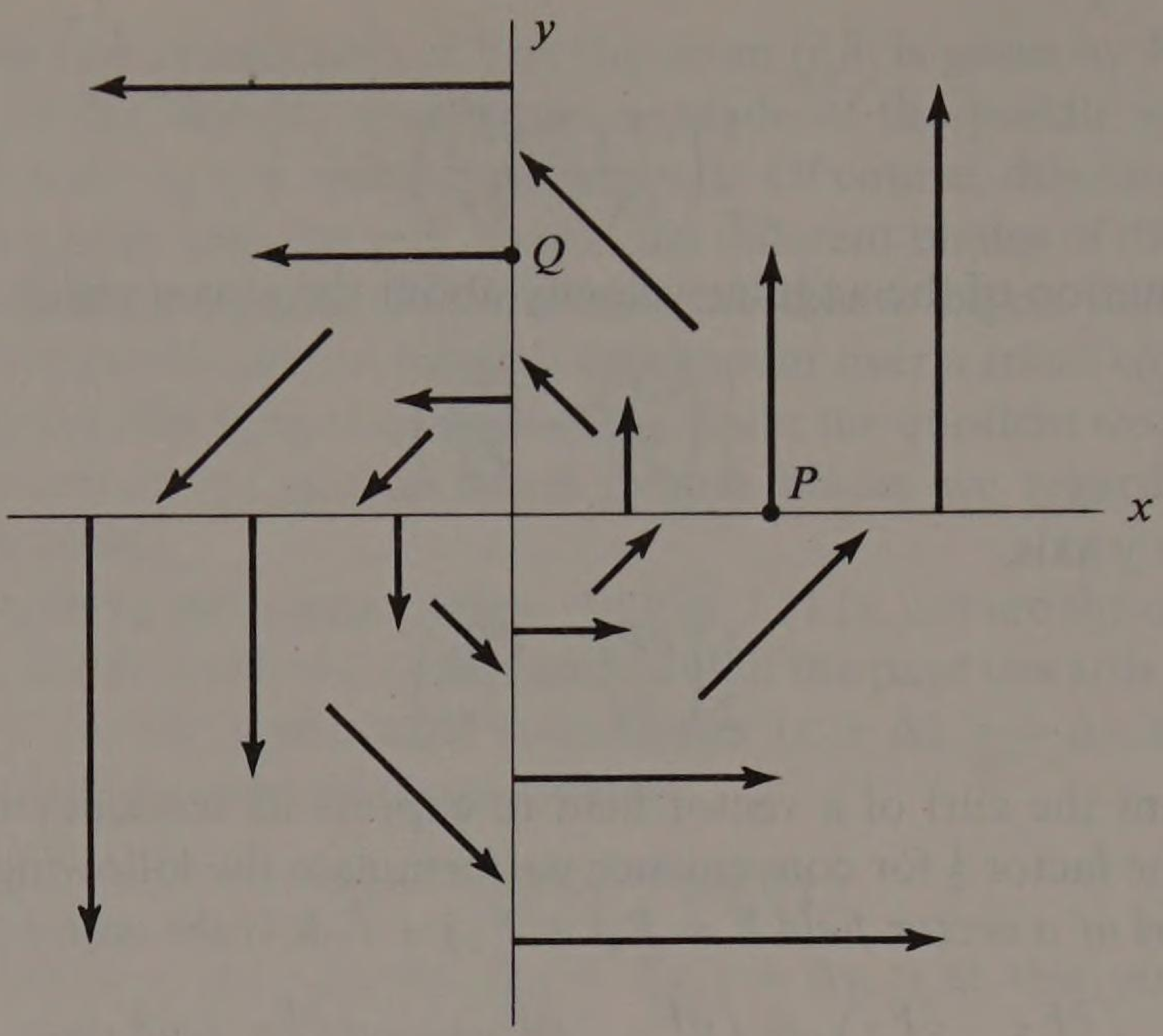

FIGURE 3.12

Solution It should be clear from our discussion that **curl F** points in the $+z$ direction. Using the formal definition, observe that at the point P, F_2 is increasing in the x direction, so $\partial F_2/\partial x$ is positive. Although F_1 is zero at P, it is positive below P and negative above, so F_1 is decreasing as we move through P in the y direction, that is, $\partial F_1/\partial y$ is negative. Since we assume F_3 identically zero, the derivatives $\partial F_3/\partial y$ and $\partial F_3/\partial x$ are also zero, and since we assume no variation in the z direction, $\partial F_2/\partial z$ and $\partial F_1/\partial z$ are zero. It follows that the only term in (3.10) that does not vanish is the last term, and that the last term is positive.

At point Q, F_2 is zero, but it is negative to the left of Q and positive to the right, hence $\partial F_2/\partial x$ is positive. F_1 is negative at Q and is becoming even more negative with increasing y, and so $\partial F_1/\partial y$ is negative. The term $(\partial F_2/\partial x - \partial F_1/\partial y)$ is therefore positive. The other derivatives in (3.10) equal zero. It follows that **curl F** at point Q is also perpendicular to the xy plane, directed toward the reader. (In fact, a little reflection will convince the reader that **curl F** at point P is equal to **curl F** at point Q.)

Example 3.21 In what direction is **curl F**, if **F** is as shown in Fig. 3.13?

Solution Since **F** is directed parallel to the y axis and appears to have magnitude proportional to x, we can guess $\mathbf{F} = Cx\mathbf{j}$, where C is a negative constant. Hence

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & Cx & 0 \end{vmatrix} = C\mathbf{k}$$

Since C is negative, the curl is directed into the page (negative z direction).

In Fig. 3.13, the paddle wheel would tend to rotate most rapidly with its axis perpendicular to the page; it will rotate because the velocity of the fluid

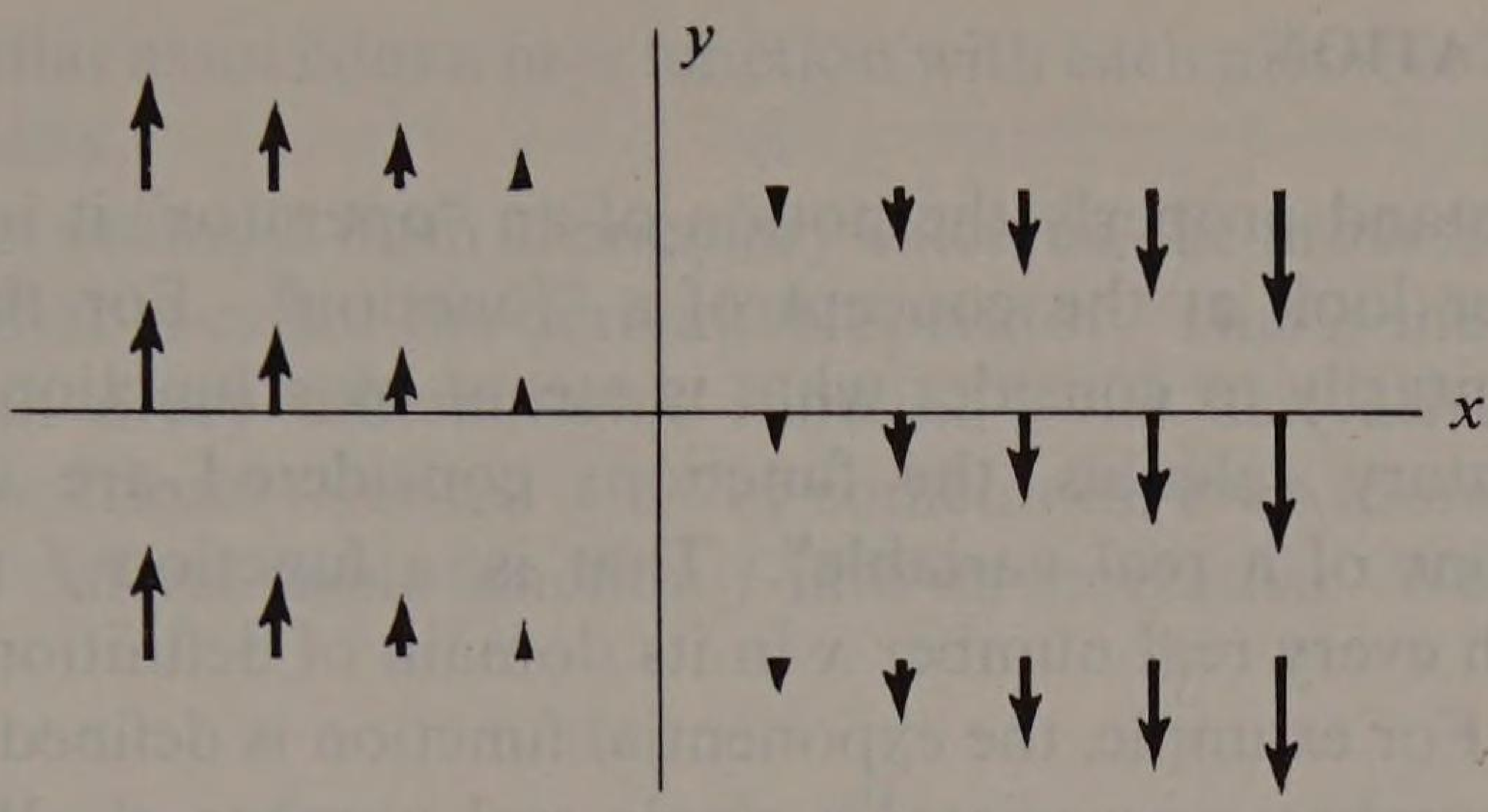

FIGURE 3.13

is greater on one side than on the other. The direction of the curl is into the page, because the paddle wheel will tend to rotate clockwise. This example shows that it is possible for a vector field to have nonzero curl even when the flow lines are straight lines; hence, to describe **curl F** as "a measure of the rate of swirling of **F**" is not completely accurate.

Example 3.22 Let us imagine that **F** represents the velocity field of a fluid of constant mass density rotating with uniform angular velocity ω about the z axis. Find **curl F**. (Assume the angular velocity vector $\boldsymbol{\omega}$ to point in the positive z direction.)

Solution Since $\boldsymbol{\omega} = \omega\mathbf{k}$, we have (Eq. 1.21) $\mathbf{F} = \omega\mathbf{k} \times \mathbf{R}$, where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Hence $\mathbf{F} = -\omega y\mathbf{i} + \omega x\mathbf{j}$. Using (3.11) we find that $\textbf{curl F} = 2\omega\mathbf{k}$. As we expected, the curl of **F** is just twice the angular velocity vector; in this situation, it is the same at every point in space.

The reader should convince himself/herself that the field in Example 3.22 is portrayed in Fig. 3.12.

Exercises

In Exercises 1 through 3, find **curl F**:

✓**1.** $\mathbf{F} = xy^2\mathbf{i} + xy\mathbf{j} + xy\mathbf{k}$

✓**2.** $\mathbf{F} = e^{xy}\mathbf{i} + \sin xy\,\mathbf{j} + \cos yz^2\,\mathbf{k}$

3. $\mathbf{F} = z^2x\mathbf{i} + y^2z\mathbf{j} - z^2y\mathbf{k}$

4. Given the vector field $\mathbf{F} = (x + xz^2)\mathbf{i} + xy\mathbf{j} + yz\mathbf{k}$,
  (a) evaluate div **F**,
  (b) evaluate **curl F**.

5. Draw a rough picture of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and, thinking of the paddle wheel interpretation of **curl F**, explain why **curl F** is identically zero in this case.

✓**6.** Give an example of a vector field with curl identically equal to $2\mathbf{i}$.

✓**7.** The flow lines of a velocity field **F** are straight lines. Does this imply that **curl F** = **0**?

✓**8.** Is it possible to tell anything about **curl F**, given only a description of the flow lines of **F**?

3.5 DEL NOTATION

To understand properly the notion of an "operator" it is necessary to take a broader look at the concept of a "function". For this reason we digress momentarily to consider what is meant by a function.

In elementary calculus, the functions considered are usually "real-valued functions of a real variable". That is, a function f is a rule that associates with every real number x in its domain of definition a single real number $f(x)$. For example, the exponential function is defined for all x, and to each real number x associates a single real number e^x. We define this function by writing $f(x) = e^x$. Other functions are defined by writing, say, $f(x) = x^2$ or $f(x) = \sin x$. Most mathematicians nowadays distinguish rather carefully between the symbol f and the notation $f(x)$. The former denotes the function and the latter denotes the value of the function (which is a number and not a function). Thus if $f(x) = x^2$, the function f is the *rule* "square the given number", but $f(3)$ is the *number* 9.

In more advanced courses we meet functions of two or three variables. In this book "scalar fields" are simply real-valued functions of three real variables. Thus, the function f defined by $f(x,y,z) = x^2y^2z^2$ says "multiply together the squares of the given numbers". When used alone, in this context, the letter f denotes this *rule*, but if we write, say, $f(2,1,3)$, we mean the *value* of the function at the point $(2,1,3)$, which in this case is the number 36.

Most of the functions we have been considering in this chapter are described by expressions involving x, y, and z. In studying vector analysis it is useful to "visualize" such functions in geometrical or physical terms. Thus, the engineering student may think of "an arbitrary function f" as meaning "an arbitrary electric potential", or "an arbitrary temperature distribution", and the mathematics major may think of this as meaning "a rule whereby we tag each point in space with a number". The student who thinks of a function as a jumble of x's, y's, and z's doesn't have much fun, and misses much of the point.

The *vector* fields we discuss are *vector-valued* functions of three real variables. But the idea is still much the same. In this context, a function $\mathbf{F}$ is a *rule* that associates with each point (x,y,z) a single vector $\mathbf{F}(x,y,z)$.

But now we come to what is a big hurdle for some students: passing to the *general* notion of a function. Much of the mystery of modern mathematics vanishes when we realize that a mathematician uses the word "function" in a much more general way, to denote any rule that associates an object with each one of a class of objects. Thus we have not only functions that associate numbers with numbers (the functions of elementary calculus), numbers with points in space (scalar fields), and vectors with points in space (vector fields), but also those that associate *functions* with *functions.*

Partly for reasons of convenience, but mainly (we suspect) because so many people have old-fashioned ideas of what the word "function" means, the latter types of functions are usually called "operators". An operator is simply a rule that associates a new function with each member of a particular class of functions.

To take an example from elementary calculus, the process of differentiation defines what is called the derivative operator. This is the operator that associates with every differentiable function f its derivative df/dx. This operator is sometimes denoted d/dx or sometimes, even more simply, D. It converts each differentiable function f into its derivative. We may formally write this

$$D(f) = \frac{df}{dx}$$

Some textbooks say that D is simply an abbreviation for d/dx, and that the symbol d/dx means nothing by itself, having meaning only when it is applied to some function f. Then we may write df/dx, which of course we all understand. Other differential operators, such as $L = (d^2/dx^2) + 2(d/dx) + 4$, are similarly interpreted as symbols that are meaningless unless followed by a function. In this case we have

$$L(f) = \frac{d^2f}{dx^2} + 2\frac{df}{dx} + 4f$$

This is to miss the whole point of the operator concept, however. It would be much better to visualize this operator as a sort of meatgrinder, into which we drop the function f, turn the handle, and out drops the function $(d^2f/dx^2) + 2(df/dx) + 4f$. There is really no insurmountable difficulty in understanding that an operator T is a *rule* that associates with a function f some other function (or possibly even the same function) $T(f)$. It is misleading to say that the symbol d/dx means nothing by itself. It means a great deal: it represents the rule whereby we associate with a differentiable function its derivative. There is no point in recounting here the basic definition of "derivative" or the innumerable techniques involved in actually computing a derivative. The point is that a differentiable function has a derivative and the derivative operator pairs the derivative with the function. (The excellent concept of "pairing" is used in many modern books in discussing the function concept. The derivative of a function is just another function, and the derivative operator is the mathematical twine that binds the two together.)

Another example of an operator is the *gradient*. We recall that the gradient of a scalar field f is a vector field **grad** f. The gradient operator may be written, symbolically,

$$\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Divergence is also an operator. It is an operator that converts a vector field into a scalar field. Similarly, *curl* is an operator, but it is an operator that changes a vector field into another vector field.

The three operators that concern us most are gradient, divergence, and curl. Although they may be written **grad**, div, and **curl**, there is a suggestive and convenient symbolic way of writing them that is commonly used. For this purpose, we introduce the symbol ∇, called "del" (sometimes "nabla"), which is an abbreviation for $\mathbf{i}(\partial/\partial x) + \mathbf{j}(\partial/\partial y) + \mathbf{k}(\partial/\partial z)$. In terms of this symbol, we can write **grad** f as ∇f. Working with ∇ purely formally, pretending for the moment it is a vector, we see that if we form the scalar product of ∇ with a vector field $\mathbf{F}$, we obtain

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\mathbf{i}F_1 + \mathbf{j}F_2 + \mathbf{k}F_3) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\end{aligned}$$

which is the divergence of $\mathbf{F}$. Similarly, if we imagine ∇ to be a vector and form the vector cross product of ∇ with $\mathbf{F}$, we obtain the curl of $\mathbf{F}$:

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (\mathbf{i}F_1 + \mathbf{j}F_2 + \mathbf{k}F_3) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \mathbf{curl}\ \mathbf{F}\end{aligned}$$

To recapitulate, ∇ is an abbreviation,

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{3.12}$$

The symbols ∇f, $\nabla \cdot \mathbf{F}$, and $\nabla \times \mathbf{F}$ are defined by

$$\nabla f = \mathbf{grad}\ f \tag{3.13}$$

$$\nabla \cdot \mathbf{F} = \operatorname{div} \mathbf{F} \tag{3.14}$$

$$\nabla \times \mathbf{F} = \mathbf{curl}\ \mathbf{F} \tag{3.15}$$

After (3.12) is memorized, formulas (3.13), (3.14), and (3.15) provide very convenient ways of remembering the expressions for gradient, divergence, and curl. We just operate with ∇ as though it were a vector. Henceforth we will use these abbreviations frequently.

Exercises

1. If $f(x,y,z) = x^2y + z$, what is $f(2,3,4)$?
2. If $f(x,y,z) = x^2y + z$, what is the value of ∇f at $(2,3,4)$?

3. If $g(t) = t^3$ and $f(x,y,z) = x^2 + y^2z$, what is $g[f(1,1,3)]$?

4. Given $\mathbf{F}(x,y,z) = x^2y\mathbf{i} + z\mathbf{j} - (x + y - z)\mathbf{k}$, find
   (a) $\nabla \cdot \mathbf{F}$
   (b) $\nabla \times \mathbf{F}$
   (c) $\nabla(\nabla \cdot \mathbf{F})$

5. If $\mathbf{F}$ is a vector field, is $\nabla \cdot (\nabla \times \mathbf{F})$ a scalar field or a vector field?

6. If $\mathbf{F}$ is a vector field, is $\nabla \times (\nabla \times \mathbf{F})$ a scalar field or a vector field?

7. Find $\nabla \cdot \mathbf{R}$ and $\nabla \times \mathbf{R}$ where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

8. If $f(x,y,z) = xyz + e^{xz}$, find $\nabla \cdot (\nabla f)$.

9. (a) Compute $\nabla \times (\nabla f)$ for the scalar field f defined in Exercise 8.
   (b) Now do the same thing for another scalar field f (use any of the scalar fields defined in preceding problems, or make one up yourself).
   (c) What can you conjecture from this?

10. (a) Compute $\nabla \cdot (\nabla \times \mathbf{F})$ for the vector field $\mathbf{F}$ defined in Exercise 4.
   (b) Do the same for a vector field $\mathbf{F}$ that you have made up yourself.
   (c) What can you conjecture from this?

3.6 THE LAPLACIAN

In electrostatics, the gradient of the electric potential is a scalar multiple of the electric field intensity, and the divergence of the electric field intensity is related to the charge density. For this and other reasons it is convenient to introduce a single operator that is the composite of the two operators **grad** and div. This operator is called the *laplacian.*

The laplacian of a scalar field f is defined to be div (**grad** f). Note that **grad** f is a vector field and the divergence of **grad** f is a scalar field; hence the laplacian of a scalar field f is a scalar field. In del notation this is $\nabla \cdot (\nabla f)$ and for simplicity is frequently written $\nabla^2 f$, or Δf.

We have

$$\text{laplacian}\,(f) = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f \tag{3.16}$$

since

$$\nabla \cdot (\nabla f) = \nabla \cdot \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The symbol ∇^2 or Δ may be considered to be simply an abbreviation for

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The equation

$$\Delta f = \nabla^2 f = 0 \tag{3.17}$$

is called *Laplace's equation.* Any function satisfying this equation in a given region is said to be *harmonic* in that region. For example, the electric potential of a static distribution of charges is harmonic in any region where the charge density is zero. A function describing the steady-state temperature distribution of a homogeneous material is harmonic in the interior of the region occupied by the material.

The laplacian operator is by far the most important differential operator in mathematical physics. In this section we discuss its intuitive meaning without giving proofs.

If f is a scalar, then $\nabla^2 f(x,y,z)$ denotes the value of $\nabla^2 f$ at the point (x,y,z). This is a number that tells us something about the behavior of the scalar field in the vicinity of (x,y,z). Roughly speaking, it provides a measure of the *difference between the average value of the field in the immediate neighborhood of the point and the precise value of the field at the point.*

(The word "average" here refers to an average over a region of space, not to a time average. Variations with time t do not enter into the calculation of $\nabla^2 f$.)

Thus, if $\nabla^2 f$ is positive at a point, and f denotes the temperature, this means that the temperature in the vicinity of the point is, on the average, greater than the temperature at the point itself. In particular, if the temperature takes its minimum value at a certain point in space, it is reasonable to expect that the value of $\nabla^2 f$ will *not* be negative at that point. In this respect, the laplacian can be viewed as a sort of three-dimensional generalization of the ordinary operator d^2/dx^2, which is used in elementary calculus to test extreme points to see if they represent maxima or minima.

If $\nabla^2 f$ is identically zero, the average value of f throughout any sphere (or any cube) will be exactly equal to the value of f at the center of the sphere (or cube). This is an important property of harmonic functions.

Suppose that (x,y,z) is a fixed point in space, and that $\bar{f}$ denotes the mean value of f throughout the interior of a sphere (or cube) with center at (x,y,z). If this sphere (or cube) is sufficiently small, we will have (approximately)

$$\bar{f} - f(x,y,z) = K\nabla^2 f(x,y,z) \tag{3.18}$$

where K is a positive constant depending only on the dimensions of the sphere or cube. For a sphere, $K = R^2/10$, where R is the radius of the sphere. For a cube, we have $K = a^2/24$, where a is the length of the side of the cube.

Relation (3.18) is exact only in one very special instance: when $\nabla^2 f$ is a constant, independent of x, y, and z. It is approximate otherwise, but the approximation is fairly good in some sense if the sphere or cube is sufficiently small. It is very helpful in giving an intuitive meaning to expressions containing the laplacian.

The formal differential operator

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

may also be applied to *vector* fields to obtain new vector fields, since if $\mathbf{F}$ is a *vector* field, $(\partial^2\mathbf{F}/\partial x^2) + (\partial^2\mathbf{F}/\partial y^2) + (\partial^2\mathbf{F}/\partial z^2)$ makes perfectly good sense. For example, if

$$\mathbf{F} = x^2y\mathbf{i} + y^2z^3\mathbf{j} + xyz^4\mathbf{k}$$

then we have

$$\frac{\partial^2\mathbf{F}}{\partial x^2} = 2y\mathbf{i}$$

$$\frac{\partial^2\mathbf{F}}{\partial y^2} = 2z^3\mathbf{j}$$

$$\frac{\partial^2\mathbf{F}}{\partial z^2} = 6y^2z\mathbf{j} + 12xyz^2\mathbf{k}$$

whence $\nabla^2\mathbf{F} = 2y\mathbf{i} + (2z^3 + 6y^2z)\mathbf{j} + 12xyz^2\mathbf{k}$. When used in this sense, to operate on vector fields to produce vector fields, we will call ∇^2 the *vector* laplacian operator.

OPTIONAL READING: DYADICS

Notice that when ∇^2 is used as an operator on vector fields, as in $\nabla^2\mathbf{F}$, its interpretation as div **grad** is rather strained. After all, there is no meaning to **grad F** in our scheme of things.

However, in some areas of physics and engineering it proves convenient to use such strange symbols. To see how this might come about, suppose we want to express the vector component of a vector $\mathbf{F}$ in the direction of a unit vector $\mathbf{n}$. The answer is given by $\mathbf{n}(\mathbf{n} \cdot \mathbf{F})$ (recall Sec. 1.9). This formula tempts one to define an operator, the *projection operator in the direction* $\mathbf{n}$, and to denote it as $\mathbf{n}\,\mathbf{n}$; then the projection of $\mathbf{F}$ in the direction $\mathbf{n}$ is

$$(\mathbf{n}\,\mathbf{n}) \cdot \mathbf{F} \equiv \mathbf{n}\,\mathbf{n} \cdot \mathbf{F}$$

Generalizing, given any two vectors $\mathbf{A}$ and $\mathbf{B}$, one formally defines the dyadic $\mathbf{AB}$ as an operator acting as follows: for any vector $\mathbf{F}$,

$$(\mathbf{A}\,\mathbf{B}) \cdot \mathbf{F} \equiv \mathbf{A}\,\mathbf{B} \cdot \mathbf{F}$$

and

$$\mathbf{F} \cdot (\mathbf{A}\,\mathbf{B}) \equiv \mathbf{F} \cdot \mathbf{A}\,\mathbf{B}$$

Thus the dyadic $\mathbf{ii}$ projects a vector onto the x-axis. As another example, observe that the dyadic $\mathbf{ii} + \mathbf{jj} + \mathbf{kk}$ is an identity operator because

$$(\mathbf{ii} + \mathbf{jj} + \mathbf{kk}) \cdot \mathbf{F} = \mathbf{F} = \mathbf{F} \cdot (\mathbf{ii} + \mathbf{jj} + \mathbf{kk})$$

[Recall Eq. (1.14)].

In this context, we can consider **grad F**, or $\nabla\mathbf{F}$, as a dyadic:

$$
\begin{aligned}
\nabla\mathbf{F} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \\
&= \frac{\partial F_1}{\partial x}\mathbf{ii} + \frac{\partial F_1}{\partial y}\mathbf{ji} + \frac{\partial F_1}{\partial z}\mathbf{ki} \\
&\quad + \frac{\partial F_2}{\partial x}\mathbf{ij} + \frac{\partial F_2}{\partial y}\mathbf{jj} + \frac{\partial F_2}{\partial z}\mathbf{kj} \\
&\quad + \frac{\partial F_3}{\partial x}\mathbf{ik} + \frac{\partial F_3}{\partial y}\mathbf{jk} + \frac{\partial F_3}{\partial z}\mathbf{kk}
\end{aligned}
$$

(Notice that $\mathbf{ij} \neq \mathbf{ji}$). Then the dyadic interpretation of div **grad F** becomes

$$
\begin{aligned}
\nabla\cdot(\nabla\mathbf{F}) &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial F_1}{\partial x}\mathbf{i}\,\mathbf{i} + \cdots + \frac{\partial F_3}{\partial z}\mathbf{k}\,\mathbf{k}\right) \\
&= \frac{\partial^2 F_1}{\partial x^2}\mathbf{i} + \frac{\partial^2 F_2}{\partial x^2}\mathbf{j} + \frac{\partial^2 F_3}{\partial x^2}\mathbf{k} \\
&\quad + \frac{\partial^2 F_1}{\partial y^2}\mathbf{i} + \frac{\partial^2 F_2}{\partial y^2}\mathbf{j} + \frac{\partial^2 F_3}{\partial y^2}\mathbf{k} \\
&\quad + \frac{\partial^2 F_1}{\partial z^2}\mathbf{i} + \frac{\partial^2 F_2}{\partial z^2}\mathbf{j} + \frac{\partial^2 F_3}{\partial z^2}\mathbf{k}
\end{aligned}
$$

which is the same as $\nabla^2\mathbf{F}$.

Exercises

1. Find $\nabla^2 f$, given that $f(x,y,z) = x^5yz^3$.
2. Find $\nabla^2 f$, given that $f(x,y,z) = 1/(x^2 + y^2 + z^2)^{\frac{1}{2}}$.
3. Find $\nabla^2\mathbf{F}$, given that $\mathbf{F}(x,y,z) = 3\mathbf{i} + \mathbf{j} - x^2y^3z^4\mathbf{k}$.
4. Which of the following functions satisfy Laplace's equation?
   (a) $f(x,y,z) = e^z \sin y$
   (b) $f(x,y,z) = \sin x \sinh y + \cos x \cosh z$
   (c) $f(x,y,z) = \sin px \sinh qy$ (p and q are constants)
5. Tell whether each of the following is a vector field or a scalar field, given that f is a scalar field and **F** is a vector field. Two of the expressions are meaningless; determine which two.
   (a) ∇f
   (b) $\nabla\cdot\mathbf{F}$
   (c) $\nabla\times\mathbf{F}$
   (d) $\nabla\cdot(\nabla f)$
   (e) $\nabla\times(\nabla f)$
   (f) $\nabla\times f$
   (g) $\nabla^2\mathbf{F}$
   (h) $\nabla\times(\nabla^2\mathbf{F})$
   (i) $\nabla\times(\nabla^2 f)$
   (j) $\nabla(\nabla^2 f)$

6. (a) Show that, if f and g satisfy Laplace's equation, $f + g$ does also.
   (b) Find a function satisfying Laplace's equation and also the following identities: $f(0,y,z) = 0$, $f(x,0,z) = 0$, $f(\pi,y,z) = 0$, $f(x,5,z) = \sin x + \sin 2x$. [*Hint*: Use 6(a) and 4(c) to guess an answer.]

3.7 VECTOR IDENTITIES

Although we continue to use the del notation, formally manipulating

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

as though it were a vector, this practice has certain hazards. Keep in mind that the derivative operators appearing in the del operator act only on functions appearing to the right of the del operator.

For example, supposing that

$$\mathbf{F} = x^3y\mathbf{i} + y^2\mathbf{j} + x^2z\mathbf{k} \qquad \mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

let us compare the two expressions $(\nabla \cdot \mathbf{R})\mathbf{F}$ and $(\mathbf{R} \cdot \nabla)\mathbf{F}$. For the first of these we have

$$(\nabla \cdot \mathbf{R})\mathbf{F} = 3\mathbf{F} = 3x^3y\mathbf{i} + 3y^2\mathbf{j} + 3x^2z\mathbf{k}$$

On the other hand, in the second expression, $\mathbf{R}$ is to the left of ∇, and therefore the derivatives in the del operator do not act on $\mathbf{R}$. We have

$$\begin{aligned}(\mathbf{R} \cdot \nabla)\mathbf{F} &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)(x^3y\mathbf{i} + y^2\mathbf{j} + x^2z\mathbf{k}) \\ &= x(3x^2y\mathbf{i} + 2xz\mathbf{k}) + y(x^3\mathbf{i} + 2y\mathbf{j}) + z(x^2\mathbf{k}) \\ &= 4x^3y\mathbf{i} + 2y^2\mathbf{j} + 3x^2z\mathbf{k}\end{aligned}$$

Also, it is common practice to omit parentheses in a vector expression when there is only one interpretation of the expression which makes sense, in the context of ordinary vector analysis (i.e., excluding dyadics). For example, $\nabla \cdot \mathbf{R}\mathbf{F}$ and $\mathbf{R} \cdot \nabla\mathbf{F}$ must mean $(\nabla \cdot \mathbf{R})\mathbf{F}$ and $(\mathbf{R} \cdot \nabla)\mathbf{F}$, respectively, since $\nabla \cdot (\mathbf{R}\mathbf{F})$ and $\mathbf{R} \cdot (\nabla\mathbf{F})$ do not make sense in this context.

Similarly, $\nabla \cdot f\mathbf{F}$ means $\nabla \cdot (f\mathbf{F})$, simply the divergence of $f\mathbf{F}$, since $\nabla \cdot f$, and hence $(\nabla \cdot f)\mathbf{F}$, is meaningless.

In some cases where parentheses are omitted, two interpretations are possible, both of which make sense. For example, if $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ is a vector field and f is a scalar field, both $(\mathbf{A} \cdot \nabla)f$ and $\mathbf{A} \cdot (\nabla f)$ are meaningful and are sometimes written $\mathbf{A} \cdot \nabla f$. This is because both interpretations lead to exactly the same final result. We have

$$(\mathbf{A} \cdot \nabla)f = \left(A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\right)f = A_1\frac{\partial f}{\partial x} + A_2\frac{\partial f}{\partial y} + A_3\frac{\partial f}{\partial z}$$

and also

$$\mathbf{A}\cdot(\nabla f)=\mathbf{A}\cdot\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)=A_1\frac{\partial f}{\partial x}+A_2\frac{\partial f}{\partial y}+A_3\frac{\partial f}{\partial z}$$

both of them equal.

Because of the convention adopted above it is especially important to preserve order in working with ∇. For instance, $\nabla\cdot\mathbf{A}$ is a scalar field, simply the divergence of **A**, but $\mathbf{A}\cdot\nabla$ is the differential operator

$$A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}$$

a horse of quite a different color.

We now list a number of identities. Here **F** and **G** denote vector fields, ϕ denotes a scalar field, and $\mathbf{R}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Their proofs vary in character and difficulty, and we discuss them below.

$$\nabla(\phi_1\phi_2)=\phi_1\nabla\phi_2+\phi_2\nabla\phi_1 \tag{3.19}$$

$$\nabla\cdot\phi\mathbf{F}=\phi\nabla\cdot\mathbf{F}+\mathbf{F}\cdot\nabla\phi \tag{3.20}$$

$$\nabla\times\phi\mathbf{F}=\phi\nabla\times\mathbf{F}+\nabla\phi\times\mathbf{F} \tag{3.21}$$

$$\nabla f(u)=\frac{df}{du}\nabla u \tag{3.22}$$

In Eqs. (3.23) to (3.27), **A** is any constant vector.

$$\nabla\cdot(\mathbf{R}-\mathbf{A})=3 \tag{3.23}$$

$$\nabla\times(\mathbf{R}-\mathbf{A})=\mathbf{0} \tag{3.24}$$

$$\nabla(|\mathbf{R}-\mathbf{A}|^n)=n|\mathbf{R}-\mathbf{A}|^{n-2}(\mathbf{R}-\mathbf{A}) \tag{3.25}$$

$$\mathbf{F}\cdot\nabla(\mathbf{R}-\mathbf{A})=\mathbf{F} \tag{3.26}$$

$$\nabla(\mathbf{A}\cdot\mathbf{R})=\mathbf{A} \tag{3.27}$$

$$\nabla\cdot(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot(\nabla\times\mathbf{F})-\mathbf{F}\cdot(\nabla\times\mathbf{G}) \tag{3.28}$$

$$\nabla\times(\mathbf{F}\times\mathbf{G})=(\mathbf{G}\cdot\nabla)\mathbf{F}-(\mathbf{F}\cdot\nabla)\mathbf{G}+(\nabla\cdot\mathbf{G})\mathbf{F}-(\nabla\cdot\mathbf{F})\mathbf{G} \tag{3.29}$$

$$\nabla\times(\nabla\times\mathbf{F})=\nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F} \tag{3.30}$$

$$\nabla(\mathbf{F}\cdot\mathbf{G})=(\mathbf{F}\cdot\nabla)\mathbf{G}+(\mathbf{G}\cdot\nabla)\mathbf{F}+\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F}) \tag{3.31}$$

$$\nabla\times\nabla(\phi)=\mathbf{0} \tag{3.32}$$

$$\nabla\cdot(\nabla\times\mathbf{F})=0 \tag{3.33}$$

$$\nabla\cdot(\nabla\phi_1\times\nabla\phi_2)=0 \tag{3.34}$$

Identities (3.19), (3.20), and (3.21) are very simple. They are based on the formula expressing the derivative of a product as the sum of two terms, each containing the derivative of one factor. Any of these is easy to verify componentwise. For instance, the z component of $\nabla \times \phi\mathbf{F}$ is

$$\frac{\partial}{\partial x}(\phi F_2) - \frac{\partial}{\partial y}(\phi F_1)$$

Breaking this up, we see that this is ϕ times the z component of $\nabla \times \mathbf{F}$, plus the z component of $(\nabla\phi) \times \mathbf{F}$.

Identity (3.22) expresses the chain rule; its x component merely says

$$\frac{\partial}{\partial x} f(u) = \frac{df}{du}\frac{\partial u}{\partial x}$$

It can be generalized to functions of more than one variable. For example, if u_1 and u_2 are functions of x, y, and z and if f is a function of u_1 and u_2, then we have

$$\nabla f(u_1,u_2) = \frac{\partial f}{\partial u_1}\nabla u_1 + \frac{\partial f}{\partial u_2}\nabla u_2$$

Identities (3.23) through (3.27), which involve the vector $\mathbf{R}$, are quite trivial but occasionally useful.

Identities (3.28) through (3.31) involve the interplay of the vector and differential properties of ∇, and they are quite complex. Any of them can be verified by laboriously working out the components, and we cheerfully invite the devoted student to do so. In the next (optional) section on tensor notation we will use some heavy notational machinery to derive these equations more efficiently. However, we would like to mention a heuristic device for guessing at the form of the identities.

Let's take identity (3.28), and go to work on

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) \tag{3.35}$$

We know that, as far as the *vector* nature of the triple scalar product is concerned, we can interchange the dot and the cross. Thus we suspect that the expression (3.35) is equal to

$$(\nabla \times \mathbf{F}) \cdot \mathbf{G} \tag{3.36}$$

However, we must interpret (3.36) in an unconventional manner, namely, the operator ∇ must continue to differentiate *both* **F** *and* **G** [and not merely **F**, as (3.36) dictates]. So to be correct we must split (3.36) into two terms, analogous to the splitting in differentiating a product. The term where **F** alone is differentiated can be expressed unambiguously as

$$\mathbf{G} \cdot (\nabla \times \mathbf{F}) \tag{3.37}$$

To get the term where **G** is differentiated, we "rewrite" (3.36) as

$$-(\nabla \times \mathbf{G}) \cdot \mathbf{F} \tag{3.38}$$

which is consistent with the *vector* nature of the triple scalar product. Clearly from (3.38) we can display the part of the formula in which **G** is differentiated as

$$-\mathbf{F}\cdot(\nabla\times\mathbf{G}) \tag{3.39}$$

Thus we are led to guess that (3.35) equals (3.37) plus (3.39), in accordance with identity (3.28)!

Let us try this out again on formula (3.29). Using our old rule for $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$, we first write, *incorrectly*,

$$\nabla\times(\mathbf{F}\times\mathbf{G}) = (\nabla\cdot\mathbf{G})\mathbf{F} - (\nabla\cdot\mathbf{F})\mathbf{G} \tag{3.40}$$

This is incorrect because we must interpret ∇ as differentiating both **F** and **G** in each expression on the right. To break up this compound derivative and get a correct expression for "$(\nabla\cdot\mathbf{G})\mathbf{F}$", we observe that $(\nabla\cdot\mathbf{G})\mathbf{F}$, interpreted conventionally, gives the term in which **G** is differentiated, while $(\mathbf{G}\cdot\nabla)\mathbf{F}$ gives the term where **G** is treated as constant and we differentiate **F**. Handling the other term in (3.40) similarly, we propose that

$$\nabla\times(\mathbf{F}\times\mathbf{G}) = (\nabla\cdot\mathbf{G})\mathbf{F} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\nabla\cdot\mathbf{F})\mathbf{G} - (\mathbf{F}\cdot\nabla)\mathbf{G}$$

This is identity (3.29).

Clearly the above reasoning is tricky, but it can be very helpful in suggesting "which way to turn" in the derivation of complicated vector equations (such as those of electromagnetic theory). Suffice it to say that one always breathes easier after verifying any such "identity" in a mathematical handbook.

Identities (3.32), (3.33), and (3.34) are based on the appearance, in each case, of differences of mixed second derivatives. For example, the z component of $\nabla\times(\nabla\phi)$ is

$$\frac{\partial}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\phi}{\partial x}$$

It is well known from advanced calculus that such mixed derivatives are the same when taken in either order; hence these terms cancel. [To be rigorous, we should stipulate that ϕ and **F** possess *continuous second derivatives* when applying (3.32), (3.33), or (3.34).] A proof of the equality of the mixed partial derivatives appears in Appendix B.

Identity (3.22) has been included because, in this book, curvilinear coordinates are not discussed in detail until Chapter 5, and many people who use this book do not proceed that far. For such readers, the labor of computing a gradient can be reduced by using (3.22) and a little common sense, and the following example should be studied carefully by anyone in this category.

Example 3.23 Find the gradient of the scalar field f given by $f(r) = 1/r$, where r is the distance from the origin, $r = |\mathbf{R}| = \sqrt{x^2 + y^2 + z^2}$.

Solution Since r is the distance from the origin, ∇r can be computed by using Properties 3.1 through 3.4. Obviously, r increases most rapidly in the direction away from the origin, so the direction of ∇r is the same as the direction of the position vector $\mathbf{R}$, which also points away from the origin. When we move in this direction, the rate of increase of r per unit distance is simply $dr/dr = 1$. So ∇r is a unit vector directed away from the origin, and hence equals $\mathbf{R}/|\mathbf{R}|$, which is the position vector divided by its own magnitude. That is,

$$\nabla r = \nabla|\mathbf{R}| = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$$

Applying (3.22) to $f(r) = 1/r$, we have $\nabla f(r) = f'(r)\,\nabla r = (-1/r^2)\,\nabla r$, and, therefore,

$$\nabla\left(\frac{1}{r}\right) = \left(-\frac{1}{r^2}\right)\nabla r = -\frac{1}{r^2}\frac{\mathbf{R}}{|\mathbf{R}|} = -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = -\frac{\mathbf{R}}{r^3}$$

in agreement with Eq. (3.25) for $n = -1$. Readers familiar with electric fields will recognize this expression. Except for some physical constants, it is the electric field intensity due to a point charge located at the origin.

As in Sec. 1.14, the reader is advised to attach a permanent bookmark to this section, for future referencing.

Exercises

1. Verify Eqs. (3.19) and (3.20).
2. Verify (3.23) through (3.27).
3. Verify (3.32), (3.33), and (3.34).
4. "Derive" (3.30) heuristically.
5. Why is the following "identity" obviously not valid? (*Hint*: Check the symmetry.)

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) + \mathbf{F} \cdot (\nabla \times \mathbf{G})$$

3.8 OPTIONAL READING: TENSOR NOTATION

The operator ∇, considered as a vector operator, has components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$. In tensor notation we adopt two conventions which enable us to absorb ∇ into our system painlessly. First, we designate coordinates by the triple (x_1,x_2,x_3) instead of (x,y,z); this makes the ith component of ∇ equal $\partial/\partial x_i$. Second, we abbreviate $\partial/\partial x_i$ by ∂_i. Now let us write down the tensor expressions for the concepts introduced in this chapter.

The ith component of the gradient of ϕ is $\partial_i\phi$.

The divergence of $\mathbf{F}$ is the scalar $\partial_i F_i$ (remember summation).

The ith component of the curl, $\nabla \times \mathbf{F}$, is $\varepsilon_{ijk}\,\partial_j F_k$ (recall the determinant expression for curl).

The laplacian of ϕ is $\partial_i \partial_i \phi$. We may write this as $\partial_i{}^2\phi$ if we stipulate that the summation convention applies to *squared* terms, since they would have repeated subscripts if written out.

Now the proof of the identities of the last section can be carried out easily. To check identity (3.21), observe that the ith component of $\nabla \times \phi\mathbf{F}$ is

$$\varepsilon_{ijk}\,\partial_j(\phi F_k) = \varepsilon_{ijk}(\partial_j\phi)F_k + \varepsilon_{ijk}\phi\,\partial_j F_k$$

These terms we identify as the ith components of $\nabla\phi \times \mathbf{F}$ and $\phi\nabla \times \mathbf{F}$.

To check (3.26), observe that the ith component of $\mathbf{F}\cdot\nabla\mathbf{R}$ is $F_j\partial_j x_i$ (summing over j). But $\partial_j x_i = \delta_{ij}$, the Kronecker delta; so this expression is $F_j\delta_{ij} = F_i$, the ith component of $\mathbf{F}$.

The proof of formula (3.29) proceeds as follows:

$$\begin{aligned}\varepsilon_{ijk}\,\partial_j(\mathbf{F}\times\mathbf{G})_k &= \varepsilon_{ijk}\,\partial_j(\varepsilon_{klm}F_l G_m)\\ &= \varepsilon_{ijk}\varepsilon_{klm}\,\partial_j(F_l G_m)\\ &= (\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl})\,\partial_j(F_l G_m)\\ &= \partial_j(F_i G_j) - \partial_j(F_j G_i)\\ &= G_j\partial_j F_i + (\partial_j G_j)F_i - (\partial_j F_j)G_i - F_j\partial_j G_i\\ &= (\mathbf{G}\cdot\nabla)\mathbf{F}_i + (\nabla\cdot\mathbf{G})\mathbf{F}_i - (\nabla\cdot\mathbf{F})\mathbf{G}_i - (\mathbf{F}\cdot\nabla)\mathbf{G}_i\end{aligned}$$

The proof of (3.31) is rather complicated. We begin by developing the obvious expression for the ith component of $\nabla(\mathbf{F}\cdot\mathbf{G})$:

$$\partial_i(F_j G_j) = F_j\partial_i G_j + G_j\partial_i F_j \tag{3.41}$$

Now we are stumped; the terms on the right seem to have no vector analogs. How can we identify the right-hand side of identity (3.31) here? The clue lies in the tensor expression for $\mathbf{F}\times(\nabla\times\mathbf{G})$; its ith component is

$$\begin{aligned}\varepsilon_{ijk}F_j(\varepsilon_{klm}\,\partial_l G_m) &= \varepsilon_{ijk}\varepsilon_{klm}F_j\,\partial_l G_m\\ &= (\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl})F_j\,\partial_l G_m\\ &= F_j\partial_i G_j - F_j\partial_j G_i\end{aligned}$$

We observe the appearance of one of these "mystery" terms, $F_j\partial_i G_j$, plus the ith component of $-(\mathbf{F}\cdot\nabla)\mathbf{G}$. Transposing, we see that $F_j\partial_i G_j$ is the ith component of $\mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{F}\cdot\nabla\mathbf{G}$. Putting this into Eq. (3.41) above, and using a similar expression for $G_j\partial_i F_j$, we get

$$\nabla(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{F}\cdot\nabla\mathbf{G} + \mathbf{G}\times(\nabla\times\mathbf{F}) + \mathbf{G}\cdot\nabla\mathbf{F}$$

We have derived the identity.

The equality of mixed second derivatives of any (twice continuously differentiable) function can be expressed in tensor notation by the equation

$$\partial_j\partial_i\phi = \partial_i\partial_j\phi$$

or, simply

$$\partial_j\partial_i = \partial_i\partial_j$$

That is, the components of ∇ *commute* with each other. (Of course, they do not commute with functions: $\partial_i\phi$ is very different from $\phi\,\partial_i\cdot$.) This makes the verification of identities (3.32), (3.33), and (3.34) simple. For (3.34), we use ψ and χ for ϕ_1 and ϕ_2, in order not to confuse subscripts. We then have

$$\partial_i[\varepsilon_{ijk}(\partial_j\psi)(\partial_k\chi)] = \varepsilon_{ijk}(\partial_i\partial_j\psi)(\partial_k\chi) + \varepsilon_{ijk}(\partial_j\psi)(\partial_i\partial_k\chi)$$

Because of the antisymmetric nature of ε_{ijk}, as we sum over i and j the terms $\partial_i\partial_j\psi$ and $\partial_j\partial_i\psi$ come in with opposite signs for $i \neq j$, and with coefficient zero if $i = j$. Thus all the addends in the first term cancel, as do those in the second, and we get zero, in accordance with the identity (3.34).

Exercises

Using the tensor notation, prove the following vector identities:

1. $\nabla \cdot \phi\mathbf{F} = \phi\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla\phi$
2. $\nabla(\mathbf{A} \cdot \mathbf{R}) = \mathbf{A}$ if $\mathbf{A}$ is constant.
3. $\nabla \times \mathbf{R} = \mathbf{0}$
4. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
5. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$
6. $\nabla \times (\nabla\phi) = \mathbf{0}$
7. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

Supplementary Problems

1. Let $T(x,y,z) = x^2 + 2y^2 + 3z^2$ and let S be the isotimic surface: $T = 1$. Find all points (x,y,z) on S that have tangent planes with normals (1,1,1).
2. Find the direction of maximal increase of the function

$$f(x,y,z) = e^{-xy}\cos z$$

at the point (1,1,0).

3. If $\phi(x,y,z) = x^2y + zy + z^3$, find:
   (a) the gradient of ϕ, and
   (b) the equation of the plane passing through the point $(1,-1,1)$ and tangent to the level surface of ϕ at that point.
4. Let S_1 and S_2 be the surfaces with equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$$

Show that, if $a^2B^2 - b^2A^2 = 0$, then the curve of intersection of S_1 and S_2 must be parallel to the xy plane.

5. Evaluate div $\mathbf{F}$ and **curl** $\mathbf{F}$ when

$$\mathbf{F} = e^x \cos yz\,\mathbf{i} + e^y \cos xz\,\mathbf{j} + e^z \cos xy\,\mathbf{k}$$

6. If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is a constant vector, and if $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |R|$, show that

$$\nabla \cdot \left|\frac{\mathbf{A} \times \mathbf{R}}{r}\right| = 0$$

7. Find:
   (a) the divergence and the curl of **F** when

$$\mathbf{F} = e^{y+z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$$

   (b) the direction of maximal increase of the function f with

$$f(x,y,z) = \ln\left[(x+y)^2 + (y+z)^2 + (z+x)^2\right]$$

   at the point $(2,-1,1)$, and
   (c) the directional derivative at the function f of part (b), at the point $(2,-1,1)$ in the direction $(0,-1,0)$.

8. Given $\mathbf{F} = y^2\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$, find:
   (a) the curl of **F**, and
   (b) the component of **curl F** along the tangent to the curve

$$x = \cos \pi t \qquad y = \sin \pi t \qquad z = t^2 \qquad \text{at} \quad t = 1$$

9. Let $\mathbf{F}(x,y,z)$ be a vector field defined in all space, and consider an intelligent ant living on the xy plane. Suppose all the ant knows about **F** is its values on the xy plane.
   (a) Can this ant compute $\nabla \times \mathbf{F}$? Explain briefly.
   (b) Can this ant compute $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$? Explain briefly.

10. A volcano just erupted and lava is streaming down from the mountain top. Suppose that the altitude of the mountain is given by

$$z(x,y) = he^{-(x^2+2y^2)}$$

   where h is the maximum height, and suppose also that lava flows in the direction of steepest descent (fastest change in z). Find:
   (a) the projection on the xy plane of the direction in which lava flows away from the point $(1,2,he^{-9})$, and
   (b) the projection on the xy plane of the equation of the flow line of the lava passing through the point $(1,2,he^{-9})$.

11. If $\mathbf{V}(\mathbf{R})$ can be expressed $\mathbf{V}(\mathbf{R}) = \mathbf{A}f(\mathbf{R} \cdot \mathbf{B})$, where **A** and **B** are constant, prove that **curl V** is perpendicular to both **A** and **B**.

CHAPTER 4

Line and Surface Integrals

4.1 LINE INTEGRALS

In this section we are going to study a construction that has found considerable utility in mathematics and physics. It is the operation of integrating a vector field along a curve in space.

Let us give some thought to the meaning of integration along a curve. By analogy with the theory of (Riemann) integration in elementary calculus, one would suspect that the curve is partitioned into short arcs, then some sort of sum is formed over the partition; and, finally, the "integral" emerges as the limit of these sums as the partitions are made finer and finer.

In fact, we have already gained some experience with this type of process in Sec. 2.2, where we computed arc length for a smooth arc. Figure 4.1 (a replica of Fig. 2.11, repeated for convenience) illustrates how the points Q_0, $Q_1, \ldots, Q_n$, (with position vectors $\mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_n$, respectively) partition the curve and generate the inscribed polygonal path, whose length we calculate by summing the lengths of the sides $|\Delta\mathbf{R}_k| = |\mathbf{R}_k - \mathbf{R}_{k-1}|$. The length of the *curve* is then taken to be the limit of these sums, as the partitions are refined in such a manner that the largest length $|\Delta\mathbf{R}_k|$ goes to zero.

In that same section we saw how this limit can be evaluated if the curve is parametrized by $\mathbf{R} = \mathbf{R}(t)$, $a \le t \le b$. Recall the essentials of the technique: The interval $[a,b]$ is partitioned $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ to correspond with the points $\mathbf{R}_k = \mathbf{R}(t_k)$, and the approximation

$$\Delta\mathbf{R}_k \approx \frac{d\mathbf{R}}{dt}\Delta t_k$$

is used to argue that

$$\int_C |d\mathbf{R}| \equiv \lim \sum_{k=1}^{n} |\Delta\mathbf{R}_k| = \lim \sum_{k=1}^{n} \left|\frac{d\mathbf{R}}{dt}\Delta t_k\right| = \int_a^b \left|\frac{d\mathbf{R}}{dt}\right| dt$$

as the Δt_k go to zero. This final expression is an ordinary integral.

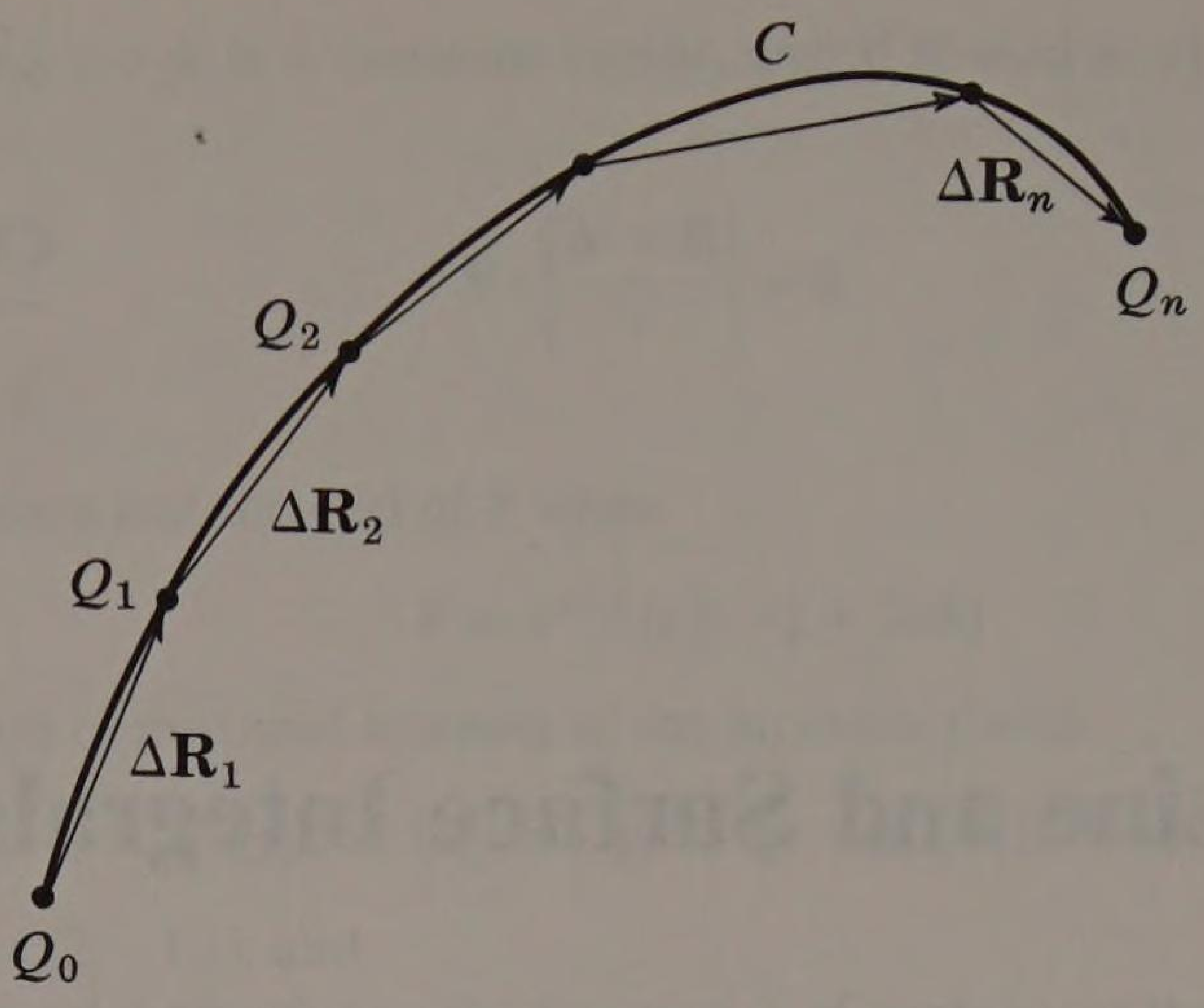

FIGURE 4.1

Using this as a model, we now turn to the definition of the line integral. To form a line integral we start with a smooth, oriented space curve C in a region in which there is defined a continuous vector field $\mathbf{F}$. Let us subdivide C into smaller arcs, and approximate it by a polygonal path, as in Fig. 4.1. Let $\mathbf{F}_k$ denote the value of $\mathbf{F}$ at the point Q_k, and form the sum $\sum_{k=1}^{n} \mathbf{F}_k \cdot \Delta\mathbf{R}_k$, a scalar. We define

$$\int_C \mathbf{F} \cdot d\mathbf{R} \tag{4.1}$$

to be the limit of sums of this form, when the approximating polygonal paths are obtained by taking increasingly small subdivisions while n increases without bound. It can be shown that the limit exists, and is independent of the particular subdivisions chosen, provided that the maximum value of the magnitudes $|\Delta\mathbf{R}_k|$ tends to zero.

Expressions like (4.1) are called *line integrals.* (This is perhaps unfortunate, since C need not be a line segment; *curve integral* would be a better term.) *The necessary ingredients for a line integral are a vector field and an oriented curve, and the result is a scalar.*

The definition as given is ambiguous unless C is *oriented.* The direction of the vectors $\Delta\mathbf{R}_k$ is taken to be consistent with the orientation of C, which in Fig. 4.1 is *from* Q_0 *to* Q_n. If C were oriented in the opposite way, from Q_n to Q_0, each of the vectors $\Delta\mathbf{R}_k$ would be chosen in the opposite direction, and the line integral would change sign.

Other notations can be used to denote line integrals. If $\mathbf{T}$ is a unit vector tangent to the path, in the direction determined by the orientation, then $\mathbf{T} = d\mathbf{R}/ds$ and Eq. (4.1) can be written

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds \tag{4.2}$$

where s, the arc length measured along C, is taken to be *increasing* in the direction determined by the orientation of C.

If $F_t = \mathbf{F} \cdot \mathbf{T}$ is the scalar component of $\mathbf{F}$ in the direction of the unit tangent, the line integral can also be written

$$\int_C F_t \, ds \tag{4.3}$$

In vector language, we sometimes speak of the *line integral of the tangential component of* **F** *over the oriented curve C.* If we wish to be sloppier, we just say "the integral of **F** along C."

In books on advanced calculus that do not use vector notation, yet another form is used:

$$\int_C (F_1 \, dx + F_2 \, dy + F_3 \, dz) \tag{4.4}$$

We obtain this from (4.1) by taking $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Then, since $d\mathbf{R} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$, we have $\mathbf{F} \cdot d\mathbf{R} = F_1 \, dx + F_2 \, dy + F_3 \, dz$.

An expression such as $F_1 \, dx + F_2 \, dy + F_3 \, dz$, where F_1, F_2, and F_3 are functions of x, y, and z, is called a *differential form.* We call (4.4) the line integral of the differential form over the oriented curve C.

Many students feel queasy about line integrals, at first, because they don't see "what good" they are. A common question is "What do you have after you have computed a line integral?" The answer is: you have a number. Depending on the type of problem, this number may represent work done, change in potential energy, total heat flow, change in entropy, circulation of a fluid, and so on, but at this point the student is advised to concentrate simply on learning *how to compute* line integrals.

In a moment we are going to see how the line integral can be easily evaluated when the curve is parametrized. Just for the experience, however, we first present an example that computes a line integral directly from the definition. Keep in mind that Example 4.1 is gimmicked to work out nicely, and is atypical in this respect.

Example 4.1 Let C be the curve $y = \sqrt{x}$ in the xy plane, extending from (0,0,0) to (1,1,0), and let $\mathbf{F} = xy^2\mathbf{i} + y^2\mathbf{k}$. Find $\int_C \mathbf{F} \cdot d\mathbf{R}$ directly from the definition of the integral as the limit of a sum.

Solution For convenience, let all Δx's equal $1/n$, so that

$$Q_k = (x_k, y_k, z_k) = \left(\frac{k}{n}, \sqrt{\frac{k}{n}}, 0\right)$$

$$\Delta\mathbf{R}_k = \frac{1}{n}\mathbf{i} + \left(\sqrt{\frac{k}{n}} - \sqrt{\frac{k-1}{n}}\right)\mathbf{j}$$

$$\mathbf{F}_k = x_k {y_k}^2\mathbf{i} + {y_k}^2\mathbf{k} = \frac{k^2}{n^2}\mathbf{i} + \frac{k}{n}\mathbf{k}$$

$$\sum_{k=1}^{n} \mathbf{F}_k \cdot \Delta\mathbf{R}_k = \sum_{k=1}^{n} \frac{k^2}{n^3} = \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\left(\frac{1}{6}n(n+1)(2n+1)\right) = \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

which tends to $\frac{1}{3}$ as $n \to \infty$. Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \tfrac{1}{3}$$

In the usual situation one has a parametrization $\mathbf{R} = \mathbf{R}(t)$, $a \le t \le b$, for the smooth curve C, and by analogy with the arc-length technique we propose

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \lim \sum_{k=1}^{n} \mathbf{F}_k \cdot \Delta\mathbf{R}_k = \lim \sum_{k=1}^{n} \mathbf{F}_k \cdot \frac{d\mathbf{R}}{dt} \Delta t_k = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{R}}{dt} dt \tag{4.5}$$

The final form is an ordinary definite integral. To see this, observe that we have the continuous vector function $\mathbf{F} = \mathbf{F}(x,y,z) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, and the continuously differentiable parametric functions $x = x(t)$, $y = y(t)$, and $z = z(t)$, so that plugging the latter into the former produces, in Eq. (4.5), an ordinary integral of a function of t:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{R} = \int_a^b \Bigg[& F_1(x(t), y(t), z(t)) \frac{dx}{dt} \\ & + F_2(x(t), y(t), z(t)) \frac{dy}{dt} \\ & + F_3(x(t), y(t), z(t)) \frac{dz}{dt} \Bigg] dt\end{aligned}$$

For instance, the curve in Example 4.1 can be parametrized $x = t$, $y = \sqrt{t}$, $z = 0$, $0 \le t \le 1$, and Eq. (4.5) becomes

$$\int_0^1 \left(xy^2 \frac{dx}{dt} + y^2 \frac{dz}{dt} \right) dt = \int_0^1 t^2 \, dt = \frac{1}{3}$$

Observe that the line integral has been defined without reference to the parametrization of the curve, so its value will depend only on the field $\mathbf{F}$ and the oriented curve C, not on the choice of the parameter t. Sometimes *arc length* is a convenient parameter, sometimes it is better to use an *angle* or the *time*, or one of the variables x, y, z. Examples are given below; study them carefully!

The integrals $\int_a^b f(x)\,dx$ that occur in elementary calculus can be regarded as very special kinds of line integrals. Indeed, let us suppose that $\mathbf{F}$ is always directed parallel to the x axis, so that $\mathbf{F} = f(x)\mathbf{i}$, and suppose C is a segment of the x axis, $a \le x \le b$, oriented in the direction of increasing x. Then $d\mathbf{R} = dx\,\mathbf{i}$, and $\int_C \mathbf{F} \cdot d\mathbf{R} = \int_a^b f(x)\,dx$. So you already have had some experience in evaluating line integrals! *Caution*: In general, line integrals are *not* interpreted to represent areas under curves and do *not* represent arc length.

Example 4.2 Compute the line integral $\int \mathbf{F} \cdot d\mathbf{R}$ from (0,0,0) to (1,2,4) if

$$\mathbf{F} = x^2\mathbf{i} + y\mathbf{j} + (xz - y)\mathbf{k}$$

(a) along the line segment joining these two points,
(b) along the curve given parametrically by $x = t^2$, $y = 2t$, $z = 4t^3$.

Solution (a) Parametric equations for the line segment joining (0,0,0) to (1,2,4) are $x = t$, $y = 2t$, $z = 4t$ (Sec. 1.8). We have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_C x^2\,dx + y\,dy + (xz - y)\,dz\\ &= \int_0^1 t^2\,dt + (2t)(2\,dt) + (4t^2 - 2t)(4\,dt)\\ &= \int_0^1 (17t^2 - 4t)\,dt = \frac{11}{3}\end{aligned}$$

(b) In this case we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_0^1 (t^4)(2t\,dt) + (2t)(2\,dt) + (4t^5 - 2t)(12t^2\,dt)\\ &= \int_0^1 (2t^5 + 4t + 48t^7 - 24t^3)\,dt = \frac{7}{3}\end{aligned}$$

Example 4.3 Find the line integral of the tangential component of $\mathbf{F} = x\mathbf{i} + x^2\mathbf{j}$ from $(-1,0)$ to $(1,0)$ in the xy plane (a) along the x axis, (b) along the semicircle $y = \sqrt{1 - x^2}$, (c) along the dotted polygonal path shown in Fig. 4.2.

Solution

(a) Along the x axis, $y = 0$, hence $dy = 0 \cdot dx$ and

$$\begin{aligned}\int \mathbf{F}\cdot d\mathbf{R} &= \int (x\,dx + x^2\,dy)\\ &= \int_{-1}^{1} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{1} = 0\end{aligned}$$

(b) Along the semicircle, a convenient parameter is the polar coordinate θ. Since the radius of the circle is unity, we have, for points (x, y) on this path, $x = \cos\theta$, $y = \sin\theta$, hence $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$, and θ runs from π down to zero.

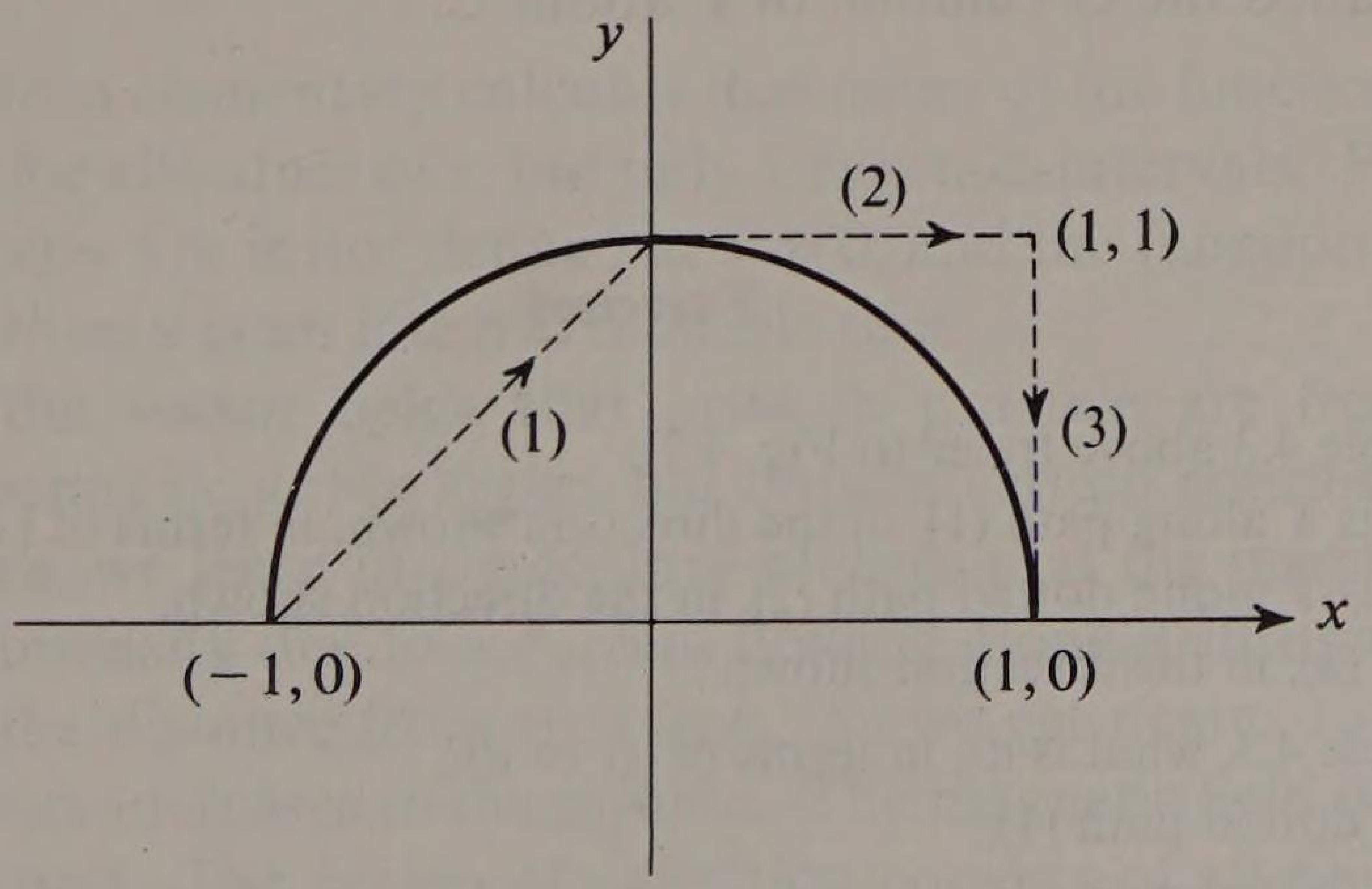

FIGURE 4.2

$$\begin{aligned}\int \mathbf{F}\cdot d\mathbf{R} &= \int (x\,dx + x^2\,dy)\\ &= \int_\pi^0 [(\cos\theta)(-\sin\theta\,d\theta) + (\cos^2\theta)(\cos\theta\,d\theta)]\\ &= \int_\pi^0 (-\sin\theta\cos\theta + \cos^3\theta)\,d\theta\\ &= \left[-\frac{\sin^2\theta}{2} - \sin\theta + \frac{\sin^3\theta}{3}\right]_\pi^0 = 0\end{aligned}$$

(c) Along the path labeled (1) in Fig. 4.1, $y = x + 1$, so that $dy = dx$ and

$$\int (x\,dx + x^2\,dy) = \int_{-1}^0 [x\,dx + x^2\,dx] = -\frac{1}{6}$$

Along path (2), $y = 1$, so that $dy = 0 \cdot dx$ and

$$\int (x\,dx + x^2\,dy) = \int_0^1 x\,dx = \frac{1}{2}$$

Along path (3), $x = 1$, so that $dx = 0 \cdot dy$ and

$$\int (x\,dx + x^2\,dy) = \int_1^0 dy = -1$$

[Note that we use y instead of x as the parameter along path (3).] The value of the integral is $-\frac{1}{6} + \frac{1}{2} - 1 = -\frac{2}{3}$.

Example 4.4 The work W done by a force $\mathbf{F}$ in moving a particle from the initial to the final point of an oriented curve C is given by

$$W = \int_C \mathbf{F}\cdot d\mathbf{R} \tag{4.6}$$

This generalizes Example 1.14, and reduces to it in the special case where **F** is a constant.

Note: If a curve is *closed*, i.e., its initial and final points coincide, the notation $\oint \mathbf{F}\cdot d\mathbf{R}$ is frequently used. The line integral of **F** around a closed curve C is called the *circulation* of **F** about C.

Exercises

1. In Example 4.3 above (refer to Fig. 4.2),
   (a) what is **T** along path (1), in the direction shown, in terms of **i** and **j**,
   (b) what is **T** along dotted path (2), in the direction shown,
   (c) along (3), in the direction shown?

2. In Example 4.3, what is ds, in terms of dx or dy,
   (a) along dotted path (1),
   (b) along dotted path (2),
   (c) along dotted path (3)?

3. Show that $d\mathbf{R} = dx\,\mathbf{i} + dy\,\mathbf{j}$ is the same as $\mathbf{T}\,ds$ in each of the three special cases referred to in the preceding two problems. (This illustrates the general rule that, in practice, it is easier to find $d\mathbf{R}$ directly than to find $\mathbf{T}$ and ds separately and multiply.)
4. Let

$$\mathbf{F} = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}$$

Find the line integral of the tangential component of $\mathbf{F}$, from $(-1,0)$ to $(1,0)$,
(a) along the semicircle $y = \sqrt{1 - x^2}$;
(b) along the dotted polygonal path shown in Fig. 4.2.
5. By changing to polar coordinates, find the answers to Exercise 4 by inspection.
6. Find $\int \mathbf{F} \cdot d\mathbf{R}$ from $(1,0,0)$ to $(1,0,4)$, if $\mathbf{F} = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$,
(a) along the line segment joining $(1,0,0)$ and $(1,0,4)$;
(b) along the helix $x = \cos 2\pi t,\ y = \sin 2\pi t,\ z = 4t$.
7. Find $\int \mathbf{R} \cdot d\mathbf{R}$ from $(1,2,2)$ to $(3,6,6)$, along the line segment joining these points,
(a) in the manner described in the text;
(b) by observing that $\mathbf{R} \cdot d\mathbf{R} = s\,ds$, where $s = (x^2 + y^2 + z^2)^{1/2}$ is the distance from the origin, and computing $\int_3^9 s\,ds$.
8. Find the value of $\oint [(3x + 4y)\,dx + (2x + 3y^2)\,dy]$ around the circle $x^2 + y^2 = 4$.
9. Find the line integral $\int \mathbf{F} \cdot d\mathbf{R}$ along the line segment from $(1,0,2)$ to $(3,4,1)$ where $\mathbf{F} = 2xy\mathbf{i} + (x^2 + z)\mathbf{j} + y\mathbf{k}$.
10. Find the integral $\oint \mathbf{F} \cdot d\mathbf{R}$ around the circumference of the circle $x^2 - 2x + y^2 = 2$, $z = 1$, where $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + xyz^2\mathbf{k}$.
11. Find $\int \mathbf{F} \cdot d\mathbf{R}$ where $\mathbf{F} = x^2\mathbf{i} + \mathbf{j} + yz\mathbf{k}$, along C: $x = t,\ y = 2t^2,\ z = 3t,\ 0 \le t \le 1$.
12. Let $\mathbf{F} = \boldsymbol{\omega} \times \mathbf{R}$, where $\boldsymbol{\omega}$ is a constant. (Recall Example 3.22.)
(a) Compute $\int \mathbf{F} \cdot d\mathbf{R}$ along the straight line from $(0,0,0)$ to $(2,2,2)$. (*Hint*: Use a little thought, and you can avoid any work.)
(b) Compute the same line integral along the path $z = (x^2 + y^2)/4$ in the plane $x = y$.

4.2 DOMAINS; SIMPLY CONNECTED DOMAINS

We recall from elementary calculus that many of the functions that arise are not defined for all values of x, but only for certain intervals. For example, the function $f(x) = 1/x$ is not defined at $x = 0$, and the function $f(x) = \csc x$ is not defined when x is an integral multiple of π.

Similarly, the vector fields that arise in practice are frequently not defined at all points (x,y,z) in space, but only in certain regions of space.

For instance, we learn in elementary physics that the magnitude of the magnetic field intensity due to a current flowing along a straight line varies inversely with the distance from that line. As we get nearer to the line the magnetic intensity increases in magnitude. The magnetic field is not defined along the line itself. The region of definition consists of all points in space except those along the line.

Similarly, the electric intensity due to a system of n point charges is defined everywhere in space except at the n points in question.

To be sure, the fields that arise in elementary physics are rather hypothetical (Is a charge really concentrated at a point?), but they are useful in theoretical discussions and their study is essential to more advanced work.

The reader with limited knowledge of electric or magnetic field theory may imagine instead that the fields we consider are the velocity fields of fluids that are in some container. Obviously, it is nonsense to speak of the velocity vector at any point outside the container. The region of definition in this case consists of all points within the container.

The vector fields that usually arise, both in theory and in practice, have two important properties. First, such a field is defined in the interior of a given region but not on the boundary of the region. Secondly, if the field is defined at two points P and Q, it is possible to find a smooth arc C joining P to Q along which the field is everywhere defined.

For instance, the velocity of a fluid in a container is not defined for points on the surface of the container, but only for points in the interior of the container. Moreover, it is unusual to consider a container with separate compartments; we usually assume that if there is fluid at two points P and Q, it is possible to move from P to Q without passing through any separating walls. Motivated by these ideas, we now give several precise definitions.

If P is any given point and ε is any positive number (zero is excluded), we say that an *ε neighborhood of P* is the set of all points that are *less* than ε in distance away from P. Thus, if we are speaking of points in the plane, an ε neighborhood of a point P consists of all points in the interior (but not on the circumference) of a circle of radius ε and center at P. If we are speaking of points in space, an ε neighborhood of P consists of all points in the interior (but not on the surface) of a sphere of radius ε and center at P.

Given a region R, we say that P is an *interior point* of R if it is possible to find an ε neighborhood of P that lies completely within R. We say that P is a *boundary point* of R if, no matter how small we take the positive number ε, the ε neighborhood of P contains at least one point in R and one point not in R. So, by definition, an interior point cannot be a boundary point, nor can a boundary point be an interior point.

A region is said to be *open* if every point in the region is an interior point of the region. Thus, if the region of definition of a vector field is an open region, we can say: if the field is defined at a point P, it will also be defined in some ε neighborhood of P. Of course, if P is very near the boundary of the region, ε may have to be very small.

By definition, an open region does not include its boundary. (For example, the set of all points *within* a cube is an open region in space, but the set consisting of all those points either within or on the surface of a cube is not an open region.) If we say an arc C lies in an open region, then by definition C cannot intersect or even touch the boundary of the region.

Henceforth, we shall consider only open regions.

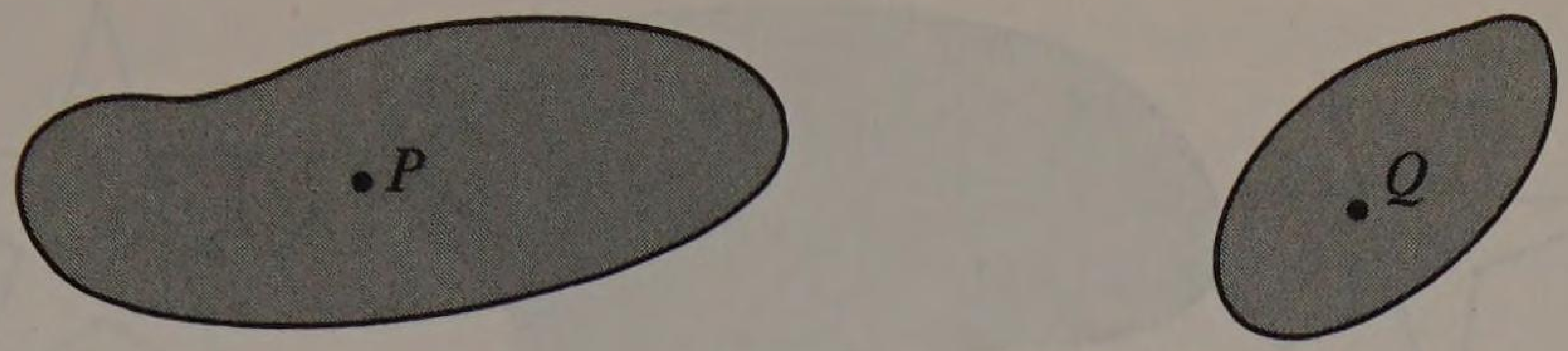

FIGURE 4.3

An open region R is said to be *connected* if, given any two points P and Q in R, there can be found a smooth arc in R that joins P to Q.

In Fig. 4.3 we show a region in the plane that is *not* connected. Obviously we cannot join P to Q by a smooth arc that lies completely within the region. We will have no occasion to consider such regions; henceforth we consider only connected regions.

A region that is both open and connected is called a *domain.*

The region of definition of the magnetic field due to a steady current flowing along the z axis consists of all points except those on the z axis. The region of definition of the electric field due to a system of n fixed point charges consists of all points other than the given n points. It is easy to see that in either case the region is both open and connected, so that the word "domain" applies.

In Fig. 4.4 we give an example of a region in the plane. If we let D denote the set of points within the shaded region, not including any points on either of the curves C_1 and C_2, then D is a domain. The points on the curves C_1 and C_2 constitute the boundary of the domain. In the figure we give an example of a smooth arc joining two points P and Q.

Of special importance are those domains that are simply connected. In Fig. 4.5 we show a region in the plane that is simply connected. The regions indicated in Figs. 4.4 and 4.6 are not simply connected.

Roughly speaking, a domain is said to be *simply-connected* if every closed curve lying in the domain can be continuously shrunk to a point in the domain without any part of the curve passing through regions outside

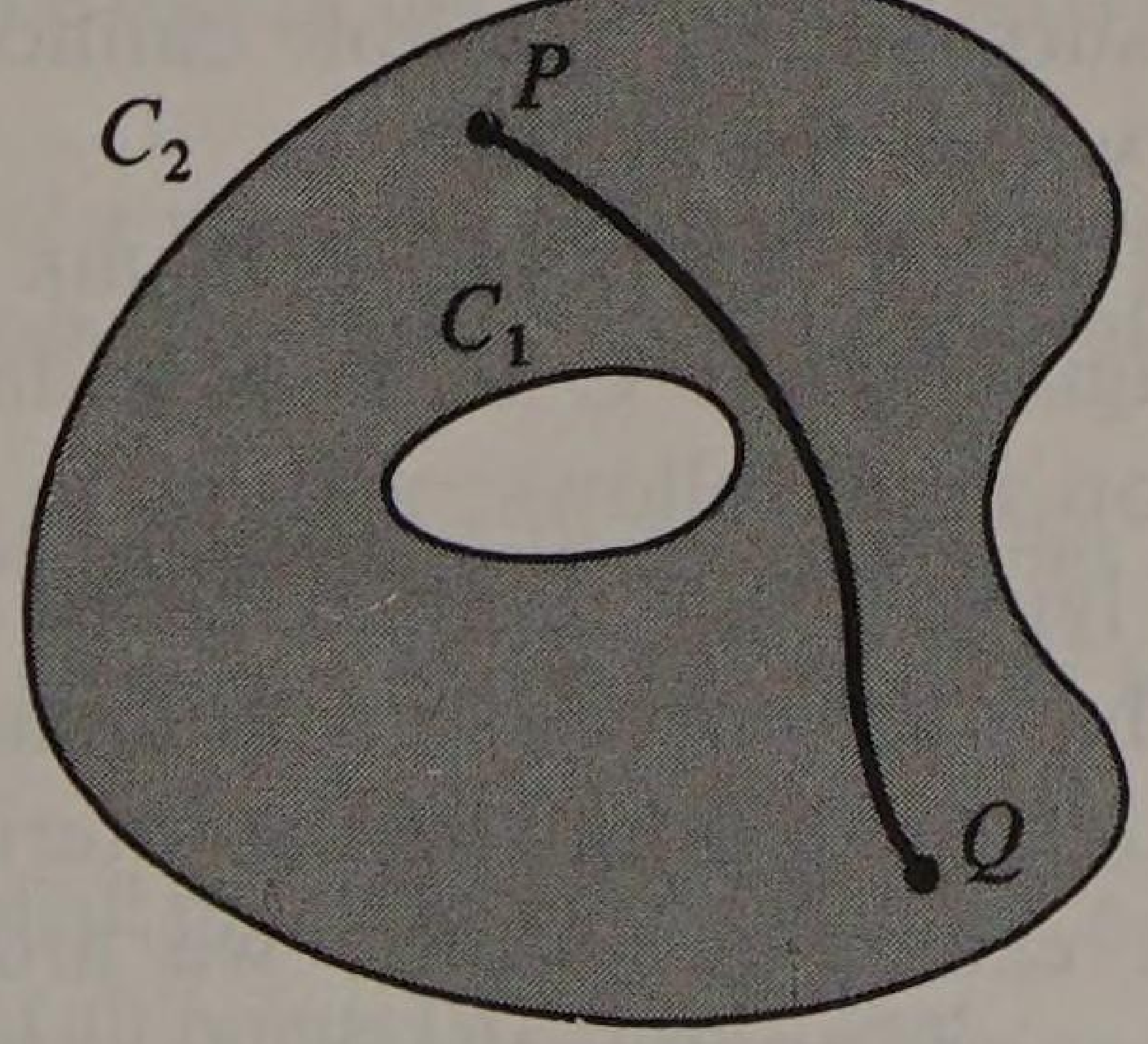

FIGURE 4.4

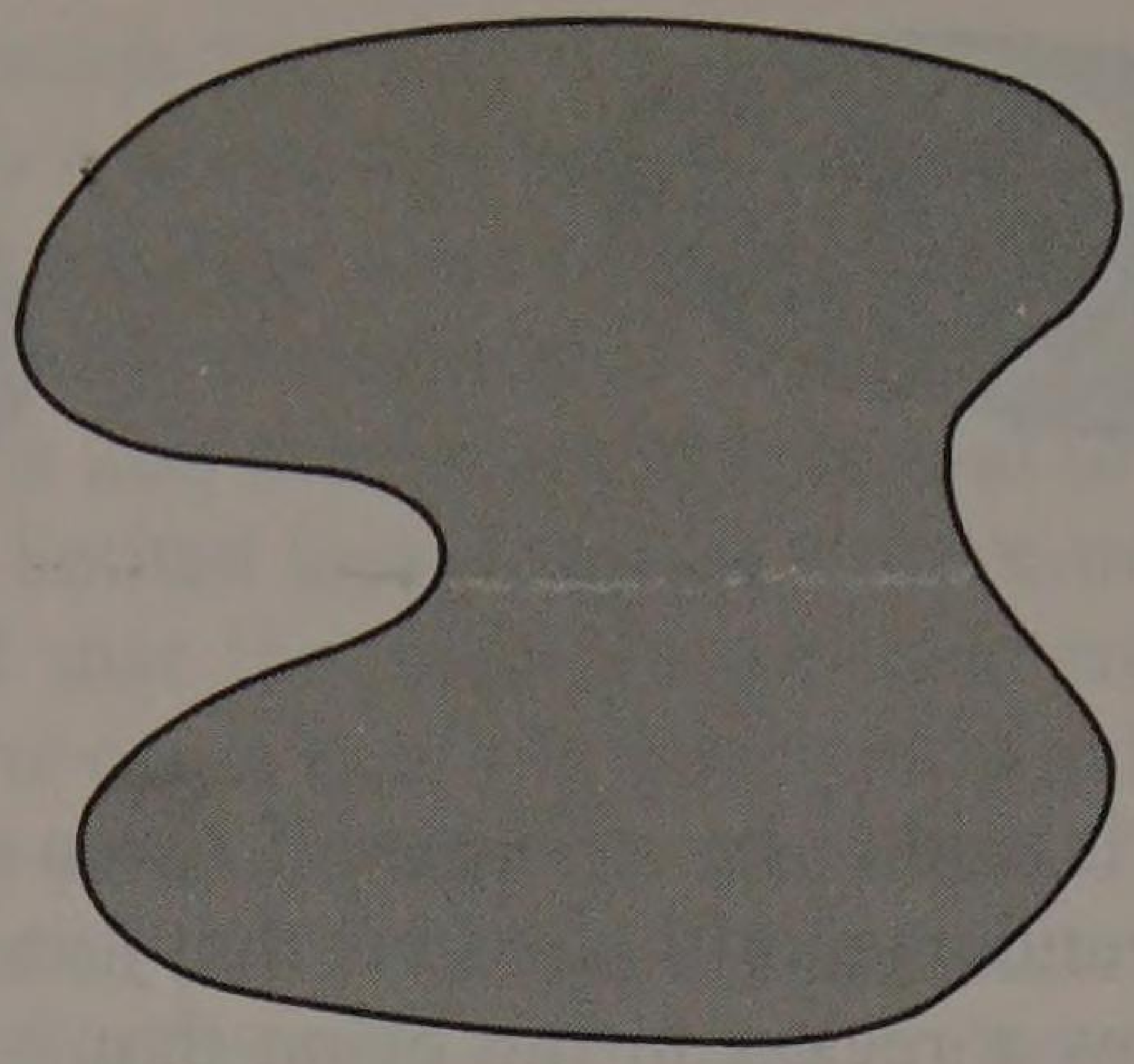

FIGURE 4.5

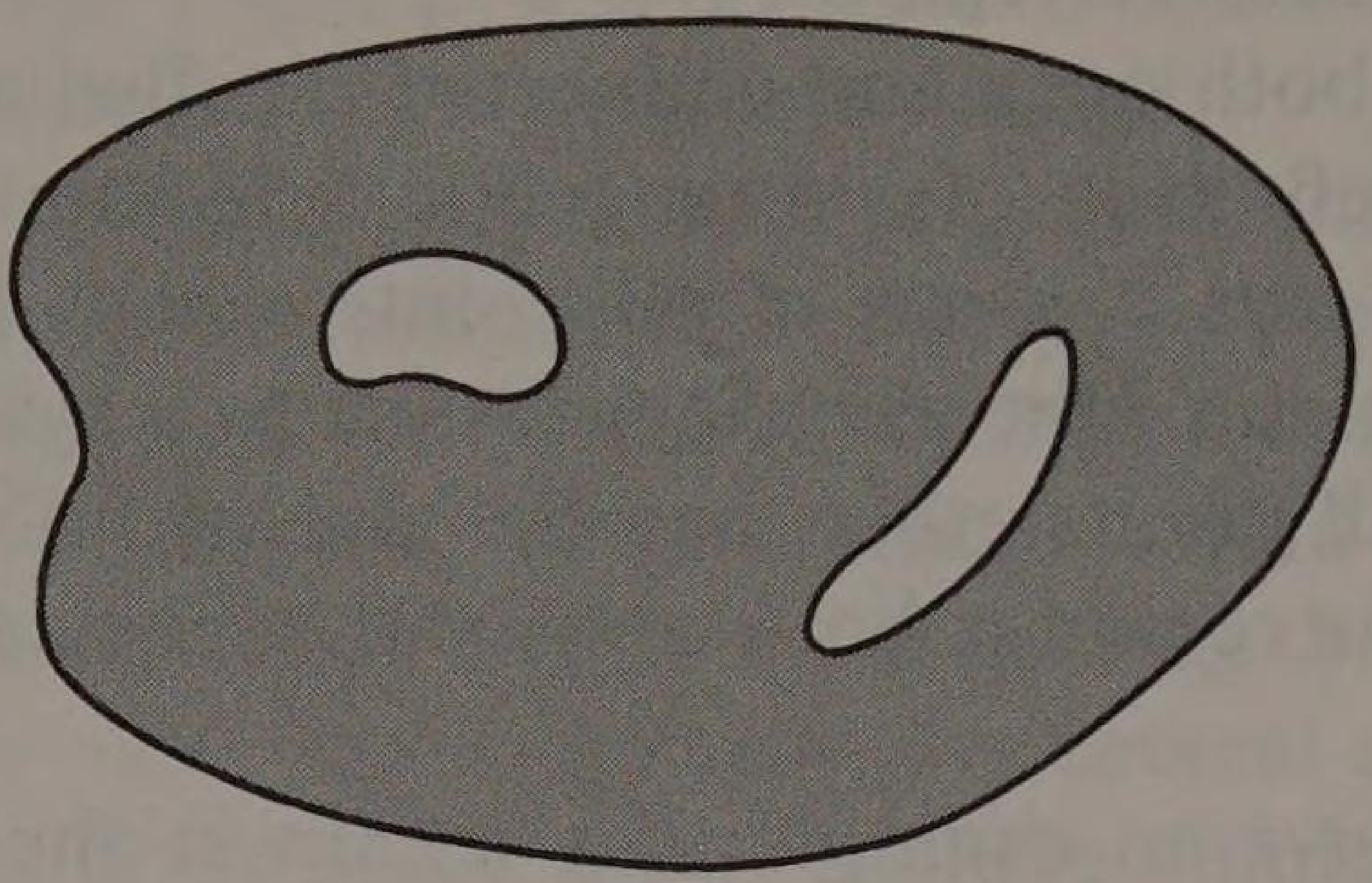

FIGURE 4.6

the domain. The plane regions indicated in Figs. 4.4 and 4.6 are not simply connected because no closed curve surrounding one of the "holes" could be shrunk to a point while still always remaining in the domain. Thus, in the special case of a domain of points in the plane, this simply means that, given any closed curve in the domain, all points within the closed curve are also in the domain. In other words, there are no "holes" in the domain.

Simply connected domains in space are, very roughly speaking, those domains through which no holes have been bored. Thus, the set of points in the interior of a torus (doughnut) is not simply connected, since a closed curve within the torus surrounding the hole cannot be shrunk to a point while remaining always within the torus.

A closed curve C, in the process of being shrunk to a point, will generate a surface having the original curve C as its boundary. Thus, another way of wording the definition is as follows: a domain is simply connected if, given any closed curve lying in the domain, there can be found a surface within the domain that has that curve as its boundary.

The domain consisting of all points in the interior of a sphere is simply connected. As another example, suppose we are given two concentric spheres; then the set of points outside the inner sphere but inside the outer sphere comprises a simply connected domain.

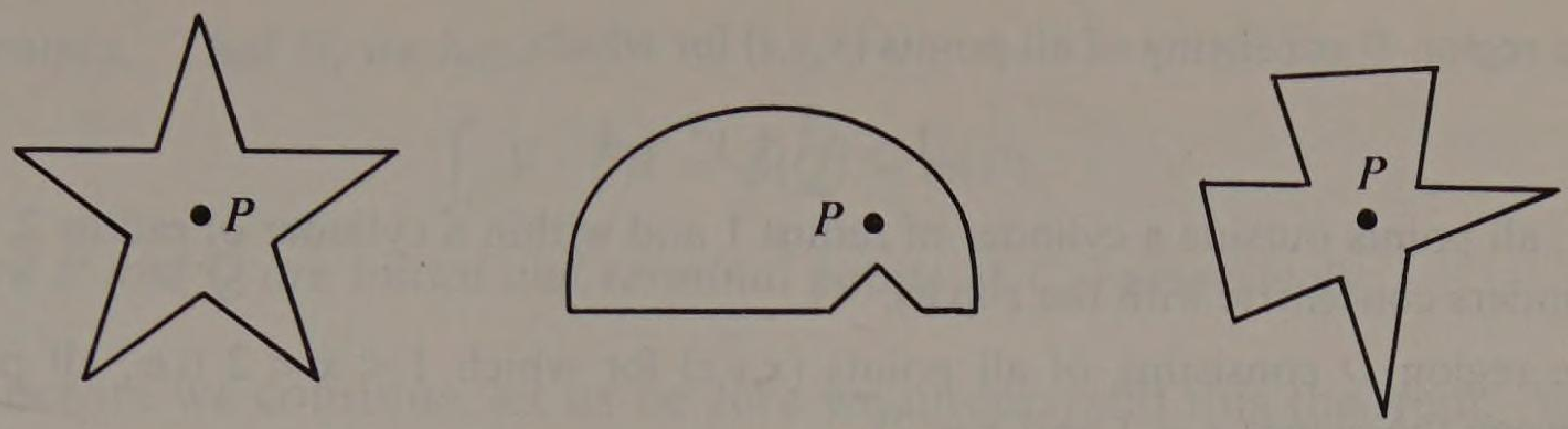

FIGURE 4.7

As a further example, consider the cylinder $x^2 + y^2 = 1$. This is a cylinder of radius 1, concentric with the z axis. Every point outside the cylinder has coordinates (x,y,z) satisfying the inequality $x^2 + y^2 > 1$ (z arbitrary), and the set of all such points is a domain that *is not* simply connected. The set of points in the interior of the cylinder, $x^2 + y^2 < 1$, is simply connected.

Vector fields defined in simply connected regions have much simpler properties, in general, than those having domains of definition that are not simply connected. Domains that are not simply connected may be very complicated; the reader may wish to contemplate the region of space within an old-fashioned steam radiator, which is very far indeed from being simply connected.

In this chapter we shall have occasion to refer to a *star-shaped domain.* A domain is called star-shaped if there is a point P in the domain such that, if Q is any other point in the domain, then the entire line segment PQ lies in the domain. Sometimes we say the domain is star-shaped *with respect to P.* Fig. 4.7 illustrates some star-shaped domains.

A star-shaped domain is simply connected; indeed, any curve can be shrunk to the point P.

Exercises

In each of the following cases, a region D is defined. Tell whether the region is a domain. If it is a domain, determine whether or not it is simply connected. If it is not a domain, explain why not.

1. The region of definition of a magnetic field due to a steady current flowing along the z axis [in other words, the region consisting of all points (x,y,z) such that $x^2 + y^2 > 0$].

2. The region of definition of an electric field due to n point charges.

3. The region consisting of all points above the xy plane (i.e., all points (x,y,z) such that $z > 0$).

4. The region D consisting of all points (x,y,z) for which $z \geq 0$.

5. The region D consisting of all points (x,y,z) such that

$$x^2 + y^2 + z^2 > 4$$

6. The region D consisting of all points (x,y,z) for which
$$1 < x^2 + y^2 < 4$$
(i.e., all points outside a cylinder of radius 1 and within a cylinder of radius 2, both cylinders concentric with the z axis).
7. The region D consisting of all points (x,y,z) for which $1 < x < 2$ (i.e., all points between the planes $x = 1$ and $x = 2$).
8. The region D consisting of all points (x,y,z) for which $z \neq 0$.

4.3 CONSERVATIVE FIELDS

In this section we let $\mathbf{F}$ denote a vector field that is defined and continuous throughout a domain D. Then

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \tag{4.7}$$

where F_1, F_2, and F_3 are scalar-valued functions, each of which is continuous throughout D. If these three functions have partial derivatives (there will be nine such derivatives, $\partial F_1/\partial x$, $\partial F_1/\partial y, \ldots, \partial F_3/\partial z$) all of which are continuous throughout D, then $\mathbf{F}$ is said to be *continuously differentiable* in D. It follows from these definitions that, if $\mathbf{F}$ is continuously differentiable in D, then **curl** $\mathbf{F}$ is a vector field that is continuous in D, and div $\mathbf{F}$ is a scalar field that is continuous in D.

A vector field $\mathbf{F}$ is said to be *conservative* in a domain D if there can be found some scalar field ϕ defined in D such that $\mathbf{F} = \mathbf{grad}\ \phi$. If this is possible, then ϕ is called a *potential function* or simply a *potential* for $\mathbf{F}$.

Notice that the potential function for a conservative field is not unique, since one can always add an arbitrary constant to ϕ to obtain a new potential whose gradient is also $\mathbf{F}$. (Physicists conventionally choose potentials to satisfy certain natural boundary conditions; for instance, they may choose the constant so that the potential function for a gravitational field is zero along the laboratory floor, or so that the potential function for an electric field tends to zero at infinity.)

In some physical applications, a different definition of potential is used, so that one has $\mathbf{F} = -\mathbf{grad}\ \phi$ instead of $\mathbf{F} = \mathbf{grad}\ \phi$. The difference is one of sign, and this will give the student no difficulty when he is thoroughly familiar with the basic ideas involved.

The following theorem may indicate why conservative fields are so important:

THEOREM 4.1 *A vector field* $\mathbf{F}$ *continuous in a domain D is conservative if and only if the line integral of the tangential component of* $\mathbf{F}$ *along every regular curve in D depends only on the endpoints of the curve. In that case, the line integral is simply the difference in potential of the* *endpoints. That is, we have*

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(Q) - \phi(P)$$

where P and Q are initial and terminal points of C, respectively.

Before we continue, let us be sure we understand this theorem. We are given a vector field **F** defined and continuous in a domain D. The theorem says this field is conservative if and only if the following condition holds: that if we are given any two points P and Q in D, and any regular curve C within the domain extending from P to Q, then

$$\int_P^Q \mathbf{F} \cdot d\mathbf{R}$$

depends only on the location of the endpoints P and Q and not in any way on the choice of the curve C that joins them. (We summarize this condition by saying "the line integral is independent of path.") Moreover, if this condition holds, then we can *evaluate* this line integral by first finding a function ϕ such that $\mathbf{F} = \mathbf{grad}\ \phi$, and then subtracting the value of ϕ at P from its value at Q.

This is the first theorem of any depth that has been stated in this book. We strongly urge the student to study the following outline of the proof.

Proof The phrase "if and only if" requires that we prove the implication in both directions. We break the proof up into four steps. First, we assume that the line integral of **F** depends only on the endpoints, and (*i*) define a function ϕ in a certain manner, (*ii*) show that ϕ is a potential for **F**, and (*iii*) show that

$$\int_P^Q \mathbf{F} \cdot d\mathbf{R} = \phi(Q) - \phi(P)$$

Finally, we complete the argument by proving the converse; (iv) assuming that **F** is conservative, we show that the line integral is given by $\phi(Q) - \phi(P)$ and hence is independent of path. Here we go:

(*i*) *Definition of the function*

We choose, once and for all, an arbitrary point (x_0, y_0, z_0) in D, which we call the "point of zero potential." Given any other point (x, y, z) in D, we choose some smooth arc C_1 in D extending from (x_0, y_0, z_0) to (x, y, z); this is possible since we assume D is a domain. We define $\phi(x, y, z)$ to be

$$\phi(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{R}$$

where we integrate along C_1. By hypothesis, this integral is independent of path, and so this definition of $\phi(x, y, z)$ does not depend on the particular arc C_1 that we choose. In other words, we have defined ϕ in an unambiguous manner.

(*ii*) *Proof that* $\mathbf{F} = \nabla\phi$

We begin by computing $\partial\phi/\partial x$ at (x,y,z). By definition this is

$$\lim_{\Delta x\to 0} \frac{\phi(x+\Delta x, y, z) - \phi(x,y,z)}{\Delta x} \tag{4.8}$$

Since D is open (every domain is) there is some ε neighborhood of (x,y,z) that is within D. Let us consider a line segment, parallel to the x axis and passing through (x,y,z), that is within this ε neighborhood. For a point $(x+\Delta x, y, z)$ along this line segment, let C_2 denote that part of the segment extending from (x,y,z) to $(x+\Delta x, y, z)$. Then C_2, being a line segment, is *a fortiori* a smooth arc, and the path from (x_0, y_0, z_0) to $(x+\Delta x, y, z)$ obtained by joining C_2 to C_1 consists of two smooth arcs and is therefore a regular curve (Fig. 4.8). We integrate along this curve to find $\phi(x+\Delta x, y, z)$ by first integrating along C_1 and then along C_2: since the first integral gives $\phi(x,y,z)$, we have

$$\phi(x+\Delta x, y, z) = \phi(x,y,z) + \int_{(x,y,z)}^{(x+\Delta x,\, y,\, z)} \mathbf{F}\cdot d\mathbf{R}$$

from which it follows that the numerator of (4.8) is simply the integral

$$\int_{(x,y,z)}^{(x+\Delta x,\, y,\, z)} \mathbf{F}\cdot d\mathbf{R}$$

taken along C_2. Since y and z are constant along this line segment, we have $d\mathbf{R} = dx\,\mathbf{i}$, and hence $\mathbf{F}\cdot d\mathbf{R} = F_1\,dx$. Thus (4.8) becomes

$$\lim_{\Delta x\to 0} \frac{\int_{(x,y,z)}^{(x+\Delta x,\, y,\, z)} F_1\,dx}{\Delta x} \tag{4.9}$$

Only one variable is involved in (4.9) since y and z are constant along C_2; in other words, one can treat the numerator just like any integral one meets in elementary calculus. The reader will recognize this inte-

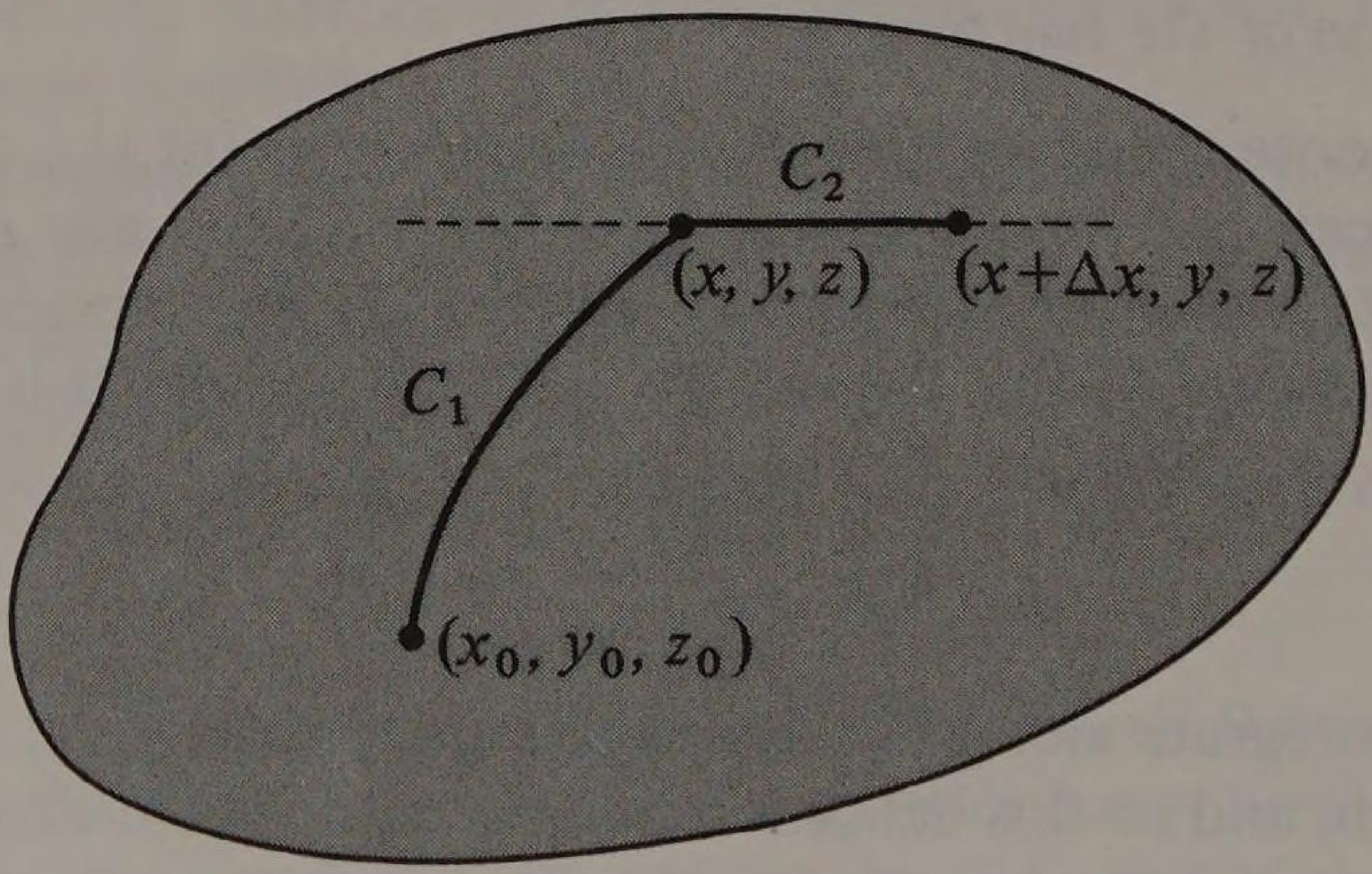

FIGURE 4.8

gral, divided by Δx, as simply the average value of F_1 along the line segment C_2. Since F_1 is continuous, this average value tends to $F_1(x, y, z)$ as Δx tends to zero. (This is a consequence of the fundamental theorem of calculus.) It follows that, at any point (x, y, z), we have $\partial\phi/\partial x = F_1$.

Similarly, we can show (taking line segments parallel to the y and z axes respectively) that $\partial\phi/\partial y = F_2$ and $\partial\phi/\partial z = F_3$. Therefore

$$\mathbf{grad}\ \phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \mathbf{F},$$

proving that ϕ is a potential function for $\mathbf{F}$.

(*iii*) *Proof that* $\int_P^Q \mathbf{F}\cdot d\mathbf{R} = \phi(Q) - \phi(P)$

Let P and Q be two distinct points in D, and let C denote any regular curve extending from P to Q. Let C_1 be a smooth arc extending from (x_0, y_0, z_0) to P. Since the integral is independent of path, $\phi(Q)$ must equal the integral taken along the regular curve obtained by attaching C_1 and C together. Thus

$$\begin{aligned}\phi(Q) &= \int_{C_1} \mathbf{F}\cdot d\mathbf{R} + \int_C \mathbf{F}\cdot d\mathbf{R} \\ &= \phi(P) + \int_C \mathbf{F}\cdot d\mathbf{R}\end{aligned}$$

from which it follows that

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(Q) - \phi(P)$$

(*iv*) *The converse*

To prove the converse, we assume $\mathbf{F}$ to be conservative, i.e. that there exists ϕ such that $\mathbf{F} = \mathbf{grad}\ \phi$. Then along any smooth arc we have $\mathbf{F}$ and $d\mathbf{R}$ expressed in terms of some parameter t and its differential dt.

$$\begin{aligned}\int_P^Q \mathbf{F}\cdot d\mathbf{R} &= \int_P^Q \left[\frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz\right] \\ &= \int_P^Q \left(\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\right)dt \\ &= \int_P^Q \frac{d\phi}{dt}\,dt = \phi(Q) - \phi(P)\end{aligned}$$

Here we made use of the fact that, if ϕ is a function having continuous partial derivatives with respect to x, y, and z, where x, y, and z are differentiable functions of a single parameter t, then

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$$

The above equations may be written in simplified notation:

$$\int_P^Q \mathbf{F}\cdot d\mathbf{R} = \int_P^Q d\phi = \phi(Q) - \phi(P)$$

where

$$\mathbf{F}\cdot d\mathbf{R} = d\phi = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz$$

is the total differential of ϕ.

This completes the proof. It will be noticed that if the path C is *closed*, i.e., if P and Q coincide, then

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = 0$$

since $\phi(P) - \phi(P) = 0$. Conversely, if

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = 0$$

around every regular closed curve in the domain, then $\mathbf{F}$ must be conservative (see Exercise 1 for the proof).

THEOREM 4.2 *A vector field* **F** *continuous in a domain D is conservative if, and only if, around every regular closed curve in D the line integral of the tangential component of* **F** *is zero.*

Example 4.5 Show that $\mathbf{F} = xy^2\mathbf{i} + x^3y\mathbf{j}$ is not conservative.

Solution A quick way of solving such problems will be given in the next example. However, we can prove that a field is not conservative by showing that its line integral does depend on the path. In this case, for instance, let us compute the integral along two paths joining (0,0) to (1,1) in the xy plane (Fig. 4.9). Along the line $y = x$ we have

$$\int_{(0,0)}^{(1,1)} (xy^2\,dx + x^3y\,dy) = \int_{x=0}^{x=1} (x^3 + x^4)\,dx = \frac{9}{20}$$

Now let us move along the regular path consisting of two line segments, the first joining (0,0) to (1,0) and the second joining (1,0) to (1,1). Along the first line segment $y = 0$, so that the line integral is zero. Along the second line segment $x = 1$, so that $dx = 0$ and the integral becomes

$$\int_{y=0}^{y=1} y\,dy = \frac{1}{2}$$

The total of the two integrals is thus $\frac{1}{2}$, differing from $\frac{9}{20}$. Hence the field is not conservative.

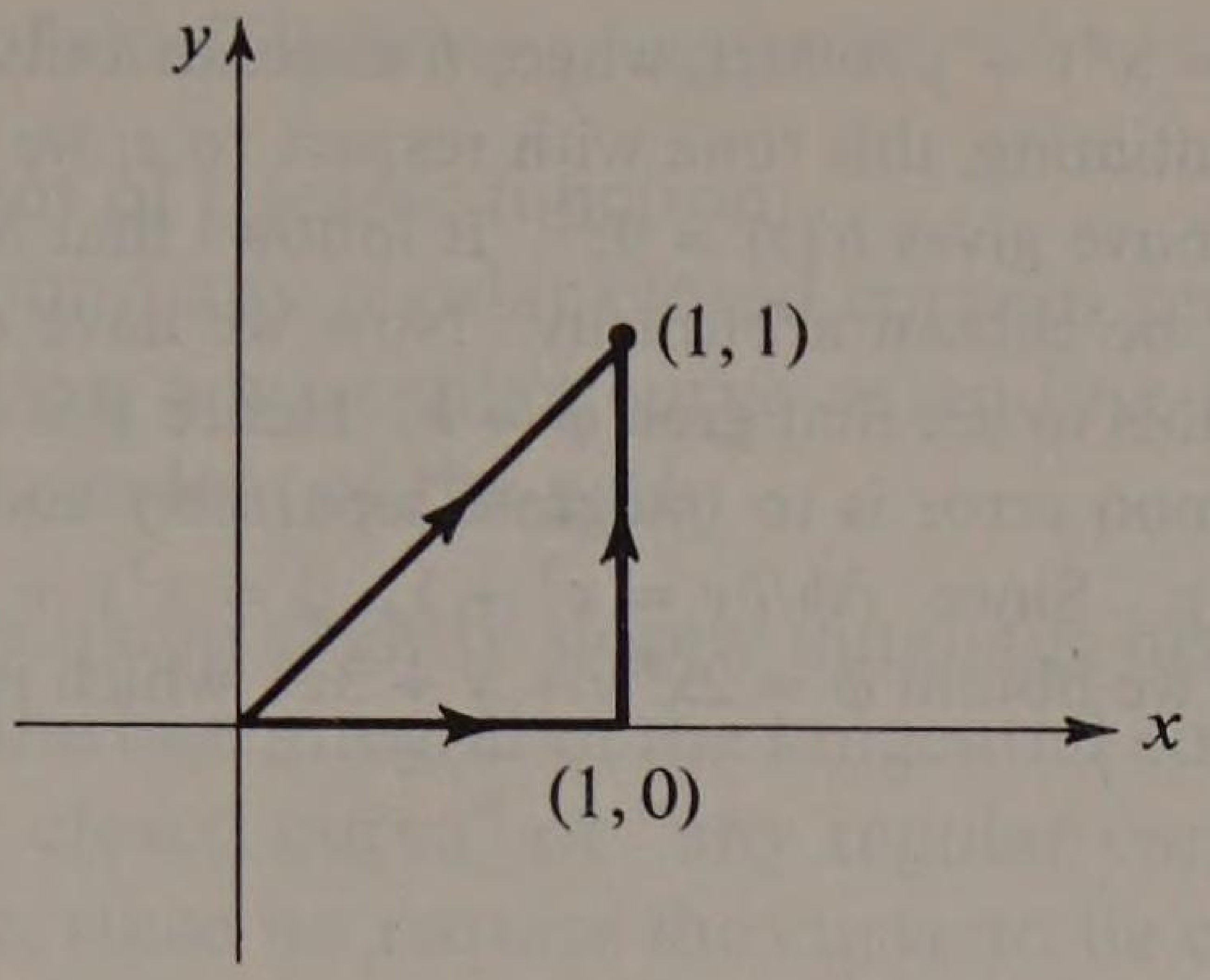

FIGURE 4.9

It is important to notice that if these two line integrals had turned out to be equal, we would not have been able to draw any conclusions from that alone. Such a result could have happened by coincidence even though the field **F** was not conservative. Since it is obviously impossible to compute $\int \mathbf{F} \cdot d\mathbf{R}$ along every conceivable regular curve, the theorem does not provide a practical way of showing that a given field is conservative.

Example 4.6 Show that $\mathbf{F} = xy^2\mathbf{i} + x^3y\mathbf{j}$ is not conservative, without computing any integrals.

Solution This can be done by contradiction. Suppose **F** were conservative. Then $\mathbf{F} = \mathbf{grad}\ \phi$ for some function ϕ. Since

$$\mathbf{grad}\ \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

we must have $\partial\phi/\partial x = xy^2$ and $\partial\phi/\partial y = x^3y$. But this is impossible, since the mixed derivatives $\partial^2\phi/\partial y\,\partial x$ and $\partial^2\phi/\partial x\,\partial y$ would be $2xy$ and $3x^2y$ respectively, whereas the theory of partial differentiation requires these derivatives to be equal. This contradiction shows that such a function ϕ cannot exist, and so **F** is not conservative.

Example 4.7 Show that $\mathbf{F} = 3x^2y\mathbf{i} + (x^3 + 1)\mathbf{j} + 9z^2\mathbf{k}$ is conservative.

Solution Again, a routine way of solving such problems will be given later. At this point, we have no alternative but to try to find a function ϕ such that $\mathbf{F} = \mathbf{grad}\ \phi$. As we have remarked already, the theorems of this section are not useful in proving that a field *is* conservative since we would have to compute an infinite number of integrals. (If we were to take two points and compute line integrals along a dozen or so paths joining these points, the equality of these numbers might lead us to suspect the field to be conservative, but the experiment would not provide a rigorous proof.)

If $\mathbf{F} = \mathbf{grad}\ \phi$, then $\partial\phi/\partial x = 3x^2y$, $\partial\phi/\partial y = x^3 + 1$, and $\partial\phi/\partial z = 9z^2$. In computing $\partial\phi/\partial x$ one differentiates while holding y and z constant, and so evidently

$$\phi = x^3y + (\text{either a constant term or a term involving only } y \text{ and } z)$$

Let us write this as $\phi = x^3y + g(y,z)$, where g is a function not yet determined. Differentiating, we have $\partial\phi/\partial y = x^3 + (\partial g/\partial y)$. Comparing this with $\partial\phi/\partial y$ above, we see that $\partial g/\partial y = 1$. Since g is a function of y and z, evidently

$$g(x,y) = y + (\text{either a constant term or a term involving } z \text{ alone})$$

Therefore, we have $\phi = x^3y + y + h(z)$, where h depends only on z (or may possibly be a constant). Differentiating, this time with respect to z, we have $\partial\phi/\partial z = h'(z)$, and comparison with the above gives $h'(z) = 9z^2$. It follows that $h(z) = 3z^3 + C$, where C is a constant that may be chosen arbitrarily. Now we have $\phi = x^3y + y + 3z^3 + C$, and it is easy to check this to see that **grad** ϕ = **F**. Hence **F** is conservative.

Remark: A common error is to integrate separately and add the results. Since $\partial\phi/\partial x = 3x^2y$, $\phi = x^3y$. Since $\partial\phi/\partial y = x^3 + 1$, $\phi = x^3y + y$. Since $\partial\phi/\partial z = 9z^2$, $\phi = 3z^3$. Adding these we obtain $\phi = 2x^3y + y + 3z^3$, which is incorrect.

Exercises

1. Show that, if $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$ for every regular closed curve C, then for any two points P and Q,

$$\int_P^Q \mathbf{F} \cdot d\mathbf{R}$$

is independent of path. (*Hint*: Let C_1 and C_2 be two paths extending from P to Q, and construct a closed curve out of these.)

2. Using the method of Example 4.5, or some similar method, show that the following fields are not conservative:

(a) $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$

(b) $\mathbf{F} = y\mathbf{i} + y(x-1)\mathbf{j}$

(c) $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$ [*Suggestion*: Consider two different paths extending from (0,0,0) to (1,1,1).]

(d) $\mathbf{F} = z\mathbf{i} + z\mathbf{j} + (y-1)\mathbf{k}$

(e) $\mathbf{F} = \dfrac{x\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$ (not defined at the origin)

3. Using methods similar to that of Example 4.6, show that the fields of Exercise 2 are not conservative.

4. Compute $\oint \mathbf{F} \cdot d\mathbf{R}$ around the closed path consisting of a circle of radius r, centered at the origin, in the xy plane, taking $\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)$. (*Hint*: Change to polar coordinates.)

5. If you worked correctly, you obtained a nonzero answer to Exercise 4. Yet it appears that $\mathbf{F} = \mathbf{grad}\,\phi$ where $\phi = \tan^{-1}(y/x)$, and this would contradict Theorem 4.2. Investigate this mystery.

6. Find a potential for the force field

$$\mathbf{F} = (y + z\cos xz)\mathbf{i} + x\mathbf{j} + (x\cos xz)\mathbf{k}$$

7. Show that the field $\mathbf{F} = 2xy\mathbf{i} + (x^2 + z)\mathbf{j} + y\mathbf{k}$ is conservative.

4.4 CONSERVATIVE FIELDS (CONTINUED)

In the preceding section, we saw that a continuously differentiable vector field **F** defined in a domain D is conservative if, and only if, it possesses any one (and hence all) of the following properties:

(*i*) It is the gradient of a scalar function.
(*ii*) Its integral around any regular closed curve is zero.
(*iii*) Its integral along any regular curve extending from a point P to a point Q is independent of the path.

Note that we are using slightly sloppy language here. When we say "its integral" we mean "the line integral of the tangential component," and when we say "any regular closed curve" or "any regular curve" we really do not mean *any* such curve, since we require the curve to lie completely within the domain D.

If the domain D in which $\mathbf{F}$ is defined is *simply-connected*, we can add a fourth property, equivalent to any one of the other three:

$$\mathbf{curl\ F} = \mathbf{0} \tag{4.10}$$

This is of practical utility since, if we are given a vector field $\mathbf{F}$ defined in a simply-connected domain D, we can quickly test to determine whether it is conservative by computing its curl. In terms of components, the test to determine whether

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$$

is conservative consists of checking to see whether *all* the following equations are valid:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \tag{4.11}$$

Equations (4.11) will be valid if and only if $\mathbf{curl\ F} = \mathbf{0}$, as one sees easily from the definition of $\mathbf{curl\ F}$.

Some of the problems of the preceding section may be solved quite easily by using this test. For instance, consider the vector field $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$. Equations (4.11) written out are

$$\frac{\partial}{\partial y}(y) = \frac{\partial}{\partial x}(x) \qquad \frac{\partial}{\partial z}(x) = \frac{\partial}{\partial y}(x^2) \qquad \frac{\partial}{\partial z}(y) = \frac{\partial}{\partial x}(x^2)$$

The first two of these equations are valid but the third is not, and so the vector field is not conservative.

A vector field whose curl vanishes everywhere is said to be *irrotational*. Now let us turn to the theorem that justifies our claim.

THEOREM 4.3 A vector field $\mathbf{F}$ defined and continuously differentiable in a simply connected domain D is conservative if, and only if, $\mathbf{curl\ F} = \mathbf{0}$ throughout D.

Notice that the "only if" part of this theorem is trivial because the curl of a gradient is always zero. The "if" part, however, is harder. We shall give a proof here for the special case when D is all of space.

In Exercise 4 the reader is invited to extend the argument to spherical and rectangular domains. At the end of the section, a proof for star-shaped domains is given, as optional reading. These restricted forms suffice for most practical applications. The extension of the result to arbitrary, simply connected domains requires some topological gymnastics, and we shall not go into this.

Proof of "if" part when D is all space We are given the information that **curl F** = **0** everywhere, and our goal is to show that a scalar field ϕ exists such that **F** = **grad** ϕ. Thinking ahead, we know that the line integrals of **F** can be computed in terms of ϕ, thus, let us *define* ϕ by a line integral and then try to prove the theorem.

Specifically, we define $\phi(x,y,z)$ to be the line integral of **F** from (0,0,0) to (x,y,z) along the following curve:

(*i*) from (0,0,0) to $(x,0,0)$ along the x-axis,
(*ii*) from $(x,0,0)$ to $(x,y,0)$ parallel to the y-axis,
(*iii*) from $(x,y,0)$ to (x,y,z) parallel to the z-axis.

The parametrization is trivial and we have

$$\phi(x,y,z) = \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt \qquad (4.12)$$

At this point we cannot assume that line integrals are independent of path. The function $\phi(x,y,z)$, however, is computed in terms of a *specific* path; it therefore is well-defined.

Now we proceed to show **grad** ϕ = **F**, by components. The z-component is easy:

$$\frac{\partial\phi}{\partial z} = \lim_{\Delta z\to 0} \frac{\phi(x,y,z+\Delta z) - \phi(x,y,z)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\int_z^{z+\Delta z} F_3(x,y,t)\,dt}{\Delta z} = F_3(x,y,z)$$

reasoning as in Eq. (4.9).

For the y-component we have

$$\frac{\partial\phi}{\partial y} = \lim_{\Delta y\to 0} \frac{\phi(x,y+\Delta y,z) - \phi(x,y,z)}{\Delta y}$$

$$= \lim_{\Delta y\to 0} \frac{1}{\Delta y}\left[\int_0^{y+\Delta y} F_2(x,t,0)\,dt + \int_0^z F_3(x,y+\Delta y,t)\,dt - \int_0^y F_2(x,t,0)\,dt - \int_0^z F_3(x,y,t)\,dt\right]$$

$$= \lim_{\Delta y\to 0} \frac{\int_y^{y+\Delta y} F_2(x,t,0)\,dt}{\Delta y} + \lim_{\Delta y\to 0} \int_0^z \frac{F_3(x,y+\Delta y,t) - F_3(x,y,t)}{\Delta y}\,dt$$

$$= F_2(x,y,0) + \int_0^z \frac{\partial F_3(x,y,t)}{\partial y}\,dt$$

Now we use the second of Eqs. (4.11), keeping in mind that in the above formula the third coordinate is named t, not z:

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= F_2(x,y,0) + \int_0^z \frac{\partial F_2(x,y,t)}{\partial t}\,dt \\ &= F_2(x,y,0) + F_2(x,y,z) - F_2(x,y,0) \\ &= F_2(x,y,z)\end{aligned}$$

We sketch the computation for the x-component, allowing the reader to supply the details:

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= F_1(x,0,0) + \int_0^y \frac{\partial F_2(x,t,0)}{\partial x}\,dt + \int_0^z \frac{\partial F_3(x,y,t)}{\partial x}\,dt \\ &= F_1(x,0,0) + \int_0^y \frac{\partial F_1(x,t,0)}{\partial t}\,dt + \int_0^z \frac{\partial F_1(x,y,t)}{\partial t}\,dt \\ &= F_1(x,0,0) + F_1(x,y,0) - F_1(x,0,0) + F_1(x,y,z) - F_1(x,y,0) \\ &= F_1(x,y,z)\end{aligned}$$

This completes the proof that **grad** $\phi = \mathbf{F}$.

Example 4.8 Show that $\mathbf{F} = 2xy\mathbf{i} + (x^2 + 1)\mathbf{j} + 6z^2\mathbf{k}$ is conservative, and find a scalar potential ϕ.

Solution We use the test (4.11), which is acceptable since this field $\mathbf{F}$ is defined and continuously differentiable throughout space.

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = 2x$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} = 0$$

and

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} = 0$$

The curl is zero, hence the field is conservative.

The potential may be found by the method of Example 4.7, or we may use the line integral of Eq. (4.12). Let us try the latter technique.

Inserting the given expression for $\mathbf{F}$, we find

$$\begin{aligned}\phi(x,y,z) &= \int_0^x 0\,dt + \int_0^y (x^2+1)\,dt + \int_0^z 6t^2\,dt \\ &= 0 + (x^2t + t)\Big|_0^y + 2t^3\Big|_0^z = x^2y + y + 2z^3\end{aligned}$$

As a check, let us compute ϕ by integrating $\mathbf{F}$ along the straight line segment from (0,0,0) to (x,y,z), parametrized by

$$\mathbf{R}(t) = tx\mathbf{i} + ty\mathbf{j} + tz\mathbf{k} \qquad 0 \le t \le 1$$

We find

$$\begin{aligned}\phi(x,y,z) &= \int_0^1 [2(tx)(ty)\mathbf{i} + (t^2x^2+1)\mathbf{j} + 6(t^2z^2)\mathbf{k}]\cdot[x\mathbf{i}+y\mathbf{j}+z\mathbf{k}]\,dt \\ &= 2x^2y\int_0^1 t^2\,dt + x^2y\int_0^1 t^2\,dt + y\int_0^1 dt + 6z^3\int_0^1 t^2\,dt \\ &= x^2y + y + 2z^3\end{aligned}$$

as before.

Example 4.9 Use Eq. (4.12) to find a potential for

$$\mathbf{F} = (3x^2yz + y + 5)\mathbf{i} + (x^3z + x - z)\mathbf{j} + (x^3y - y + 7)\mathbf{k}$$

which has the value 10 at the origin.

Solution By (4.12) we have

$$\begin{aligned}\phi(x,y,z) &= 10 + \int_0^x 5\,dt + \int_0^y x\,dt + \int_0^z (x^3y - y + 7)\,dt \\ &= 10 + 5t\Big|_0^x + xt\Big|_0^y + (x^3yt - yt + 7t)\Big|_0^z \\ &= 10 + 5x + xy + x^3yz - yz + 7z\end{aligned}$$

As a rule, the reader should use the *test* given in this section to determine whether a given field **F** is conservative, but use the *method* of Example 4.7 to actually construct the potential ϕ. The reason we do not advise using (4.12) of this section is that it may be a little tricky for most students to use correctly.

It is very enlightening to investigate the vector field

$$\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$$

The reader should verify that **curl F** = **0**, but **F** is not conservative since its line integrals around closed paths are not always zero, as shown in Exercise 4 of Sec. 4.3. This does not contradict the theorem, because **F** fails to be defined on the z axis, where $x^2 + y^2 = 0$, and thus the domain D is not simply connected.

In Exercise 5 of Sec. 4.3, we tried to shake the reader up by suggesting that this **F** was a gradient, namely, the gradient of $\phi = \tan^{-1}(y/x)$. This definition is subject, of course, to quadrant ambiguities, so we might try $\mathbf{F} = \mathbf{grad}\,\theta$, where θ is the polar angle defined in Sec. 2.4. However, the polar angle jumps from π to $-\pi$ as we cross the negative x axis, so it is discontinuous there and its gradient is not defined (or infinite!). Thus, the theorem remains unchallenged.

OPTIONAL READING: PROOF OF THEOREM 4.3 FOR STAR-SHAPED DOMAINS

As before, we only have to prove the "if" part. Assuming D is star-shaped with respect to the point P, we can *define* ϕ at the point Q to be the line integral of **F** taken along *the straight-line segment from P to Q*. Again notice that the function $\phi(Q)$ is computed *in terms of a specific path*; therefore, it is not ambiguous.

Now we proceed to show **grad** $\phi = \mathbf{F}$. First we parametrize the path of integration. Let (x_0, y_0, z_0) be the coordinates of P, and (x, y, z) be coordinates for Q. Since we customarily use **R** to denote the vector (x, y, z), which now is an *endpoint* in our integral, we shall write

$$\phi(x,y,z) = \int_P^Q \mathbf{F} \cdot d\mathbf{r} \tag{4.13}$$

using $\mathbf{r}(t)$ to designate the path of integration. Then a parametrization for the segment is

$$\begin{aligned}\mathbf{r}(t) &= [x_0 + t(x - x_0)]\mathbf{i} + [y_0 + t(y - y_0)]\mathbf{j} + [z_0 + t(z - z_0)]\mathbf{k} \\ &= \mathbf{R}_0 + t(\mathbf{R} - \mathbf{R}_0) \qquad (0 \le t \le 1)\end{aligned} \tag{4.14}$$

The explicit dependence of **F** on the parameter t in the integral (4.13) is given by

$$\begin{aligned}\mathbf{F} &= \mathbf{F}[x_0 + t(x - x_0), y_0 + t(y - y_0), z_0 + t(z - z_0)] \\ &= \mathbf{F}(X,Y,Z)\end{aligned}$$

where we have abbreviated the first argument, $x_0 + t(x - x_0)$, of **F** by X, the second by Y, and the third by Z.

Now we compute the gradient of ϕ. Since ∇ operates only on the variables x, y, and z

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

(and *not* on t), we can bring the differential operator inside the integral. Using identity (3.31), we find

$$\begin{aligned}\nabla\phi &= \int_0^1 \nabla\left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) dt \\ &= \int_0^1 \left[(\mathbf{F} \cdot \nabla)\frac{d\mathbf{r}}{dt} + \left(\frac{d\mathbf{r}}{dt} \cdot \nabla\right)\mathbf{F} + \mathbf{F} \times \left(\nabla \times \frac{d\mathbf{r}}{dt}\right) + \frac{d\mathbf{r}}{dt} \times (\nabla \times \mathbf{F})\right] dt\end{aligned} \tag{4.15}$$

Here we must be very careful in our interpretation. ∇ operates, as we said, on x, y, and z; but the arguments of **F** are X, Y, and Z. Thus we cannot identify, for example, $\nabla \times \mathbf{F}(X,Y,Z)$ as the curl of **F**, evaluated at (X,Y,Z). This latter would be

$$\nabla^* \times \mathbf{F}(X,Y,Z)$$

where ∇^* denotes the operator

$$\nabla^* = \mathbf{i}\frac{\partial}{\partial X} + \mathbf{j}\frac{\partial}{\partial Y} + \mathbf{k}\frac{\partial}{\partial Z}$$

However we have the following relation, because of the definition of X in terms of x;

$$\frac{\partial \mathbf{F}}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial \mathbf{F}}{\partial X} = t\frac{\partial \mathbf{F}}{\partial X} \tag{4.16}$$

Similarly,

$$\begin{aligned} \frac{\partial \mathbf{F}}{\partial y} &= t\frac{\partial \mathbf{F}}{\partial Y} \\ \frac{\partial \mathbf{F}}{\partial z} &= t\frac{\partial \mathbf{F}}{\partial Z} \end{aligned} \tag{4.16$'$}$$

It follows from these identities that

$$\nabla \times \mathbf{F}(X,Y,Z) = t\nabla^* \times \mathbf{F}(X,Y,Z) = t\,\mathbf{curl}\,\mathbf{F}(X,Y,Z)$$

By hypothesis, the curl of $\mathbf{F}$ is zero; hence, from the above,

$$\nabla \times \mathbf{F} = \mathbf{0}$$

in expression (4.15).

Furthermore, since

$$\frac{d\mathbf{r}}{dt} = \mathbf{R} - \mathbf{R}_0$$

we have

$$\nabla \times \frac{d\mathbf{r}}{dt} = \mathbf{0}$$

and

$$(\mathbf{F}\cdot\nabla)\frac{d\mathbf{r}}{dt} = \mathbf{F}$$

[Recall identities (3.24) and (3.26).]

Combining this data in (4.15), we have shown

$$\nabla\phi(x,y,z) = \int_0^1 \left[\mathbf{F} + \left(\frac{d\mathbf{r}}{dt}\cdot\nabla\right)\mathbf{F}\right]dt \tag{4.17}$$

One more simplification is possible. If we differentiate $t\mathbf{F}$ with respect to t along the curve, using the chain rule we find

$$\begin{aligned} &\frac{d}{dt}(t\mathbf{F}[x_0 + t(x - x_0), y_0 + t(y - y_0), z_0 + t(z - z_0)]) \\ &\qquad = t\frac{\partial \mathbf{F}}{\partial X}(x - x_0) + t\frac{\partial \mathbf{F}}{\partial Y}(y - y_0) + t\frac{\partial \mathbf{F}}{\partial Z}(z - z_0) + \mathbf{F} \end{aligned}$$

By Eq. (4.16), this can be written

$$\frac{d(t\mathbf{F})}{dt} = (x - x_0)\frac{\partial \mathbf{F}}{\partial x} + (y - y_0)\frac{\partial \mathbf{F}}{\partial y} + (z - z_0)\frac{\partial \mathbf{F}}{\partial z} + \mathbf{F}$$

$$= \left(\frac{d\mathbf{r}}{dt}\cdot \nabla\right)\mathbf{F} + \mathbf{F} \tag{4.18}$$

Using this in (4.17), we get

$$\nabla\phi(x,y,z) = \int_0^1 \frac{d}{dt}(t\mathbf{F})\,dt = t\mathbf{F}\Big|_0^1 = \mathbf{F}(x,y,z)$$

We have succeeded in proving that the gradient of ϕ is $\mathbf{F}$, i.e., $\mathbf{F}$ is conservative!

Exercises

1. Test the following fields to determine whether they are conservative.
   (a) $\mathbf{F} = (12xy + yz)\mathbf{i} + (6x^2 + xz)\mathbf{j} + xy\mathbf{k}$
   (b) $\mathbf{F} = ze^{xz}\mathbf{i} + xe^{xz}\mathbf{k}$
   (c) $\mathbf{F} = \sin x\mathbf{i} + y^2\mathbf{j} + e^z\mathbf{k}$
   (d) $\mathbf{F} = 3x^2yz^2\mathbf{i} + x^3z^2\mathbf{j} + x^3yz\mathbf{k}$
   (e) $\mathbf{F} = \dfrac{2x}{x^2 + y^2}\mathbf{i} + \dfrac{2y}{x^2 + y^2}\mathbf{j} + 2z\mathbf{k}$

2. For which one of the fields in Exercise 1 is the test given in this section not applicable? How, then, can you test this field to determine whether it is conservative in its domain of definition?

3. Let $\mathbf{F}$ and $\mathbf{G}$ be conservative vector fields with potentials ϕ and ψ respectively. Is the vector field $\mathbf{F} + \mathbf{G}$ conservative? If so, determine a potential for it.

4. Show that the proof of Theorem 4.3 in the text can be adapted for some other domains D, specifically: (i) the interior of a sphere and (ii) the interior of a parallelepiped with edges parallel to the axes. (*Hint*: You must verify that all the line integrals are well-defined.)

5. Show that the scalar field

$$\phi = -\frac{1}{|\mathbf{R}|}$$

which is defined everywhere except at the origin, is a potential function for the vector field $\mathbf{R}/|\mathbf{R}|^3$, where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,
   (a) by writing ϕ in terms of x, y, and z and computing its gradient;
   (b) by inspection, using the second and third fundamental properties of the gradient listed in Sec. 3.1.

6. A force field is defined by

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

at all points in space except the origin. A particle is moved along the straight line segment from the point (1,2,3) to the point (2,3,5). What is the work done by the force on the particle? [*Hint*: Avoid a lot of work (!) by making use of the statement of Exercise 5.]

7. Would your answer to Exercise 6 be any different if the path extending from (1,2,3) to (2,3,5) were not straight?

4.5 OPTIONAL READING: VECTOR POTENTIALS

In the previous section we discussed a *partial converse* to identity (3.32), which states that the curl of a gradient is zero. It is a *converse* because it states that, *if* the curl of a field is zero, the field is a gradient; but it is only a *partial* converse because it is only valid in simply connected domains.

An astute reader will wonder if there is also a converse, or at least a partial converse, to identity (3.33), which asserts that the divergence of a curl is zero. If the divergence of a vector field is zero, is that field necessarily the curl of another vector field? The answer is *yes*, provided the domain of definition is star-shaped.

A vector field whose divergence is everywhere zero is called *solenoidal*. If $\mathbf{F} = \nabla \times \mathbf{G}$, $\mathbf{G}$ is called a *vector potential* for $\mathbf{F}$. Notice that $\mathbf{G}$ is not unique; in fact, according to (3.32), we can add the gradient of any scalar to $\mathbf{G}$.

Now let us prove the statement about the existence of a vector potential.

THEOREM 4.4 A vector field $\mathbf{F}$ continuously differentiable in a star-shaped domain D is solenoidal if, and only if, there is a vector field $\mathbf{G}$ such that $\mathbf{F} = \mathbf{curl}\ \mathbf{G}$ throughout D.

Proof The "if" part follows from identity (3.33). For the "only if" statement, we assume $\mathbf{F}$ is solenoidal in a domain D that is star-shaped with respect to the point P. We wish to find $\mathbf{G}(x,y,z)$ so that $\mathbf{F} = \nabla \times \mathbf{G}$.

The proof is quite similar to that in the previous section. (In fact, both theorems are special cases of a result known as *Poincaré's lemma*). We again parametrize the straight-line segment from $P(x_0,y_0,z_0)$ to $Q(x,y,z)$ by (4.14):

$$\begin{aligned}\mathbf{r}(t) &= [x_0 + t(x - x_0)]\mathbf{i} + [y_0 + t(y - y_0)]\mathbf{j} + [z_0 + t(z - z_0)]\mathbf{k} \\ &= X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} \\ &= \mathbf{R}_0 + t(\mathbf{R} - \mathbf{R}_0)\end{aligned}$$

(using the same notation as in the previous section).

Now we define $\mathbf{G}(x,y,z)$:

$$\mathbf{G}(x,y,z) = \int_0^1 t\mathbf{F} \times \frac{d\mathbf{r}}{dt}\,dt \tag{4.19}$$

where the dependence of **F** and **r** on t is exactly as in the previous section. Equation (4.19) is, as it stands, simply the integral of a vector function of t; however, it lends an obvious interpretation to an expression like $\int_P^Q t\mathbf{F} \times d\mathbf{r}$.

We compute the curl of **G**. Again, ∇ may be taken inside the integral and by identity (3.29), we have

$$\nabla \times \mathbf{G} = \int_0^1 t\nabla \times \left(\mathbf{F} \times \frac{d\mathbf{r}}{dt}\right) dt$$

$$= \int_0^1 \left[\left(\frac{d\mathbf{r}}{dt} \cdot \nabla\right)\mathbf{F} - (\mathbf{F} \cdot \nabla)\frac{d\mathbf{r}}{dt} + \left(\nabla \cdot \frac{d\mathbf{r}}{dt}\right)\mathbf{F} - (\nabla \cdot \mathbf{F})\frac{d\mathbf{r}}{dt}\right] t\,dt \tag{4.20}$$

As in the previous section,

$$\frac{d\mathbf{r}}{dt} = \mathbf{R} - \mathbf{R}_0$$

so that

$$\nabla \cdot \frac{d\mathbf{r}}{dt} = 3$$

and

$$(\mathbf{F} \cdot \nabla)\frac{d\mathbf{r}}{dt} = \mathbf{F}$$

For the reasons stated in the previous section, $\nabla \cdot \mathbf{F}(X,Y,Z)$ is not the divergence of **F**, but because

$$\nabla \cdot \mathbf{F}(X,Y,Z) = t\nabla^* \cdot \mathbf{F}(X,Y,Z)$$

with ∇^* defined as before, we see that $\nabla^* \cdot \mathbf{F} = 0$ implies $\nabla \cdot \mathbf{F} = 0$.

Incorporating all this into (4.20), we get

$$\nabla \times \mathbf{G} = \int_0^1 \left[\left(\frac{d\mathbf{r}}{dt} \cdot \nabla\right)\mathbf{F} - \mathbf{F} + 3\mathbf{F}\right] t\,dt \tag{4.21}$$

Now using Eq. (4.18), we find that

$$\frac{d(t^2\mathbf{F})}{dt} = t\frac{d(t\mathbf{F})}{dt} + (t\mathbf{F})\frac{dt}{dt} = 2t\mathbf{F} + t\left(\frac{d\mathbf{r}}{dt} \cdot \nabla\right)\mathbf{F}$$

This is precisely what we have in (4.21); therefore

$$\nabla \times \mathbf{G} = \int_0^1 \frac{d}{dt}(t^2\mathbf{F})\,dt = t^2\mathbf{F}\Big|_0^1 = \mathbf{F}(x,y,z)$$

i.e., **F** is the curl of **G**.

Example 4.10 Find a vector potential for

$$\mathbf{F} = \boldsymbol{\omega} \times \mathbf{R}$$

where $\boldsymbol{\omega}$ is a constant vector (recall Example 3.22, where **F** was identified as a fluid velocity field with uniform angular velocity).

Solution The verification that $\nabla \cdot \mathbf{F} = 0$ is immediate. Taking P to be the origin in the above equations, we have the parametrization of the segment PQ:

$$\mathbf{r}(t) = t\mathbf{R}$$

Therefore, since $\mathbf{F} = \boldsymbol{\omega} \times \mathbf{r}(t)$ in Eq. (4.19),

$$\begin{aligned} \mathbf{G}(x,y,z) &= \int_0^1 t(\boldsymbol{\omega} \times t\mathbf{R}) \times \mathbf{R}\,dt \\ &= (\boldsymbol{\omega} \times \mathbf{R}) \times \mathbf{R} \int_0^1 t^2\,dt \end{aligned}$$

So we obtain

$$\mathbf{G}(x,y,z) = \frac{1}{3}(\boldsymbol{\omega} \times \mathbf{R}) \times \mathbf{R}$$

The reader is invited to verify that $\nabla \times \mathbf{G} = \mathbf{F}$.

We have now proved results which state that irrotational vector fields are derivable from scalar potentials and solenoidal fields are derivable from vector potentials. The natural question arises: Can an *arbitrary* vector field be expressed as a gradient of a scalar field plus a curl of another vector field, under appropriate circumstances? This conjecture is true, and is known as the *fundamental theorem of vector analysis*. Its proof, however, involves the use of potential theory. In fact, the theorem has considerably less practical utility than the two results we have demonstrated, despite its ostentatious soubriquet.

Exercises

1. Verify that $\mathbf{F} = \nabla \times \mathbf{G}$ in Example 4.10.
2. Find a vector potential for $\mathbf{F} = x\mathbf{j}$.
3. Prove: if **F** and **G** are irrotational, then $\mathbf{F} \times \mathbf{G}$ is solenoidal. Can you find the vector potential for $\mathbf{F} \times \mathbf{G}$? (*Hint*: This problem is considerably easier if you have mastered tensor notation.)

4.6 ORIENTED SURFACES

In Chapter 2 we considered, in some detail, the geometry of space curves. We now turn our attention to a study of surfaces. Just as the basic properties of curves hinge on the *tangent vectors*, the behavior of a surface is characterized in terms of the *normal vectors* at each point. (Recall that we have one method for computing normals; namely, **grad** $f(x,y,z)$ is normal to the surface $f(x,y,z) = 0$. Another method will be derived below.)

Keeping in mind that a smooth arc has a continuously turning tangent, we say that *a surface S is smooth if* it is possible to choose a unit normal vector **n** at every point of S in such a way that **n** varies continuously on S. It is said to be *piecewise smooth* if it consists of a finite number of smooth parts joined together. Thus, the surface of a sphere is smooth, whereas the surface of a cube is piecewise smooth (consisting of six smooth surfaces joined together).

At every point of a smooth surface there will, of course, be two choices for the unit normal **n**. There will therefore be two ways in which we can define a field of unit normal vectors continuous on S. [If, for instance, the surface is given by an equation of the form $f(x,y,z) = C$, then the two fields are

$$\overset{\vec{n}}{\frac{\mathbf{grad}\, f}{|\mathbf{grad}\, f|}} \quad \text{and} \quad \overset{-\vec{n}}{\frac{-\mathbf{grad}\, f}{|\mathbf{grad}\, f|}}$$

(Sec. 3.1).] In choosing one of these two possibilities we *orient* the surface. Thus, there are always two possible orientations of a smooth surface. We have already discussed orientation for the special case of a plane (Sec. 1.11). The situation is somewhat the same for more general surfaces. When a smooth surface has been oriented by choosing a particular unit normal field **n**, then a positive direction for angles is determined at each point of the surface (Fig. 4.10). If the surface is bounded by a regular closed curve C, the orientation also determines what we mean by the *positive direction* along C, by the following rule: an observer on the positive side of the surface (i.e., the side on which **n** emerges), walking in the positive direction along the boundary, always has the surface at his left.

To produce an orientation on a piecewise-smooth surface, we have to orient its smooth parts *consistently*. This means that along every edge that is shared by two smooth parts the positive direction (on the edge) relative to one of the smooth surfaces is *opposite* to the positive direction relative to the

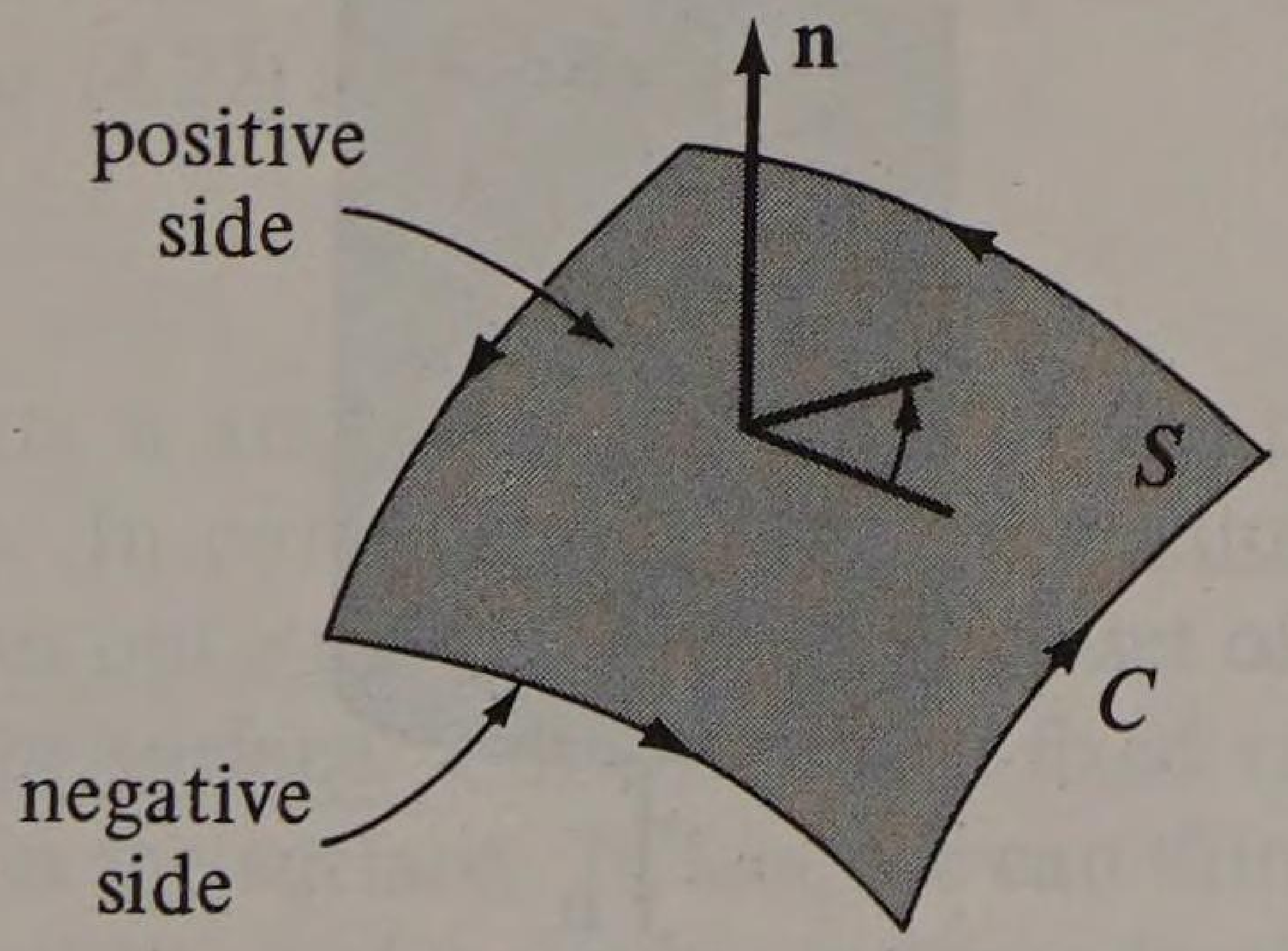

FIGURE 4.10

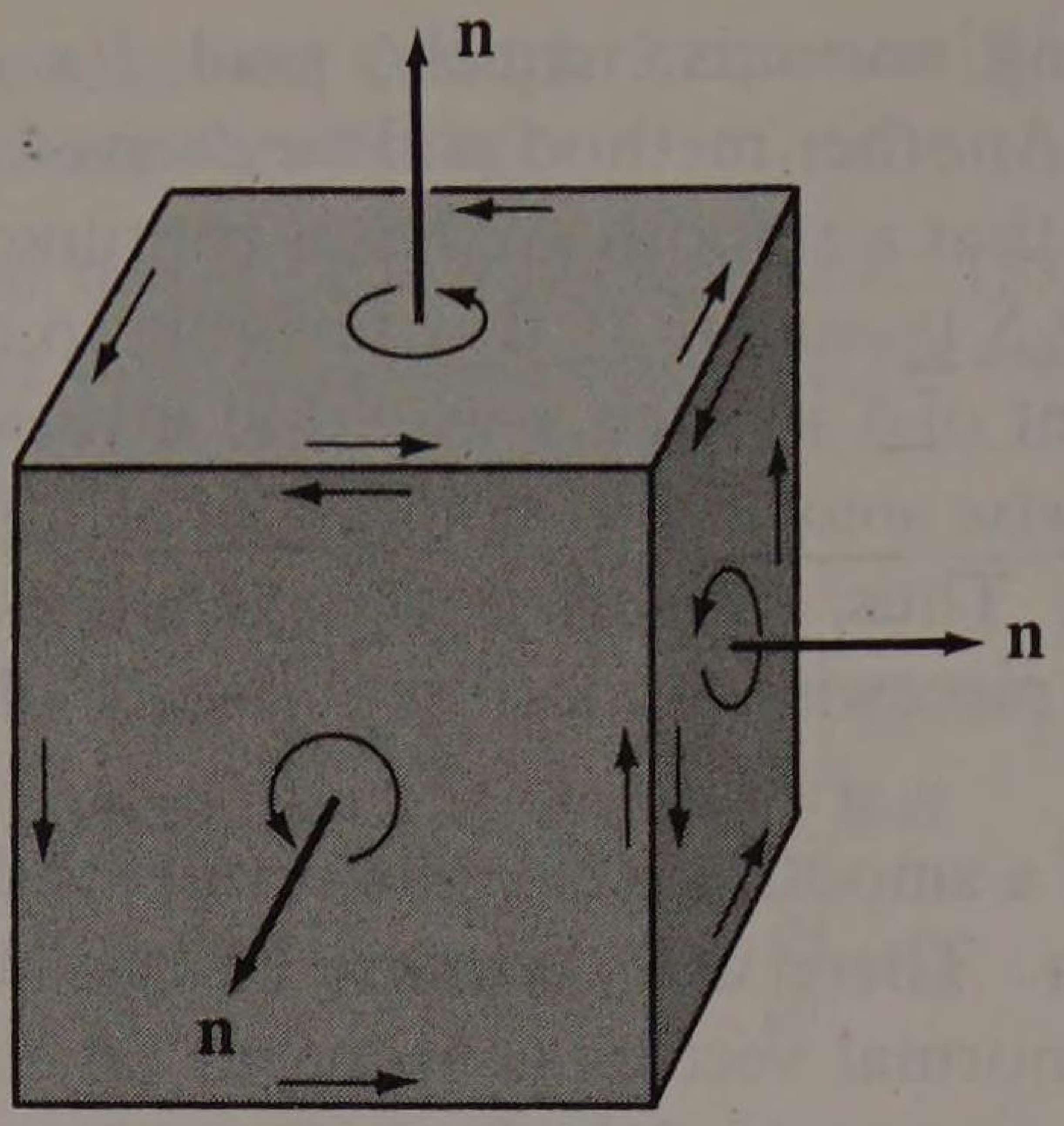

FIGURE 4.11

other. Study Figs. 4.11 and 4.12 to see why this definition is chosen. Not all piecewise-smooth surfaces can be oriented; see Exercise 1.

A *closed surface* is one that has no boundary. Thus the surfaces of Figs. 4.11 and 4.12 are closed, whereas the surface in Fig. 4.10 has a boundary and is not closed. It is conventional to take the orientation of a closed surface, which encloses a region of space, to be such that the unit normal **n** always points *away* from the enclosed region, as illustrated in Figs. 4.11 and 4.12.

A surface can be oriented only if it has two sides; the process of orientation consists essentially in choosing which side we will call "positive" and which "negative". (If the surface is closed, it is more natural to speak of the "outside" and the "inside".)

An example of a nonorientable surface is the *Möbius strip*, obtained by twisting and pasting together the ends of a strip of paper (Fig. 4.13). This

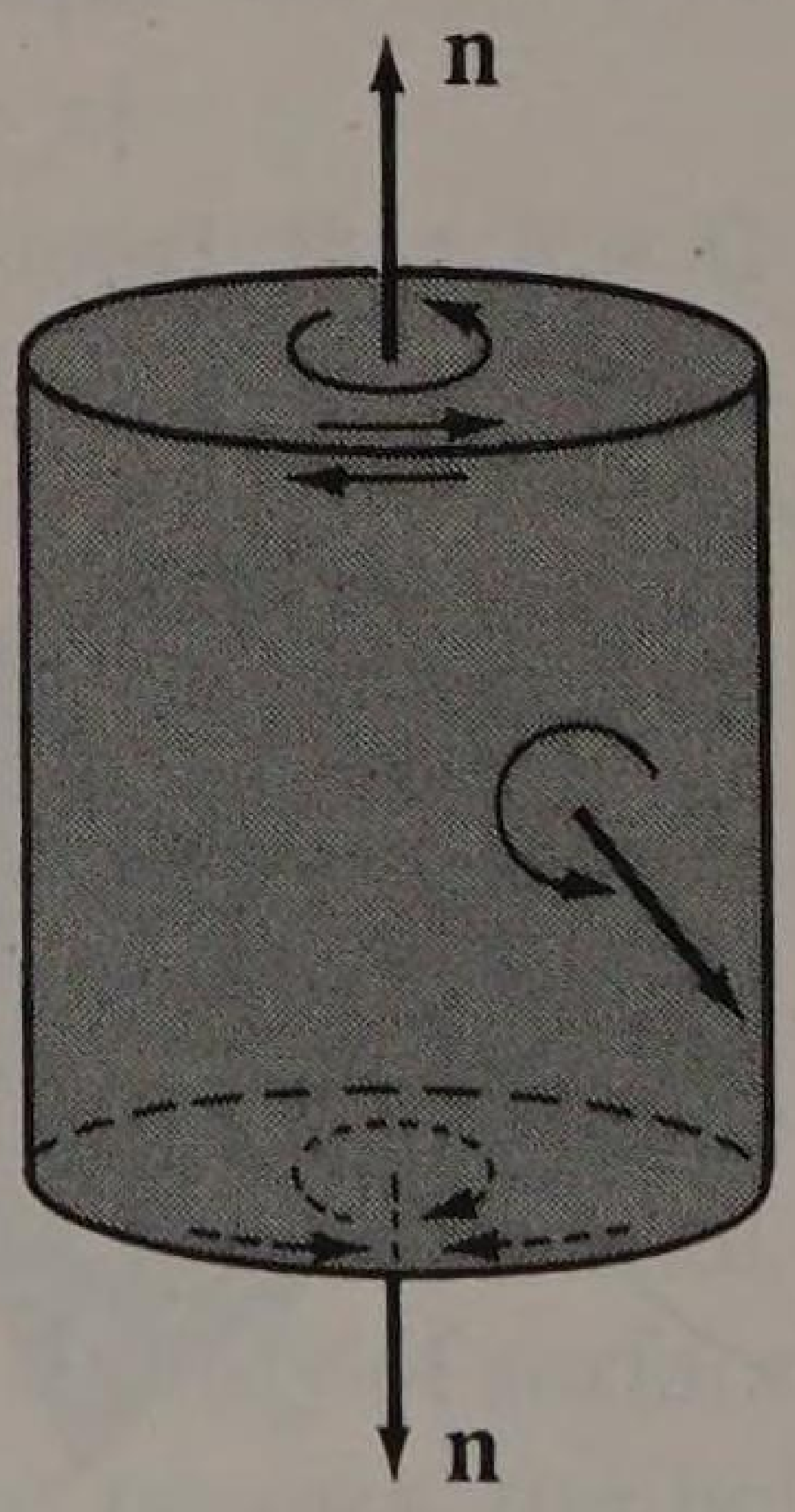

FIGURE 4.12

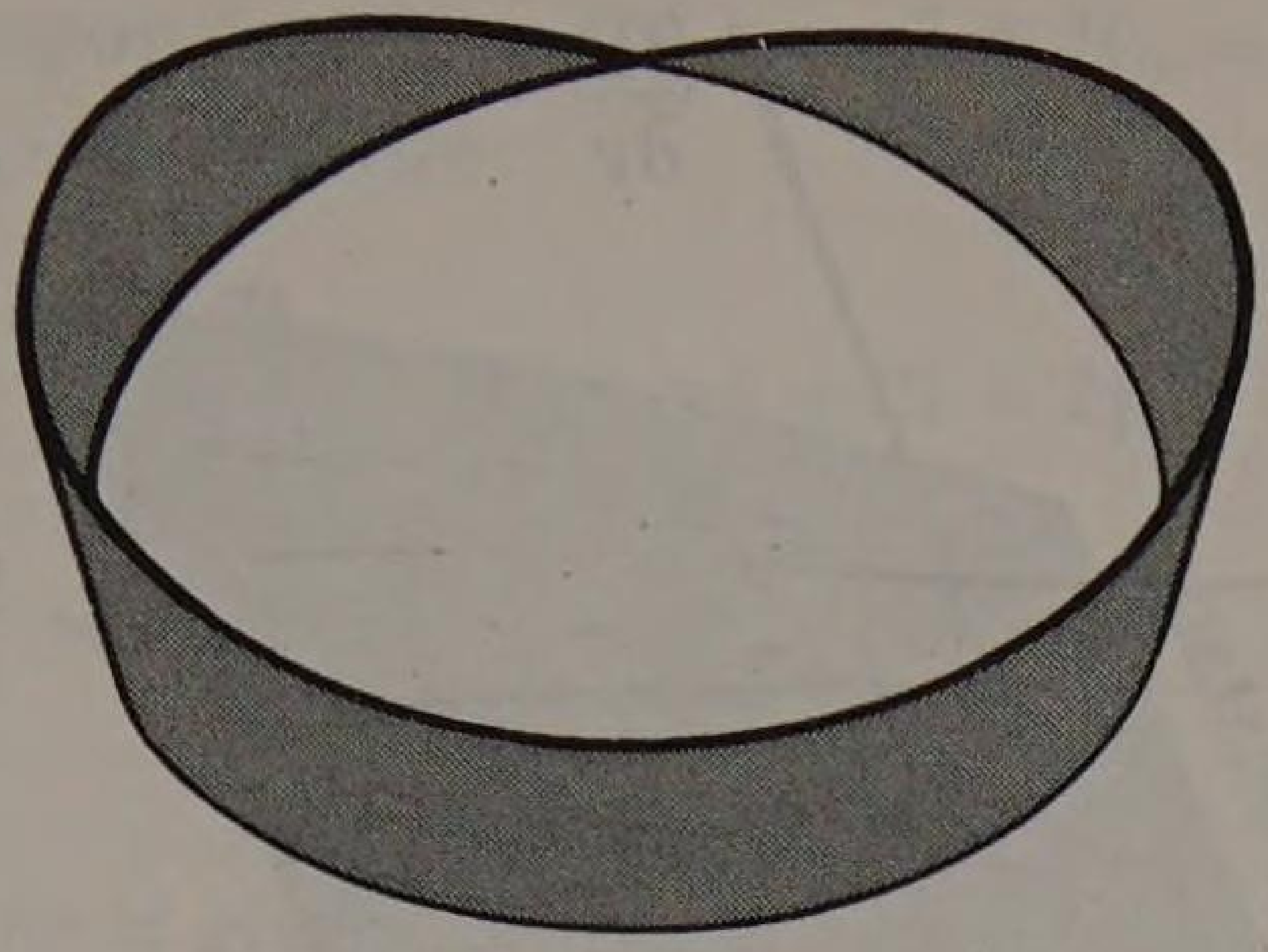

FIGURE 4.13

surface is nonorientable because it has only one side. If **n** is a unit vector normal to the surface at a point P, then as it moves around the strip its direction is reversed by the time it reaches P again. This contradicts the requirement that **n** be unambiguous at every point and still vary continuously.

The reader may amuse himself by taking two strips of paper and preparing two bands, one with a twist and one without. If the strips are long enough to dangle on the floor, no one will notice the difference between them. Have someone cut along a central line of the cylindrical band at the same time that you cut the Möbius strip. The cylindrical band will separate into two cylindrical bands, but the Möbius strip will not separate into distinct portions. Can you predict the result?

Nonorientable surfaces have other mathematical properties that are rather amazing; so amazing, in fact, that we must exclude them from further consideration. Henceforth, whenever we say "surface" we mean an orientable surface.

Just as it is possible to write the equation of a space curve in parametric form, giving x, y, and z as functions of a *single* parameter t (because the curve is a *one*-dimensional beast), it is also possible to represent these (*two*-dimensional) surfaces parametrically by giving x, y, and z as functions of two parameters u and v:

$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v) \tag{4.22}$$

We have already seen this for the case of a plane (Sec. 1.10). In vector notation, we write Eq. (4.22) as

$$\mathbf{R} = \mathbf{R}(u,v) \tag{4.22$'$}$$

As the parameters u and v vary, the tip of the position vector $\mathbf{R}(u,v)$ generates the surface. In particular, if we fix the value of, say, v, and let u vary, then $\mathbf{R}(u,v)$ traces out a *one*-dimensional subset of points in the surface, i.e., a curve lying in the surface. For a different fixed value of v, $\mathbf{R}(u,v)$ traces out a different curve in the surface. In fact, we can think of the surface itself, defined by Eq. (4.22) with u and v varying independently, as being composed of "ribs" that are the curves given by (4.22) for variable u, fixed v.

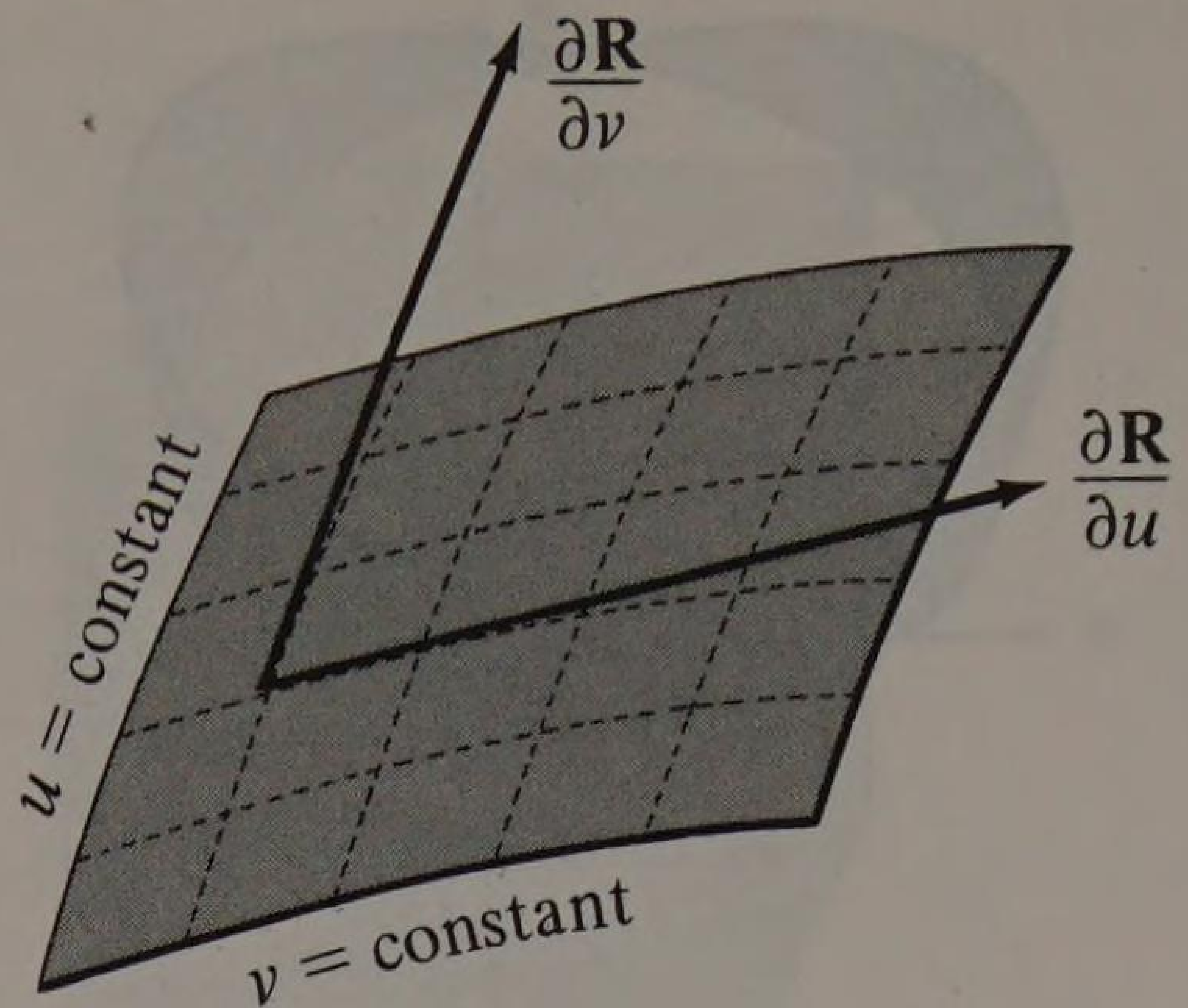
$\frac{\partial \mathbf{R}}{\partial v}$
$\frac{\partial \mathbf{R}}{\partial u}$
$u =$ constant
$v =$ constant

FIGURE 4.14

Of course, it is equally valid to picture the surface as composed of ribs $\mathbf{R}(u,v)$ with v varying, u constant. These two families criss-cross and cover the surface like a fish net (see Fig. 4.14).

This is a very useful point of view because it allows us to apply the methods of Chapter 2 to these curves, to learn about the underlying surface. For example, we know that a vector tangent to a curve with $v =$ constant is given by

$$\frac{\partial \mathbf{R}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \tag{4.23}$$

and, similarly, $\partial \mathbf{R}/\partial v$ is tangent to a curve with $u =$ constant (Fig. 4.14). We assume here that the relevant derivatives exist, that $\partial \mathbf{R}/\partial u$ and $\partial \mathbf{R}/\partial v$ are nonzero and nonparallel at every point, and that these derivatives are continuous on the surface. Since both vectors are tangent to curves in the surface, they are tangent to the surface itself. Therefore, the vector

$$\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} \tag{4.24}$$

is *normal* to the surface.

To recap, we have derived two ways of computing normals to a surface. If the surface is specified nonparametrically by $f(x,y,z) = C$, then **grad** f is a normal vector; if the surface is given parametrically through Eqs. (4.22), then (4.24) is a normal vector. In both cases a *unit* normal $\mathbf{n}$ is obtained by dividing the normal by its length, and the induced orientation is reversed by changing the sign of $\mathbf{n}$ (or, in the parametric case, using $(\partial \mathbf{R}/\partial v) \times (\partial \mathbf{R}/\partial u)$ instead of (4.24)).

Example 4.11 Write the equation for the plane tangent to the surface given by

$$x = u^2 \qquad y = uv \qquad z = v$$

at the point corresponding to $u = 1$, $v = 2$.

Solution The point corresponding to $u = 1$, $v = 2$ is $\mathbf{R}_0 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. The tangents to the constant v, constant u curves are

$$\frac{\partial \mathbf{R}}{\partial u} = 2u\mathbf{i} + v\mathbf{j} = 2\mathbf{i} + 2\mathbf{j}$$

$$\frac{\partial \mathbf{R}}{\partial v} = u\mathbf{j} + \mathbf{k} = \mathbf{j} + \mathbf{k}$$

Hence a normal to the plane is given by the cross product:

$$\mathbf{n} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

The nonparametric equation of the plane is thus

$$(\mathbf{R} - \mathbf{R}_0) \cdot \mathbf{n} = 2(x - 1) - 2(y - 2) + 2(z - 2) = 0$$

A parametric form can be written using the tangent vectors:

$$\mathbf{R}(u,v) = u(2\mathbf{i} + 2\mathbf{j}) + v(\mathbf{j} + \mathbf{k}) + \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

Any portion of a surface that can be represented by equations of the form (4.22) in a manner such that to distinct ordered pairs (u,v) there correspond distinct points (x,y,z) on the surface, and satisfying the above differentiability and continuity requirements, is called a *regular surface element.*

Recall that the arc length of a smooth arc was defined as the limit of the lengths of inscribed polygonal paths (Sec. 2.2). The surface area of a regular surface element turns out to be a slightly trickier concept, so we will be content to present a heuristic argument that leads to the correct formula. Figure 4.15 shows a small patch of the surface bounded by curves of constant u and v. Notice that, for small Δu and Δv, the patch is well approximated by a parallelogram with sides PQ and PS represented by the displacement vectors $\mathbf{R}(u + \Delta u, v) - \mathbf{R}(u,v)$ and $\mathbf{R}(u, v + \Delta v) - \mathbf{R}(u,v)$, respectively. The area of this parallelogram is given by the cross product; so if we introduce

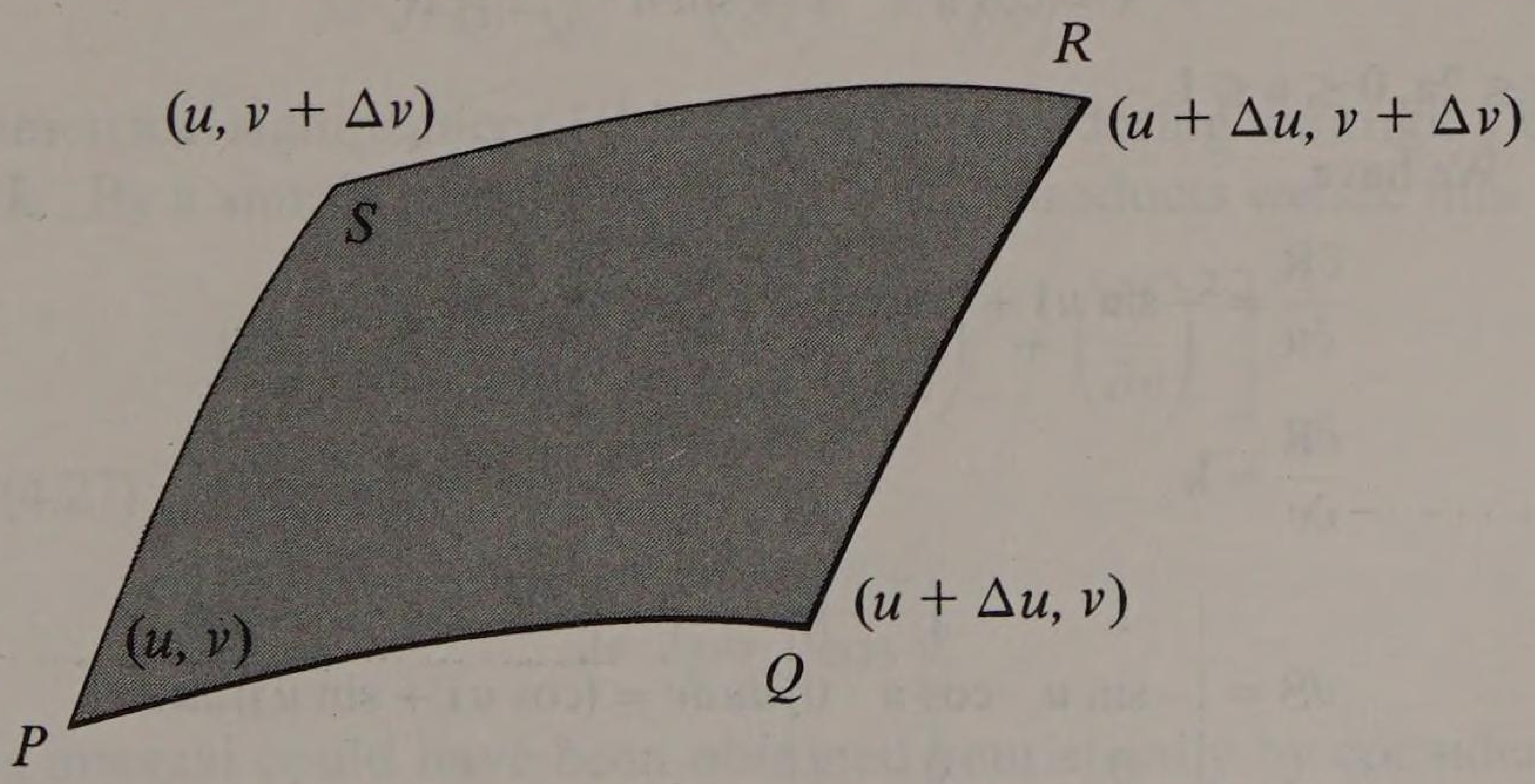

FIGURE 4.15

the further approximations

$$\mathbf{R}(u+\Delta u, v) - \mathbf{R}(u,v) \approx \frac{\partial \mathbf{R}}{\partial u}\,\Delta u$$

$$\mathbf{R}(u, v+\Delta v) - \mathbf{R}(u,v) \approx \frac{\partial \mathbf{R}}{\partial v}\,\Delta v$$

we find that the area of the patch is given approximately by

$$\Delta S = \left|\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\right| \Delta u\,\Delta v$$

Summing these up over the surface and letting Δu and Δv go to zero, we argue that *the surface area of a regular surface element is given by*

$$S = \iint \left|\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\right| du\,dv \tag{4.25}$$

If we introduce the notation

$$d\mathbf{S} = \frac{\partial \mathbf{R}}{\partial u}\,du \times \frac{\partial \mathbf{R}}{\partial v}\,dv \tag{4.26}$$

then we see that $d\mathbf{S}$ is a vector normal to the surface at P, whose magnitude $dS = |d\mathbf{S}|$ is the element of area. The integral (4.26) may be written in the alternative forms

$$\iint |d\mathbf{S}|$$

or

$$\iint dS$$

or even

$$\iint \mathbf{n}\cdot d\mathbf{S}$$

where $\mathbf{n}$ is a unit normal in the same direction as $d\mathbf{S}$.

Example 4.12 Find the surface area of the surface defined by the equations

$$x = \cos u \qquad y = \sin u \qquad z = v$$

for $0 \le u \le 2\pi$, $0 \le v \le 1$.

Solution We have

$$\frac{\partial \mathbf{R}}{\partial u} = -\sin u\,\mathbf{i} + \cos u\,\mathbf{j}$$

$$\frac{\partial \mathbf{R}}{\partial v} = \mathbf{k}$$

$$d\mathbf{S} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} du\,dv = (\cos u\,\mathbf{i} + \sin u\,\mathbf{j})\,du\,dv$$

$$\iint |d\mathbf{S}| = \int_0^1 \int_0^{2\pi} (\cos^2 u + \sin^2 u)^{\frac{1}{2}}\,du\,dv = 2\pi$$

This surface is a right circular cylinder of unit radius and unit height. If it is cut along a "seam", it unfolds into a rectangle whose dimensions are 2π by 1.

We now consider a *special case* of Eq. (4.25) that will illustrate further its geometrical significance. Let us suppose that the surface we consider is given in the form $z = f(x,y)$. In other words, we are told how far above the xy plane the surface is for each point (x,y) in the xy plane. Then it is convenient to use x and y instead of u and v as the parameters. Let us suppose that the projection of the surface element on the xy plane is bounded by the curves

$$y = y_1(x) \qquad y = y_2(x) \qquad x = a \qquad x = b$$

as shown in Fig. 4.16.

We can write $x = u$, $y = v$, and $z = f(u,v)$, in order to make use of the preceding formulas. We have

$$\frac{\partial \mathbf{R}}{\partial u} = \frac{\partial \mathbf{R}}{\partial x} = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k}$$

$$\frac{\partial \mathbf{R}}{\partial v} = \frac{\partial \mathbf{R}}{\partial y} = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

Taking the vector cross product,

$$\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{vmatrix} = \mathbf{k} - \frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j}$$

The magnitude of this vector is $\sqrt{1 + (\partial f/\partial x)^2 + (\partial f/\partial y)^2}$, so that the integral (4.25) is

$$\int_a^b \int_{y_1(x)}^{y_2(x)} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dy\, dx \tag{4.27}$$

The geometrical significance of this is seen by considering the angle γ between $d\mathbf{S}$ and $\mathbf{k}$. By a simple calculation using scalar products we see that

$$|\cos \gamma| = \frac{|d\mathbf{S} \cdot \mathbf{k}|}{|d\mathbf{S}|} = \left[1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]^{-\frac{1}{2}}$$

so that (4.27) is simply

$$\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{dx\, dy}{|\cos \gamma|} \tag{4.28}$$

This integral could have been obtained heuristically by considering the *area cosine principle* which says that, if we look at a plane area A whose normal makes an acute angle θ with the line of sight, the area we appear to

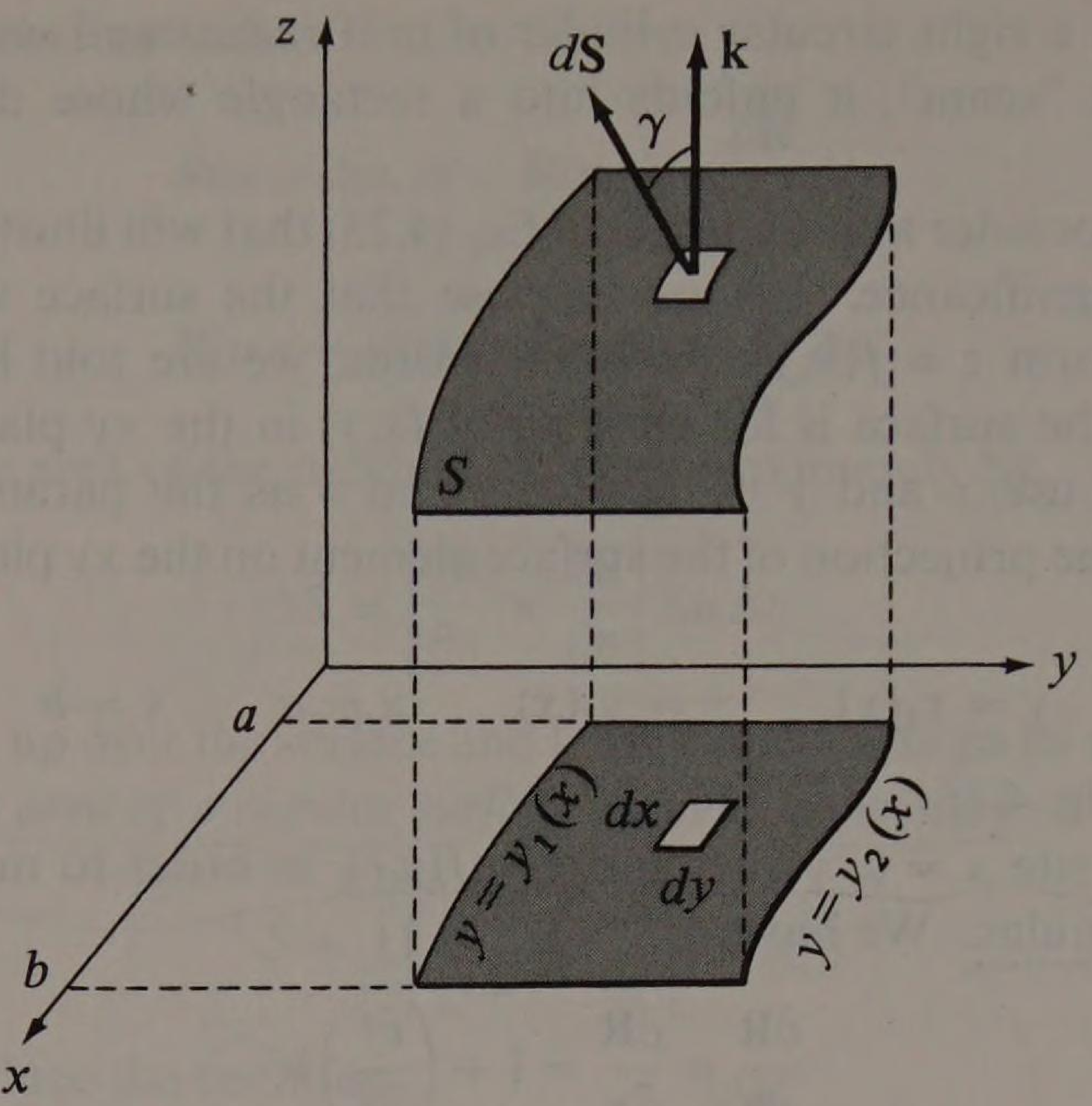

FIGURE 4.16

see is $A \cos \theta$. This is because distances in one direction will appear to be shorter by a factor of $\cos \theta$ and distances in a perpendicular direction will not change at all.

Let us digress for a moment to use this law to determine the area of the ellipse shown in Fig. 4.17. Let us pretend that this ellipse is really a circle of radius a that we are viewing at an angle. In other words, we imagine that this is a circle of radius a (area πa^2) that has been tipped in such a manner that vertical distances are shortened by a factor b/a. The area we see will be $A \cos \theta = (\pi a^2)(b/a) = \pi ab$. Thus we find the area of this ellipse to be πab, by a method that is much easier than using integral calculus.

Returning to Fig. 4.16, we can consider an element of area in S whose projection on the xy plane has area $dx\,dy$. The angle between the normal

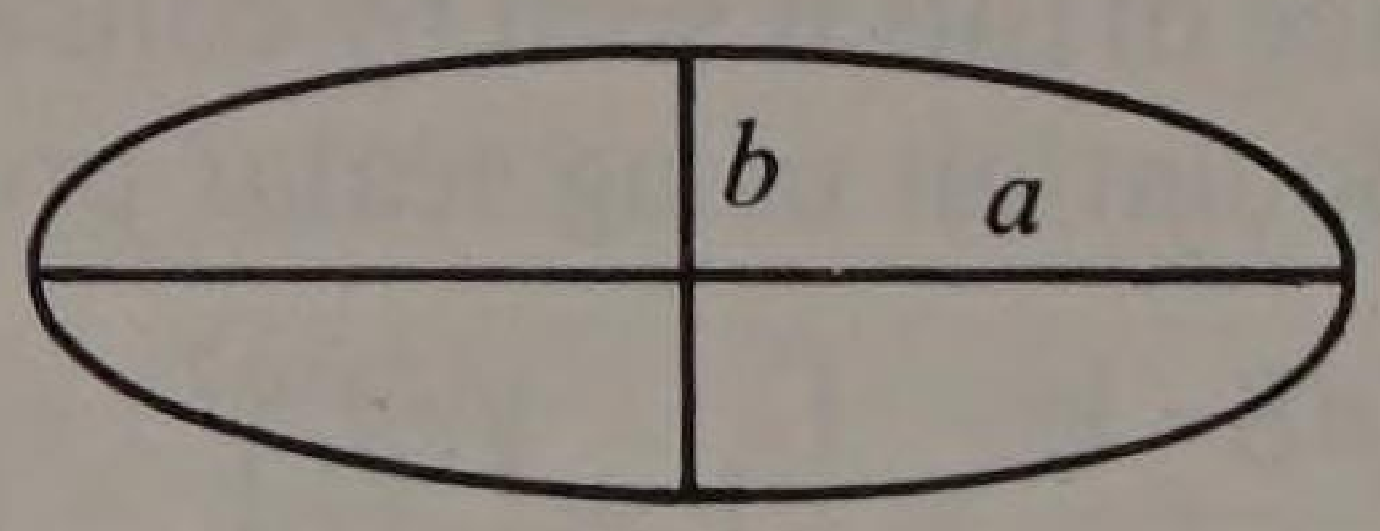

FIGURE 4.17

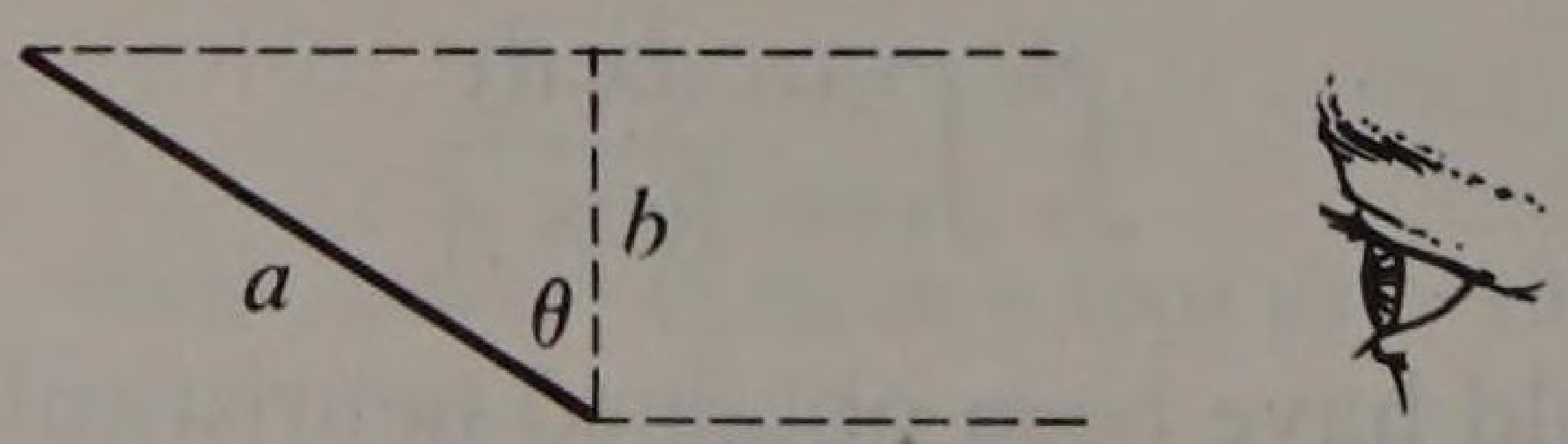

FIGURE 4.18

to this area and the "line of sight" (imagine that you are below the xy plane looking up at the surface) is γ. By the area cosine principle, the area $dx\,dy$ that we see equals $dS\,|\cos\gamma|$. It follows that

$$dS = \frac{dx\,dy}{|\cos\gamma|}$$

The absolute value is unnecessary if γ is acute.

Frequently, a judicious use of the area cosine principle makes it unnecessary to use (4.25). The cosine of the relevant angle, in this case γ, is easily computed since we can find a normal to the surface by methods we have already learned and then use scalar products to find the desired cosine. For instance, the surface $z = f(x,y)$ can be represented by the equation $z - f(x,y) = 0$, and the gradient of the function $z - f(x,y)$ is $\mathbf{k} - (\partial f/\partial x)\mathbf{i} - (\partial f/\partial y)\mathbf{j}$. This is easier than computing the vector cross product given above. [*Caution*: Using gradients to give a normal vector $\mathbf{N}$ gives a vector that does not equal $d\mathbf{S}$ but is only a scalar multiple of it. However, this makes no difference since we are only interested in computing $\cos\gamma = \mathbf{N}\cdot\mathbf{k}/|\mathbf{N}|$ when using (4.28).]

Example 4.13 Find the area of the surface defined by

$$x^2 + y^2 + z^2 = 1 \qquad x \geq 0$$

Solution The reader should be able to see that this surface is half of a unit sphere. Let us use the area cosine principle that corresponds to a line of sight along the $\mathbf{i}$ direction, projecting the area onto the yz plane. A normal is given by the gradient

$$\nabla(x^2 + y^2 + z^2) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

If α is the angle that the normal makes with $\mathbf{i}$, then

$$\cos\alpha = \frac{(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})\cdot\mathbf{i}}{(4x^2 + 4y^2 + 4z^2)^{\frac{1}{2}}} = \frac{2x}{2} = x$$

The projection of this hemisphere onto the yz plane is a unit circle; hence, the area is

$$S = \iint \frac{dy\,dz}{\cos\alpha} = \int_{-1}^{1}\int_{-(1-z^2)^{\frac{1}{2}}}^{+(1-z^2)^{\frac{1}{2}}} x^{-1}\,dy\,dz$$

Since $x = (1 - y^2 - z^2)^{\frac{1}{2}}$,

$$S = 2\pi$$

(by standard methods). This checks with our expectations; a unit hemisphere would have area $4\pi/2 = 2\pi$.

Sometimes, in practice, the formula (4.26) can be visualized. For example, suppose we parametrize a sphere of radius a by its latitude and longitude angles ϕ and θ, as in Fig. 4.19. We have

$$x = a\sin\phi\cos\theta \qquad y = a\sin\phi\sin\theta \qquad z = a\cos\phi$$
$$0 \leq \phi \leq \pi \qquad -\pi \leq \theta \leq \pi$$

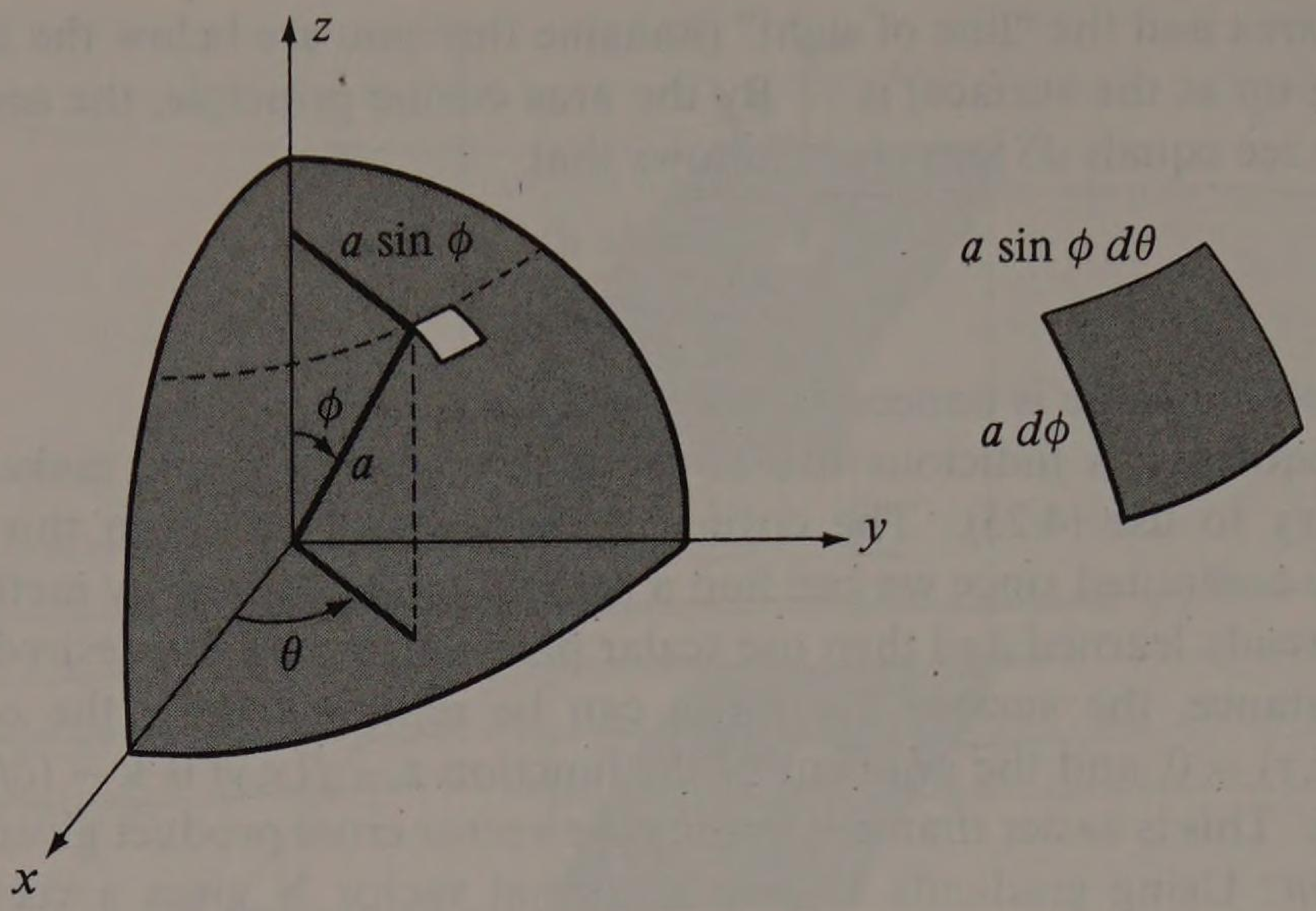

FIGURE 4.19

Using the two parameters ϕ and θ instead of u and v, we obtain

$$\frac{\partial \mathbf{R}}{\partial \phi} = a \cos \phi \cos \theta \, \mathbf{i} + a \cos \phi \sin \theta \, \mathbf{j} - a \sin \phi \, \mathbf{k} \tag{4.29}$$

$$\frac{\partial \mathbf{R}}{\partial \theta} = -a \sin \phi \sin \theta \, \mathbf{i} + a \sin \phi \cos \theta \, \mathbf{j} \tag{4.30}$$

whereupon we compute to show that

$$\frac{\partial \mathbf{R}}{\partial \phi} \times \frac{\partial \mathbf{R}}{\partial \theta} = a^2 \sin^2 \phi \cos \theta \, \mathbf{i} + a^2 \sin^2 \phi \sin \theta \, \mathbf{j} + a^2 \sin \phi \cos \phi \, \mathbf{k} \tag{4.31}$$

The magnitude of the vector is $a^2 \sin \phi$; hence it follows that

$$dS = a^2 \sin \phi \, d\phi \, d\theta \tag{4.32}$$

This result can be visualized from Fig. 4.19. Holding θ fixed and varying ϕ by an amount $d\phi$, we trace out an arc of length $a \, d\phi$. Holding ϕ fixed and varying θ, we trace out an arc of a circle of radius $a \sin \phi$, the length of this arc being $a \sin \phi \, d\theta$. For small $d\phi$ and $d\theta$, this gives us very nearly a rectangle with area $a^2 \sin \phi \, d\phi \, d\theta$.

Exercises

1. Show that the Möbius strip is piecewise smooth, and show why the smooth parts cannot be oriented consistently.
2. Draw a diagram similar to those of Figs. 4.11 and 4.12 for the surface of a tetrahedron.

3. Consider the triangle with vertices (1,0,0), (0,1,0), and (0,0,1).

(a) Find a unit vector **n** normal to this triangle, pointing away from the origin.

(b) Determine $\cos \gamma$ for this vector.

(c) Supply the appropriate limits for the integral

$$\iint \frac{dx\,dy}{|\cos \gamma|}$$

if it is to represent the area of this triangle.

(d) Evaluate the integral.

(e) Obtain the same answer by applying the area cosine principle to the projection of this triangle on the yz plane.

4. (a) Derive Eq. (4.31) from (4.29) and (4.30).

(b) Show that the magnitude of this vector is $a^2 \sin \phi$.

5. Determine the element of surface area dS for a right circular cylinder:

$$x = a \cos u \qquad y = a \sin u \qquad z = v$$

Interpret geometrically. [See Fig. 5.1, interpreting (u,v) as (θ,z).]

6. Determine the element of surface area dS in the special case of the surface $z = x^2 + y^2$.

7. Find the area of the section of the surface

$$x = u^2 \qquad y = uv \qquad z = \frac{1}{2}v^2$$

bounded by the curves $u = 0$, $u = 1$, $v = 0$, and $v = 3$.

8. Derive the identity

$$dS = (EG - F^2)^{\frac{1}{2}}\,du\,dv$$

where

$$E = \left|\frac{\partial \mathbf{R}}{\partial u}\right|^2 \qquad F = \frac{\partial \mathbf{R}}{\partial u} \cdot \frac{\partial \mathbf{R}}{\partial v} \qquad G = \left|\frac{\partial \mathbf{R}}{\partial v}\right|^2$$

(The quantities E, F, and G are employed in differential geometry in developing the theory of surfaces. They constitute the "Second Fundamental Form.")

4.7 SURFACE INTEGRALS

Let S denote a smooth surface and let $f(x,y,z)$ be a function defined and continuous on S. The surface integral of f over S, denoted

$$\iint f\,dS$$

is defined by a construction that the reader can, no doubt, anticipate by now. We imagine the surface cut up into n pieces having area $\delta S_1, \delta S_2, \ldots, \delta S_n$. In each piece we choose a point (x_i,y_i,z_i), evaluate $f(x_i,y_i,z_i)$, and form

$f(x_i,y_i,z_i)\,\delta S_i$. We sum these numbers:

$$\sum_{i=1}^{n} f(x_i,y_i,z_i)\,\delta S_i \tag{4.33}$$

In this way we obtain a single number. Now let n tend to infinity, at the same time letting the pieces grow smaller so that the maximum dimension of the areas $\delta S_1, \delta S_2, \ldots, \delta S_n$ tends to zero. In other words, we are dividing the surface into smaller and smaller elements of area, each time forming a sum of form (4.33). If these sums tend to a limit, independent of the way we form the repeated subdivisions, that limit is called the *surface integral of f over S*:

$$\iint_S f(x,y,z)\,dS = \lim_{\substack{\max \delta S_i \to 0 \\ n\to\infty}} \sum_{i=1}^{n} f(x_i,y_i,z_i)\,\delta S_i \tag{4.34}$$

In most situations the function f arises from a scalar product involving a vector field $\mathbf{F}$. We define the *flux* of $\mathbf{F}$ through the surface S to be the surface integral

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS \tag{4.35}$$

where, at any point on the oriented surface, $\mathbf{n}$ is the unit normal to S. (Thus, $f = \mathbf{F}\cdot\mathbf{n}$ in this circumstance.) The physical meaning of the flux integral will be discussed shortly, but first let us see how to compute it.

Using the notation of the previous section, we can write the flux as

$$\iint_S \mathbf{F}\cdot d\mathbf{S} \tag{4.35$'$}$$

Applying Eq. (4.26), we convert this to a workable formula when the surface is parametrized by $\mathbf{R}(u,v)$:

$$\iint \mathbf{F}\cdot\frac{\partial\mathbf{R}}{\partial u}\times\frac{\partial\mathbf{R}}{\partial v}\,du\,dv \tag{4.35$''$}$$

If the surface is specified by giving z as a function of x and y, we would use

$$\iint \mathbf{F}\cdot\mathbf{n}\frac{dx\,dy}{|\cos\gamma|} \tag{4.35$'''$}$$

where the limits of integration are determined by the projection of S onto the xy plane [with an obvious modification for surfaces described by, say, $x = x(y,z)$].

If the surface S is only piecewise smooth, we integrate over each smooth part separately and add the numbers obtained.

Example 4.14 Compute the flux of the vector field $\mathbf{F} = \mathbf{i} + xy\mathbf{j}$ across the surface given by

$$x = u + v \qquad y = u - v \qquad z = u^2$$
$$0 \le u \le 1 \qquad 0 \le v \le 1$$

Solution Using Eq. (4.35″),

$$\iint \mathbf{F} \cdot \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\, du\, dv = \int_0^1 \int_0^1 \begin{vmatrix} 1 & u^2 - v^2 & 0 \\ 1 & 1 & 2u \\ 1 & -1 & 0 \end{vmatrix} du\, dv$$

$$= \int_0^1 \int_0^1 (2u^3 - 2uv^2 + 2u)\, du\, dv$$

$$= \int_0^1 \left(\frac{1}{2} - v^2 + 1\right) dv = \frac{7}{6}$$

Example 4.15 Compute

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 4$ and

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Solution We recall that at a point (x,y,z) the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ points directly away from the origin. The outward normal $\mathbf{n}$ to this sphere also points away from the origin, since the center of the sphere is at the origin. Hence for points on the surface

$$\mathbf{F} \cdot \mathbf{n} = |\mathbf{F}|\, |\mathbf{n}| \cos \theta = |\mathbf{F}| = (x^2 + y^2 + z^2)^{\frac{1}{2}} = 2$$

and
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S 2\, dS = 2\,(\text{total surface area})$$

$$= 2(4\pi r^2) = 32\pi$$

since $r = 2$ is the radius of the sphere.

Note that, in the above example, no integration was needed, since $\mathbf{F} \cdot \mathbf{n}$ was constant over the entire surface.

Example 4.16 Compute

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where S is the surface of the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$, and $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Solution We see from Fig. 4.20 that the unit normal to the front face of the cube is $\mathbf{n} = \mathbf{i}$, so

$$\mathbf{F} \cdot \mathbf{n} = \mathbf{i} \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = x = 1$$

on this face. It follows that the integral over this face is

$$\iint \mathbf{F} \cdot d\mathbf{S} = \iint \mathbf{F} \cdot \mathbf{n}\, dS = \iint dS = 1$$

since the area of this face is unity. On the opposite face (in the yz plane) $\mathbf{n} = -\mathbf{i}$ so that $\mathbf{F} \cdot \mathbf{n} = -x$, but $x = 0$ for all points in this face and hence

$$\iint \mathbf{F} \cdot \mathbf{n}\, dS = 0$$

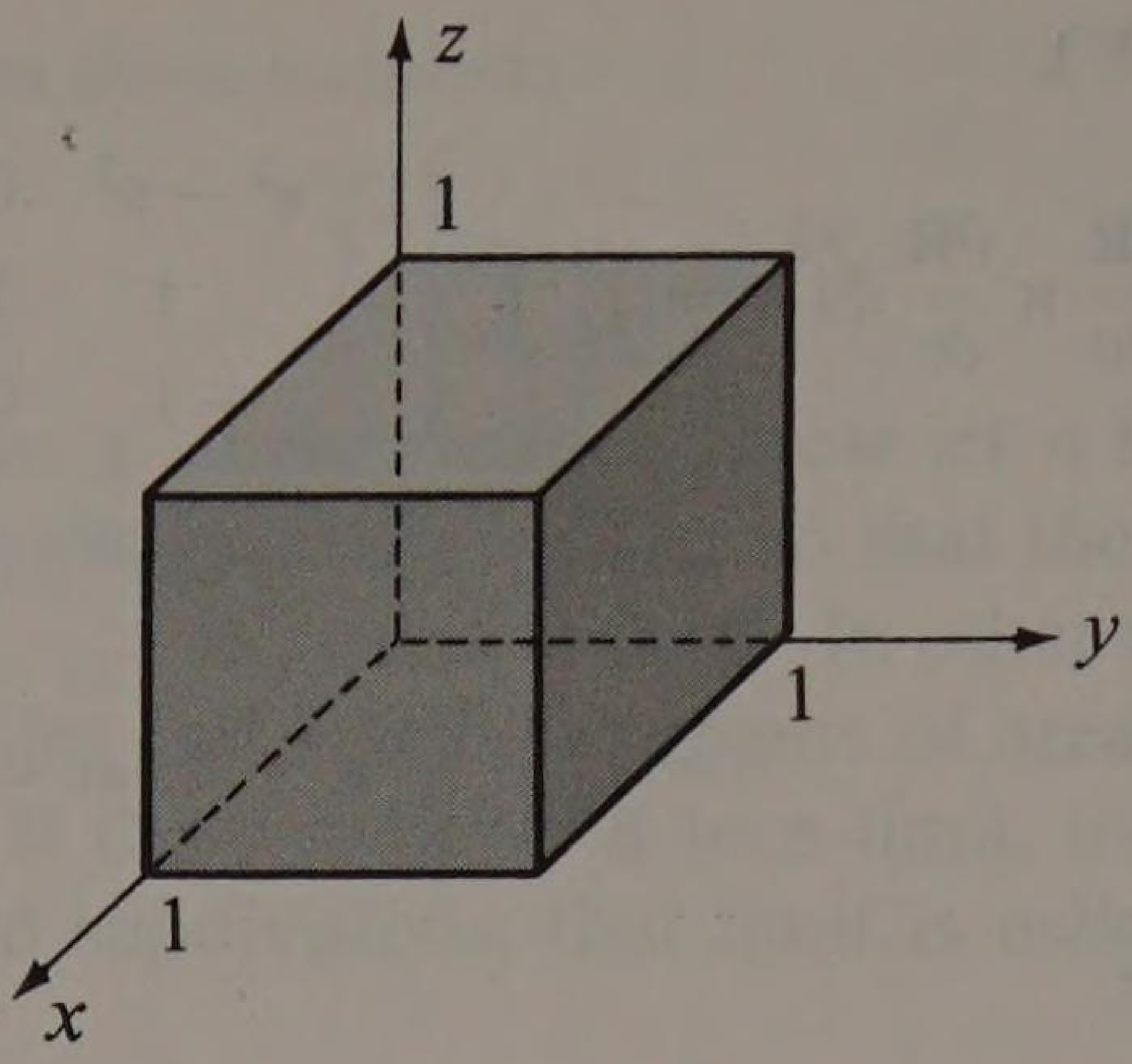

FIGURE 4.20

On the top of the cube we have

$$\iint \mathbf{F}\cdot\mathbf{n}\,dS = \iint \mathbf{F}\cdot\mathbf{k}\,dS = \iint z\,dS = \iint dS = 1$$

and on the bottom we have

$$\iint \mathbf{F}\cdot\mathbf{n}\,dS = \iint (-z)\,dS = 0$$

since $z = 0$ in the xy plane. Along the right side we have $\mathbf{n} = \mathbf{j}$ so that the normal component of $\mathbf{F}$ is unity and the integral over this face is unity. Along the left side $\mathbf{n} = -\mathbf{j}$ and $\mathbf{F}\cdot\mathbf{n} = -y = 0$, so that the contribution to the integral is zero. Summing, we find that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 3$$

Example 4.17 Compute the surface integral of the normal component of $\mathbf{F} = x^2\mathbf{i} + yx\mathbf{j} + zx\mathbf{k}$ over the triangle with vertices (1,0,0), (0,2,0), (0,0,3). Consider the triangle oriented so that its positive side is that away from the origin (Fig. 4.21).

Solution By the methods of Chap. 1 we find easily that $\mathbf{n} = \frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}$. Hence $\mathbf{F}\cdot\mathbf{n} = \frac{6}{7}x^2 + \frac{3}{7}yx + \frac{2}{7}zx$ and

$$\cos\gamma = \mathbf{n}\cdot\mathbf{k} = \frac{2}{7}$$

Using (4.35‴) we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^{2-2x} \frac{7}{2}\left(\frac{6}{7}x^2 + \frac{3}{7}yx + \frac{2}{7}zx\right)dy\,dx$$

$$= \int_0^1\int_0^{2-2x}\left(3x^2 + \frac{3}{2}yx + zx\right)dy\,dx$$

On S we have $z = 3 - 3x - \frac{3}{2}y$ so that $zx = 3x - 3x^2 - \frac{3}{2}yx$, and the integral becomes

$$\int_0^1\int_0^{2-2x} 3x\,dy\,dx = \int_0^1 3x(2-2x)\,dx = 3x^2 - 2x^3\Big|_0^1 = 1$$

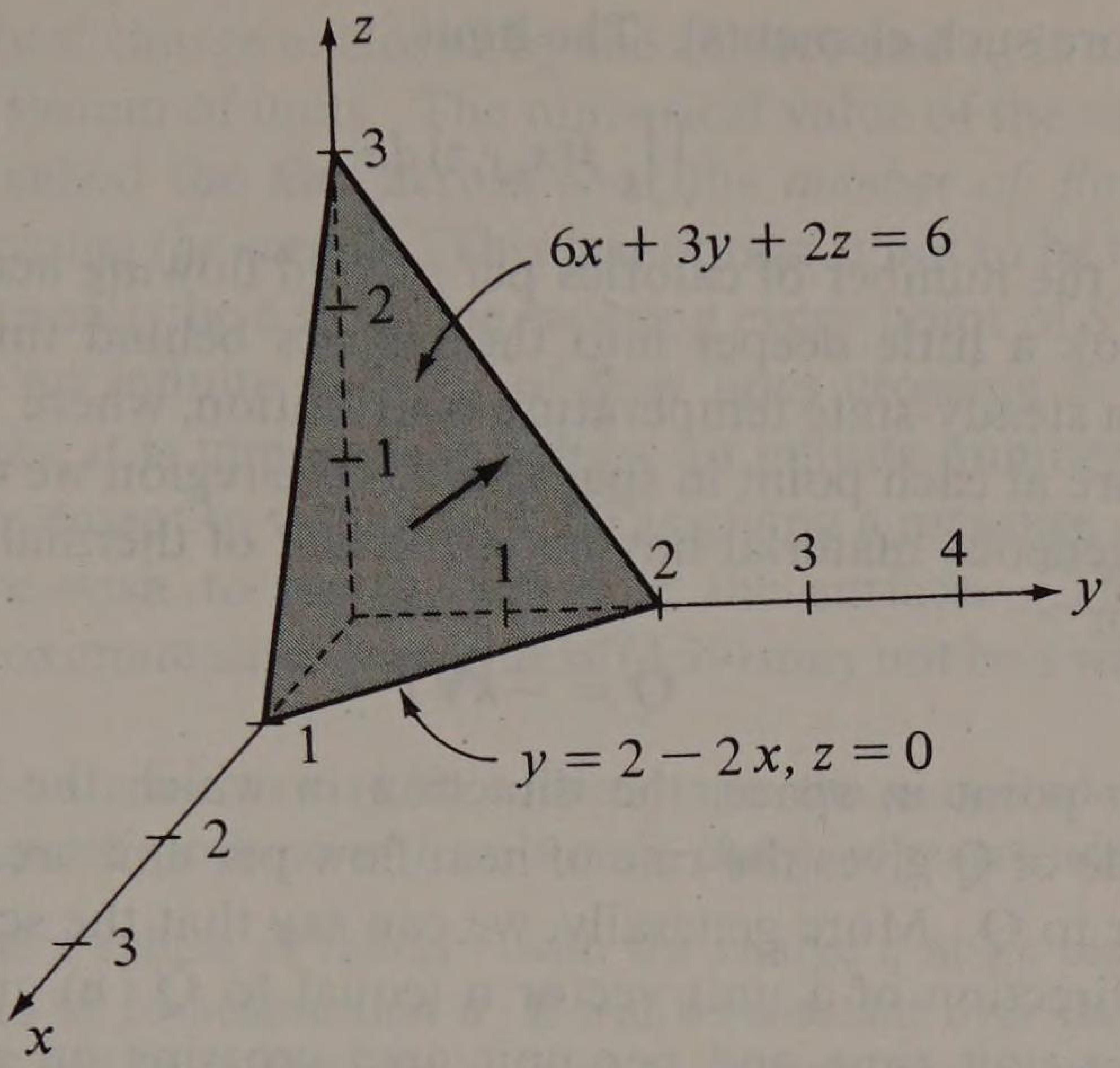

FIGURE 4.21

Example 4.18 Compute

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

over the surface of the tetrahedron with vertices (1,0,0), (0,2,0), (0,0,3), (0,0,0), where $\mathbf{F} = x^2\mathbf{i} + yx\mathbf{j} + zx\mathbf{k}$ (Fig. 4.21).

Solution We have already computed the integral over one surface. Along the bottom face we have $\mathbf{n} = -\mathbf{k}$ and, hence, $\mathbf{F} \cdot \mathbf{n} = -zx$, but since $z = 0$ the integral over the bottom face is zero. On the fact at the left we have $\mathbf{n} = -\mathbf{j}$ and $\mathbf{F} \cdot \mathbf{n} = -yx$, which is also zero since $y = 0$ there. On the rear face, in the yz plane, $\mathbf{n} = -\mathbf{i}$ and $\mathbf{F} \cdot \mathbf{n} = -x^2 = 0$. It follows that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 1$$

the only nonzero contribution being the integral already computed in Example 4.16.

Note that in Examples 4.15, 4.16, and 4.18 we took $\mathbf{n}$ to be the *outward* normal, the usual convention for closed surfaces.

Now let's consider some physical examples of surface integrals. Suppose, for instance, that at any point (x,y,z) on a surface S, $f(x,y,z)$ gives the rate of flow of heat per unit area at that point, in units (say) of calories per second per square centimeter. Then $f(x_i,y_i,z_i)\,\delta S_i$ gives, approximately, the number of calories per second flowing across the element of area δS_i and the sum (4.33) approximates the total number of calories per second flowing across the entire surface S. If $f(x,y,z)$ varies from point to point on the surface, this approximation can be improved by taking smaller elements of area (and hence more such elements). The limit

$$\iint_S f(x,y,z)\,dS$$

gives, exactly, the number of calories per second flowing across the surface.

Let us look a little deeper into the physics behind this phenomenon. If we assume a steady-state temperature distribution, where $T(x,y,z)$ denotes the temperature at each point in space, and if the region we consider is filled with a homogeneous material having coefficient of thermal conductivity k, then the vector

$$\mathbf{Q} = -k\nabla T \tag{4.36}$$

gives, at each point in space, the direction in which the heat is flowing. The magnitude of $\mathbf{Q}$ gives the rate of heat flow per unit area across an area perpendicular to $\mathbf{Q}$. More generally, we can say that the scalar component of $\mathbf{Q}$ in the direction of a unit vector $\mathbf{n}$ (equal to $\mathbf{Q}\cdot\mathbf{n}$) gives the number of calories per unit time and per unit area crossing an element of area perpendicular to $\mathbf{n}$.

It follows that the function f is given by $(-k\nabla T)\cdot\mathbf{n}$, and the total number of calories per second flowing across a surface S equals

$$\iint_S (-k\nabla T)\cdot\mathbf{n}\,dS \tag{4.37}$$

The reason for the negative sign in Eqs. (4.36) and (4.37) is that the temperature gradient ∇T points in the direction of maximum rate of increase of the temperature, whereas heat flows in the opposite direction, from hot to cold.

Let us consider another situation in physics in which surface integrals arise. If $\mathbf{F}$ denotes the velocity field of a fluid and ρ its density, then as we saw in Sec. 3.3, the amount of fluid crossing a patch of surface with area δS and unit normal $\mathbf{n}$, per unit time, is approximately $\rho\mathbf{F}\cdot\mathbf{n}\,\delta S$. This formula becomes exact as δS goes to zero. Thus we can see that

$$\iint_S \rho\mathbf{F}\cdot\mathbf{n}\,dS \tag{4.38}$$

gives the rate of flow of liquid across the surface S, expressed as mass per unit time.

As yet another example, consider an electrostatic field $\mathbf{E}$ defined in a region of space. One can form

$$\iint_S \mathbf{n}\cdot\mathbf{E}\,dS$$

which is the surface integral of the normal component of $\mathbf{E}$ over the surface S. This integral arises in connection with Gauss's law of electrostatics which states that if S is a closed surface,

$$\iint_S \mathbf{n}\cdot\mathbf{E}\,dS = \frac{q}{\varepsilon_0} \tag{4.39}$$

where q is the total charge enclosed by the surface and ε_0 is a constant that depends on the system of units. The numerical value of the surface integral in Eq. (4.39) is called the *flux* across S or the *number of flow lines of the vector field* **E** *crossing the surface.* This last phrase is not to be taken literally, since there will usually be a flow line crossing every point of S and therefore there are really an infinite number of flow lines crossing S. However, in drawing diagrams, it is impossible to draw an infinite number of flow lines, so it may be convenient to visualize (4.39) as giving a measure of the number of flow lines we wish to picture crossing the surface. [This number is necessarily approximate since the value of (4.39) may not be a whole number.]

Example 4.19 Use Gauss's law (4.39) to determine the magnitude of the electric field intensity at a point r units away from a point charge of magnitude q.

Solution Let S be a sphere of radius r with the charge q at its center. Symmetry considerations lead us to believe that $\mathbf{n}\cdot\mathbf{E}$ will be constant over the surface of this sphere, and that **E** will be normal to the surface. Hence, we can bring $\mathbf{n}\cdot\mathbf{E}$ outside the integral, and we obtain

$$\iint \mathbf{n}\cdot\mathbf{E}\,dS = \mathbf{n}\cdot\mathbf{E}\iint dS = 4\pi r^2(\mathbf{n}\cdot\mathbf{E})$$

It then follows from (4.39) that $\mathbf{n}\cdot\mathbf{E} = q/4\pi\varepsilon_0 r^2$. Hence, if the charge is positive, $|\mathbf{E}| = q/4\pi\varepsilon_0 r^2$ and **E** is directed away from q. If q is negative, **E** will be directed towards the charge q.

Example 4.20 Use Gauss's law (4.39) to determine the magnitude of the electric field intensity at a point r units away from an infinite plate carrying a charge of density σ (charge per unit area).

Solution Let S be the surface of a right circular cylinder of length $2r$ and base area A, bisected by the charged sheet. We take the bases parallel to the sheet, so by symmetry we expect **E** to be perpendicular to the bases (Fig. 4.22).

The charge within S is $q = \sigma A$. On each of the two bases we have $\mathbf{n}\cdot\mathbf{E} =$ constant by symmetry (since we assume the charged sheet infinite in extent), and there will be

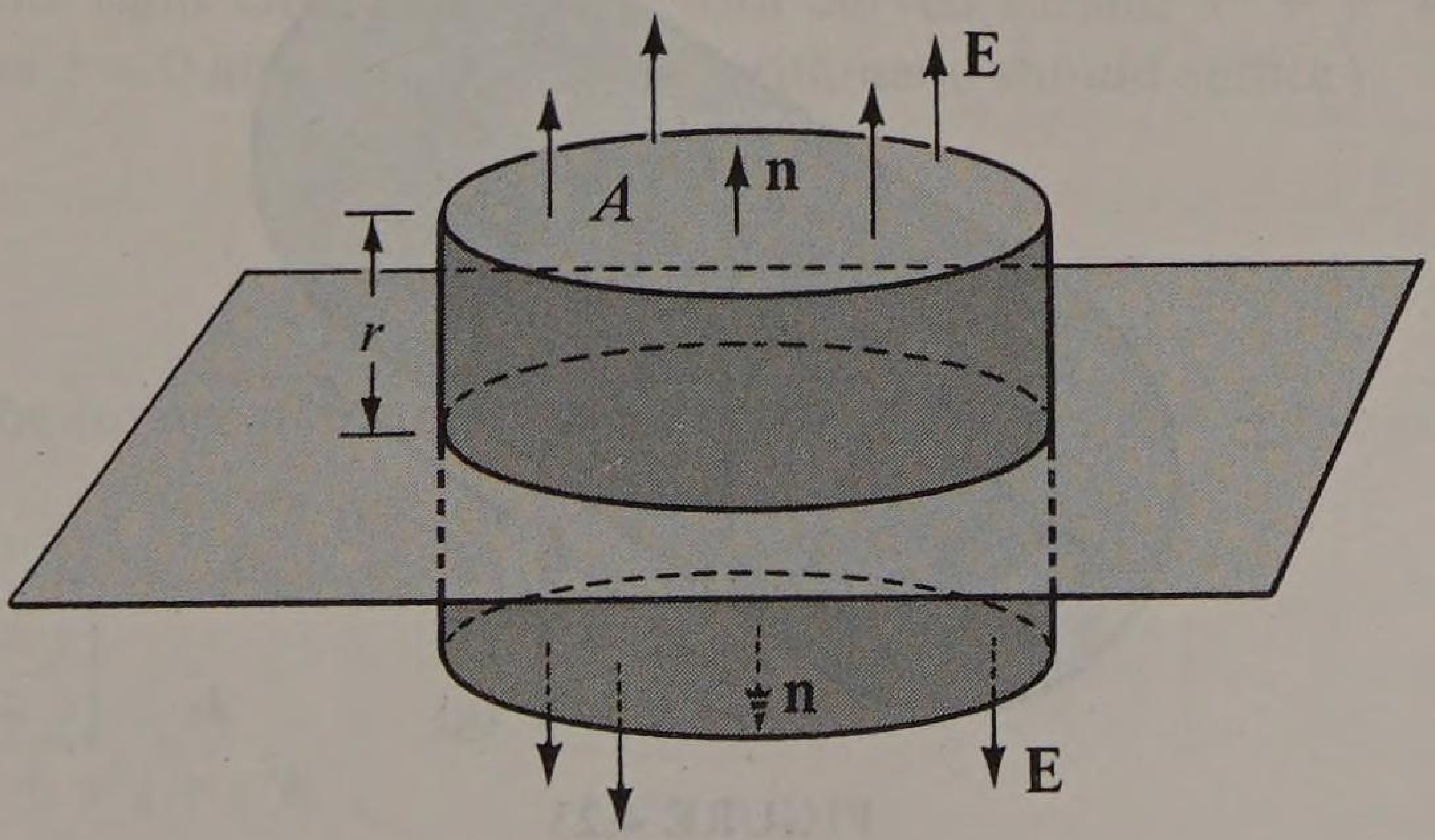

FIGURE 4.22

no contribution to the integral around the curved surface of the cylinder because $\mathbf{E}$ is parallel to this surface; therefore

$$\iint_S \mathbf{n}\cdot\mathbf{E}\,dS = \mathbf{n}\cdot\mathbf{E}\iint_S dS = (\mathbf{n}\cdot\mathbf{E})(2A)$$

By (4.39) we have $(\mathbf{n}\cdot\mathbf{E})(2A) = \sigma A/\varepsilon_0$, so $\mathbf{n}\cdot\mathbf{E} = \sigma/2\varepsilon_0$, If σ is positive, this shows that $\mathbf{E}$ is in the same direction as $\mathbf{n}$ and $|\mathbf{E}| = \sigma/2\varepsilon_0$, independent of r.

Example 4.21 Consider a cylindrical heat insulator surrounding a steampipe. Let the inner and outer radii of the insulator be $r = a$ and $r = b$ respectively, and let T_a and T_b be the temperatures, respectively, of the inner and outer surfaces of the insulator. Find the temperature T within the insulator as a function of r (Fig. 4.23).

Solution A section of the insulator, of length L, is shown in the figure. By symmetry, we assume that T is a function of r alone, so that $\nabla T = \mathbf{grad}\ T$ is directed radially towards the center of the pipe, with magnitude $-dT/dr$. (On the assumption that the pipe is hotter than the surroundings, dT/dr will be negative.) Let S be a cylindrical surface of radius r and length L within the insulator. By Eq. (4.36), we have

$$\mathbf{Q}\cdot\mathbf{n} = (-k\nabla T)\cdot\mathbf{n} = -\frac{k\,dT}{dr}$$

(as usual, we take $\mathbf{n}$ to be outward, so $\nabla T\cdot\mathbf{n} = |\nabla T|\,|\mathbf{n}|\cos 180^\circ = -|\nabla T| = dT/dr$).

Assuming steady-state heat flow, the number of calories of heat flowing across any such surface S will be the same as that across any other such surface, since otherwise the temperature would change with time. The quantity of heat flow per unit time across any such surface is

$$\begin{aligned} H &= \iint_S \mathbf{Q}\cdot\mathbf{n}\,dS = \iint_S -k\frac{dT}{dr}dS = -k\frac{dT}{dr}\iint_S dS \\ &= -k\frac{dT}{dr}(2\pi Lr) \end{aligned}$$

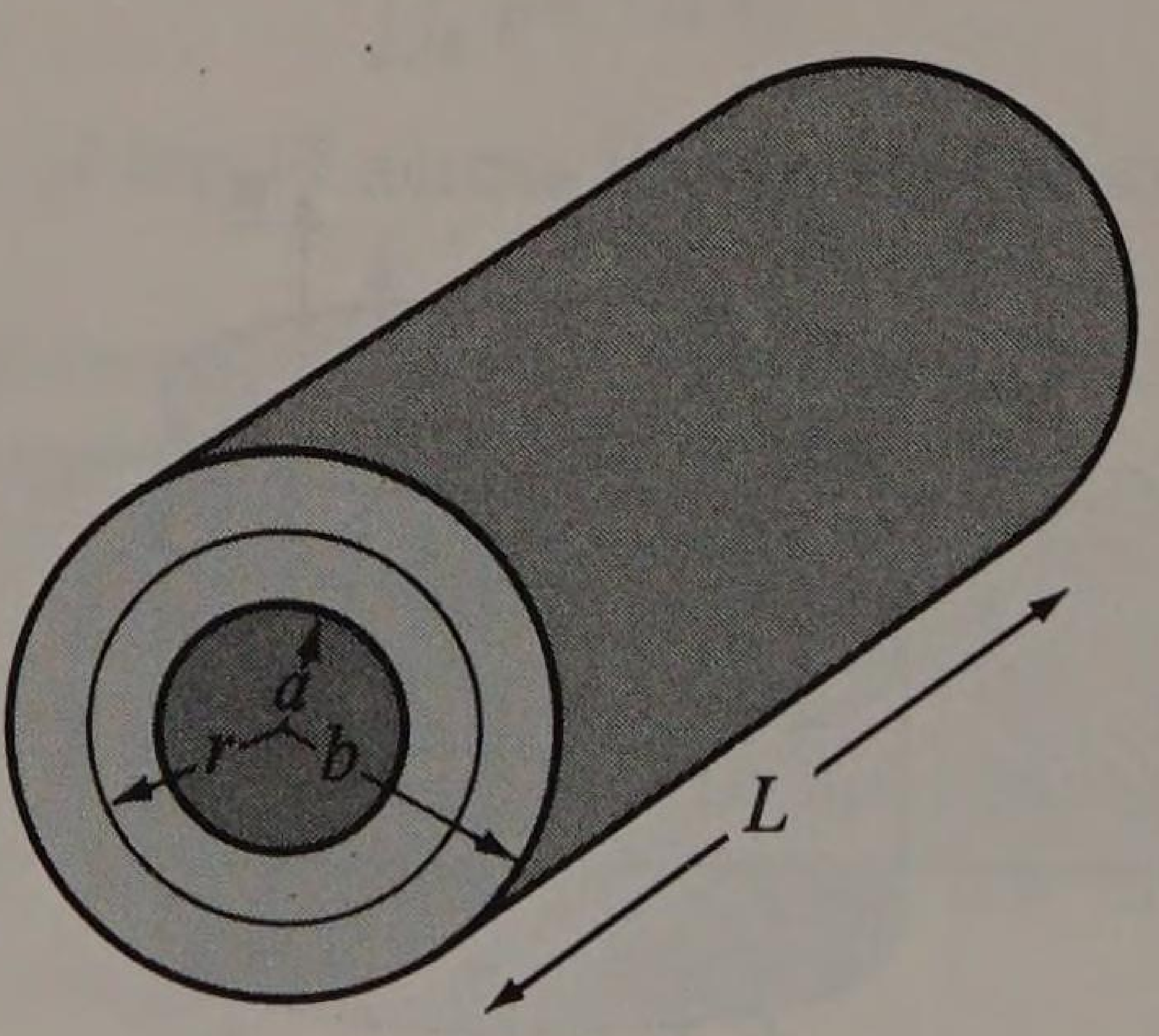

FIGURE 4.23

Here again we are using symmetry considerations in assuming that dT/dr is constant along any one surface S (but not necessarily the same as for surfaces with different r), and so dT/dr can be brought outside the integral sign.

Since H is independent of r, we treat it as a constant in solving the differential equation

$$H = -2\pi kLr\frac{dT}{dr}$$

Separating variables,

$$H\frac{dr}{r} = -2\pi kL\,dT$$

we integrate

$$H\int_a^b \frac{dr}{r} = -2\pi kL\int_{T_a}^{T_b} dT$$

which ultimately yields

$$H = \frac{2\pi Lk(T_a - T_b)}{\ln(b/a)}$$

Substituting this value of H and integrating

$$H\int_a^r \frac{dr}{r} = -2\pi kL\int_{T_a}^{T} dT$$

we finally obtain

$$T = T_a - (T_a - T_b)\frac{\ln(r/a)}{\ln(b/a)}$$

Exercises

1. If $\mathbf{F} = z\mathbf{k}$, find the surface integral of the normal component of $\mathbf{F}$ over the closed surface of the right circular cylinder with curved surface $x^2 + y^2 = 9$ and bases in the planes $z = 0$ and $z = 2$. (Mental arithmetic should suffice.)
2. Compute

$$\iint \mathbf{F}\cdot d\mathbf{S}$$

where S is the surface of the cube bounded by the planes $x = \pm 1$, $y = +1$, $z = \pm 1$, if

(a) $\mathbf{F} = x\mathbf{i}$
(b) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$
(c) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
(d) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$
(e) $\mathbf{F} = y\mathbf{i}$
(f) $\mathbf{F} = z\mathbf{i}$
(g) $\mathbf{F} = z^2\mathbf{i}$

3. Compute the surface integral of the normal component of $\mathbf{F} = x\mathbf{i}$ over the triangle with vertices (1,0,0), (0,2,0), (0,0,3), taking the normal on the side away from the origin.

4. Use Gauss's law to determine the magnitude of the electric field intensity at a point r units away from an infinitely long thin wire carrying a charge of λ units per unit length. (Consider a cylinder of length L and radius r concentric with the wire.)

5. Consider a hollow sphere of homogeneous material, with inner radius a and outer radius b, and inner temperature T_a and outer temperature T_b.

(a) Find the steady-state temperature as a function of the distance r from the center, for values of r between a and b.

(b) For a value of r halfway between a and b, is T halfway between T_a and T_b?

6. Given $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$, find the value of

$$\iint \mathbf{F} \cdot \mathbf{n}\, dS$$

over the closed surface bounded by the planes $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 = a^2$, where $\mathbf{n}$ is the unit outward normal,

(a) by direct calculation (*Hint*: The element of area is

$$dS = a\, d\theta\, dz$$

in cylindrical coordinates on the curved surface.);

(b) by symmetry considerations, without changing to cylindrical coordinates.

7. Given $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (z^2 - 1)\mathbf{k}$, find

$$\iint \mathbf{F} \cdot \mathbf{n}\, dS$$

over the closed surface bounded by the planes $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 = a^2$, where $\mathbf{n}$ is the unit outward normal.

8. Given that $\mathbf{F} = y\mathbf{i} + \mathbf{k}$, find the surface integral of the normal component of $\mathbf{F}$ over the box shown in Fig. 4.24, taking $\mathbf{n}$ to be the unit outward normal. Assume this box to have a bottom but no top, i.e., roughly like a shoe. (*Note*: Later on you will be asked to do the same problem by mental arithmetic, as a demonstration of the power of the divergence theorem. Take a furtive peek ahead at Exercise 7, Sec. 4.9.)

9. Let D be the region $x \geq 0, y \geq 0, z \geq 0, x + \frac{1}{2}y + \frac{1}{3}z \leq 1$.

(a) Is this region a domain?

(b) Is this region simply-connected?

(c) If $\mathbf{F} = 2x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, find the surface integral of the normal component of $\mathbf{F}$ over the boundary of this region, oriented by selecting the outward normal.

10. Calculate $\iint \mathbf{F} \cdot d\mathbf{S}$ over the section of surface described in Exercise 7, Sec. 4.6, for the vector field

$$\mathbf{F} = y\mathbf{i} - x\mathbf{j} + xy\mathbf{k}$$

11. Let us suppose that a field $\mathbf{F}$ due to a point "source" of "strength" q located at a point P has potential ϕ at a point Q given by $\phi(Q) = q/r$, where r is the distance from P to Q. Except when $r = 0$, ϕ is a harmonic function, that is, $\nabla^2\phi = 0$. Now suppose that the source is not concentrated at a single point but is distributed uniformly

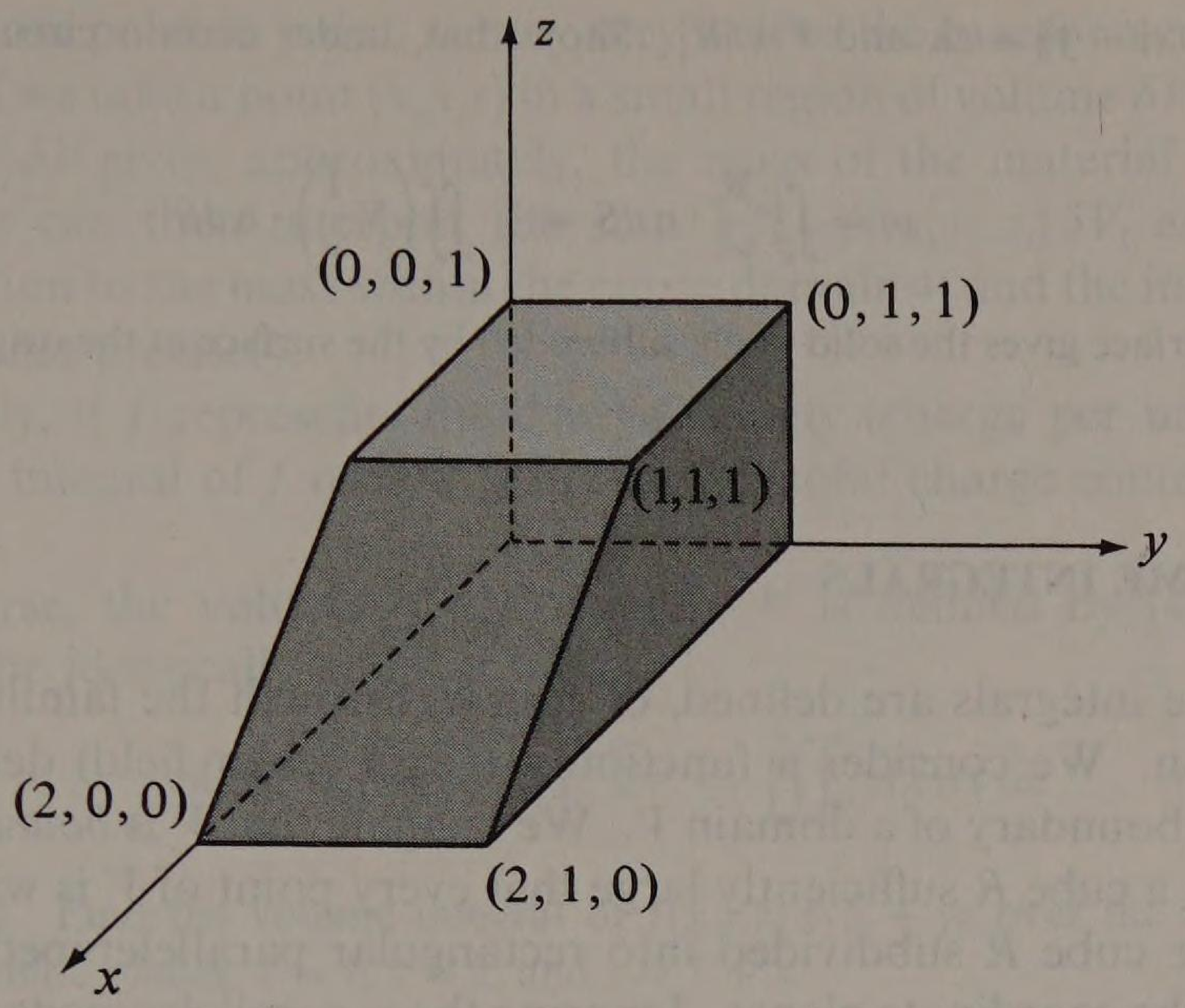

FIGURE 4.24

with density σ (source strength per unit area) over the surface of a sphere of radius a. If Q is the point (x_0, y_0, z_0), the potential at Q must then be found by integration,

$$\phi(x_0, y_0, z_0) = \iint \frac{\sigma\, dS}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}$$

where the integral is over the surface of the sphere. Give a heuristic line of reasoning to show that the potential is

(a) constant within the sphere, equal to $4\pi a\sigma$;

(b) equal to $4\pi a^2\sigma/b$ at any point outside the sphere a distance b from the center of the sphere. (*Hint*: Think of **F** as a scalar multiple of the electric field intensity due to a charge distribution and use Gauss's law.)

12. By interpreting the following integrals as potentials, find their values. Take the surface S to be the sphere $x^2 + y^2 + z^2 = 4$.

(a)
$$\iint_S \frac{dS}{\sqrt{(x-1)^2 + y^2 + z^2}}$$

(b)
$$\iint_S \frac{dS}{\sqrt{(x-3)^2 + y^2 + z^2}}$$

13. Evaluate

$$\iint_S \frac{dS}{\sqrt{(x-3)^2 + (y-2)^2 + z^2}}$$

over the surface $x^2 + y^2 + z^2 = 25$ by interpreting the integral as a potential.

14. Let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{R}|$. Show that, under certain circumstances, the integral

$$\omega = \iint \frac{\mathbf{R}}{r^3} \cdot \mathbf{n}\, dS = -\iint \left(\nabla \frac{1}{r}\right) \cdot \mathbf{n}\, dS$$

over a surface gives the solid angle subtended by the surface at the origin.

4.8 VOLUME INTEGRALS

Volume integrals are defined, of course, through the familiar partition construction. We consider a function f (i.e., a scalar field) defined within and on the boundary of a domain V. We imagine that V is *bounded*, i.e., that there exists a cube R sufficiently large that every point of V is within R. We imagine the cube R subdivided into rectangular parallelepipeds by planes parallel to the coordinate planes. Ignoring those parallelepipeds that contain no points of V, we let the volumes of the parallelepipeds that do overlap V be denoted $\delta V_1, \delta V_2, \ldots, \delta V_n$, and in each parallelepiped select a point (x_i, y_i, z_i) in V. We form the sum $\sum_{i=1}^{n} f(x_i, y_i, z_i)\,\delta V_i$ and define the volume integral of f over V, if it exists, to be

$$\iiint_V f(x,y,z)\, dV = \lim \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\delta V_i \tag{4.40}$$

taking the limit as the dimensions of each volume δV_i tends to zero (which also makes n tend to infinity). For Eq. (4.40) to make sense in an unambiguous way, we require that the limit exist independently of the particular manner of subdivision. It can be shown that this is the case if f is continuous within and on the boundary of V; we omit the proof.

Since the volume of a rectangular parallelepiped with edges dx, dy, and dz is $dV = dx\,dy\,dz$, one sometimes writes

$$\iiint_V f(x,y,z)\, dx\, dy\, dz$$

instead of

$$\iiint_V f(x,y,z)\, dV$$

This suggests, and it can be proved, that a volume integral can be evaluated by triple integration; that is, by successively integrating with respect to x, then y, then z (the obvious extension of a double integral). In fact, a volume integral is almost always evaluated in this way—as an iterated integral. The only tricky part of volume integration is in supplying the limits of the "partial integrals"; we will give examples below.

One obvious application is that in which the function to be integrated is the mass density of a material. Let $\rho(x,y,z)$ denote the mass density of a material, say in grams per cubic centimeter, at a point (x,y,z). If ρ is a constant, the mass of any material occupying a volume δV is precisely $\rho\,\delta V$.

If ρ varies from point to point, as may very well be the case for a compressible fluid, then if we take a point (x, y, z) in a small region of volume δV we can say that $\rho(x,y,z)\,\delta V$ gives, approximately, the mass of the material within this region. We can then interpret the sum $\sum_{i=1}^{n} f(x_i, y_i, z_i)\,\delta V_i$ as giving an approximation to the mass within the entire domain V, and the integral (4.40) gives this mass precisely.

Similarly, if f represents the charge density (charge per unit volume), the volume integral of f over V gives the net total charge contained in the region V.

Of course, the volume of the domain V is defined by (4.40), taking $f(x, y, z)$ to be identically equal to unity:

$$\text{volume of } V = \iiint_V dV = \iiint_V dx\,dy\,dz \tag{4.41}$$

Example 4.22 Find the volume integral of $f(x,y,z) = x + yz$ over the box bounded by the coordinate planes, $x = 1$, $y = 2$, and $z = 1 + x$.

Solution The region is illustrated in Fig. 4.25*a*; it can be described as a "four-walled house with a slant roof." The reader should take the time to identify the sides corresponding to $x = 0$, $y = 0$, $z = 0$, $y = 2$, $z = 1 + x$, and the wall, $x = 1$, which has been "cut away."

Let us get the limits of integration. Consider a typical point (x,y,z) somewhere in the middle of the region. If we hold x and y fixed, z can slide down to 0 (the floor) and up to $1 + x$ (the slant roof), tracing out a *column*. If we now let x vary and hold y fixed, the columns trace out a *slice*; x goes back and forth from 0 to 1. The columns and slice are depicted in Fig. 4.25*b*. Now if we vary y from 0 to 2, these slices stack up and fill out the region. Integrating in this order we obtain

$$\begin{aligned}\iiint f\,dz\,dx\,dy &= \int_0^2 \int_0^1 \int_0^{1+x} (x + yz)\,dz\,dx\,dy \\ &= \int_0^2 \int_0^1 \left(xz + \frac{1}{2}yz^2\right)\Big|_0^{1+x} dx\,dy \\ &= \int_0^2 \int_0^1 \left(x + x^2 + \frac{1}{2}y + yx + \frac{1}{2}yx^2\right) dx\,dy \\ &= \int_0^2 \left(\frac{5}{6} + \frac{7}{6}y\right) dy = 4\end{aligned}$$

An alternate way of choosing limits arises if we let y vary before x. Then the vertical columns would trace out a slice from left to right, for $0 \le y \le 2$, and the slices would stack up as x goes from 0 to 1. This produces

$$\begin{aligned}\int_0^1 \int_0^2 \int_0^{1+x} (x + yz)\,dz\,dy\,dx &= \int_0^1 \int_0^2 \left(x + x^2 + \frac{1}{2}y + yx + \frac{1}{2}yx^2\right) dy\,dx \\ &= \int_0^1 \left(xy + x^2y + \frac{1}{4}y^2 + \frac{1}{2}y^2x + \frac{1}{4}y^2x^2\right)\Big|_0^2 dx \\ &= \int_0^1 (1 + 4x + 3x^2)\,dx = 4\end{aligned}$$

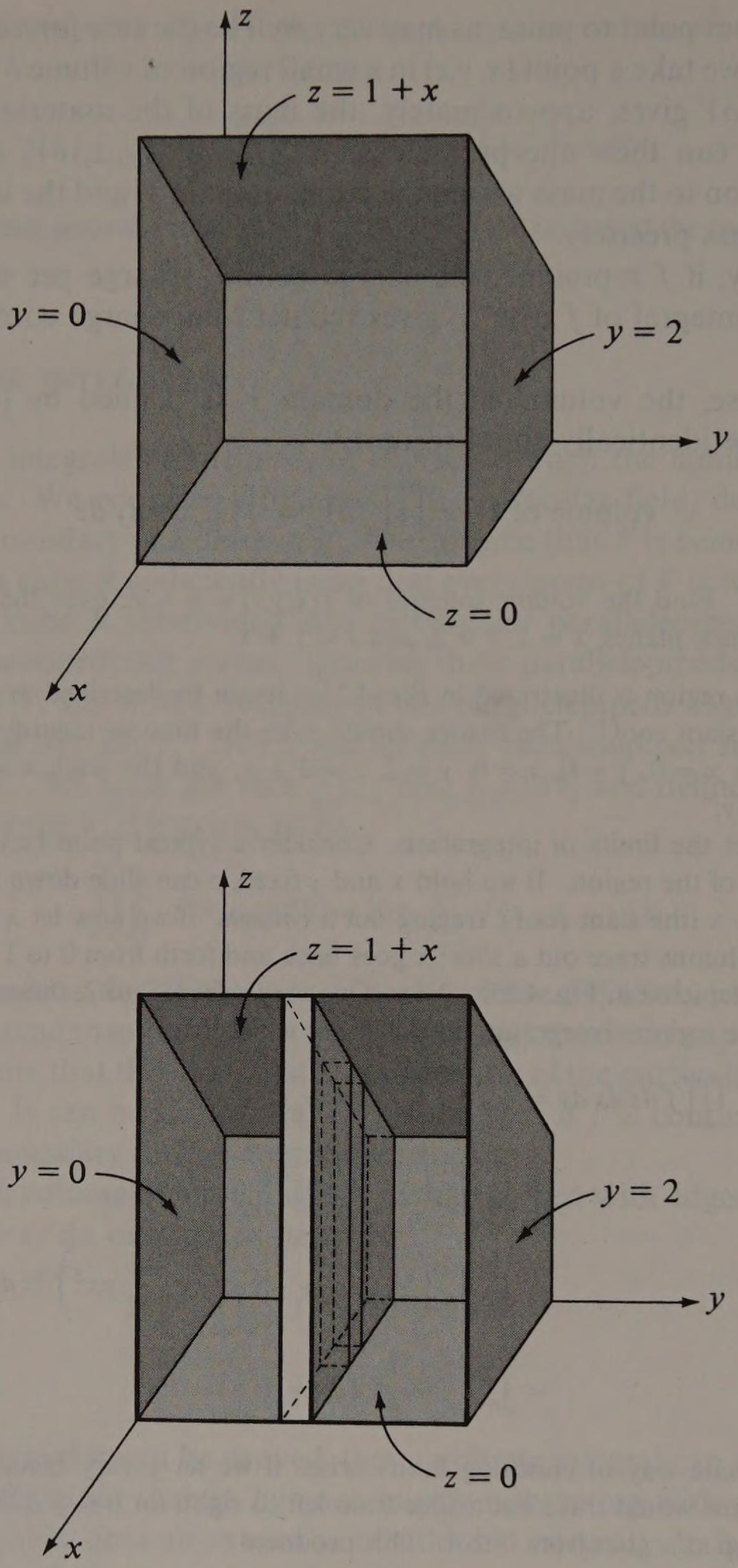
z
z = 1 + x
y = 0
y = 2
y
z = 0
x
z
z = 1 + x
y = 0
y = 2
y
z = 0
x

FIGURE 4.25

Suppose, starting from our "typical point in the middle" (x,y,z), we let y vary first, holding x and z constant. Then we would trace out a horizontal column from left $(y = 0)$ to right $(y = 2)$. If we next vary z, holding x constant, these columns fill out a slice parallel to the yz plane, running from the floor $(z = 0)$ to the ceiling $(z = 1 + x)$. Stacking these slices for x between 0 and 1 fills out the region and we have

$$\begin{aligned}\int_0^1 \int_0^{1+x} \int_0^2 (x + yz)\,dy\,dz\,dx &= \int_0^1 \int_0^{1+x} \left(xy + \frac{1}{2}y^2z\right)\Big|_0^2 dz\,dx \\ &= \int_0^1 \int_0^{1+x} (2x + 2z)\,dz\,dx \\ &= \int_0^1 (2xz + z^2)\Big|_0^{1+x} dx \\ &= \int_0^1 (1 + 4x + 3x^2)\,dx = 4\end{aligned}$$

What if, starting from our typical point in the middle, we let x vary first, holding y and z constant, which produces columns coming "out of the page" in Fig. 4.25? Then we have a complication. In the main part of the house, $z \le 1$, the column runs from the back wall $(x = 0)$ to the front $(x = 1)$, but in the "attic", $z \ge 1$, the column only goes back to the slant roof, where $x = z - 1$. (Where does this equation come from?) Thus we have to break up the region, below and above the level $z = 1$, in order to establish consistent limits of integration. In both sections the columns can run left to right, $0 \le y \le 2$, generating horizontal slices that can be stacked, $0 \le z \le 1$ in the lower part and $1 \le z \le 2$ in the attic. This produces

$$\begin{aligned}&\int_0^1 \int_0^2 \int_0^1 (x + yz)\,dx\,dy\,dz + \int_1^2 \int_0^2 \int_{z-1}^1 (x + yz)\,dx\,dy\,dz \\ &= \int_0^1 \int_0^2 \left(\frac{1}{2} + yz\right) dy\,dz + \int_1^2 \int_0^2 \left(2yz - \frac{1}{2}z^2 + z - yz^2\right) dy\,dz \\ &= \int_0^1 (1 + 2z)\,dz + \int_1^2 (6z - 3z^2)\,dz = 2 + 2 = 4\end{aligned}$$

The above example demonstrates that volume integrals can be iterated in any order, but that some orders may be more complicated than others.

Example 4.23 Find the volume of the region of space above the xy plane and beneath the plane $z = 2 + x + y$, bounded by the planes $y = 0$, $x = 0$, and the surface $y = 1 - x^2$.

Solution Let us try to visualize the region. Its base, in the xy plane, is shown in Fig. 4.26*a*. Each of the shaded points is the base of a column reaching up to the "slanted roof", $z = 2 + x + y$ (Fig. 4.26*b*).

To find the limits of integration, we start with the "typical point in the middle". If we let z vary first, with x and y fixed, we trace out a vertical column from $z = 0$ to $z = 2 + x + y$. If, instead, we hold y and z fixed and vary x, the horizontal columns are troublesome; they run from 0 to $(1 - y)^{\frac{1}{2}}$ throughout most of the region, but in the upper right-hand corner (Fig. 4.26*b*) there is an "attic" where the slant roof cuts off the columns, on their left end. The same happens if we vary y first. Taking the easy way out, we choose the vertical columns.

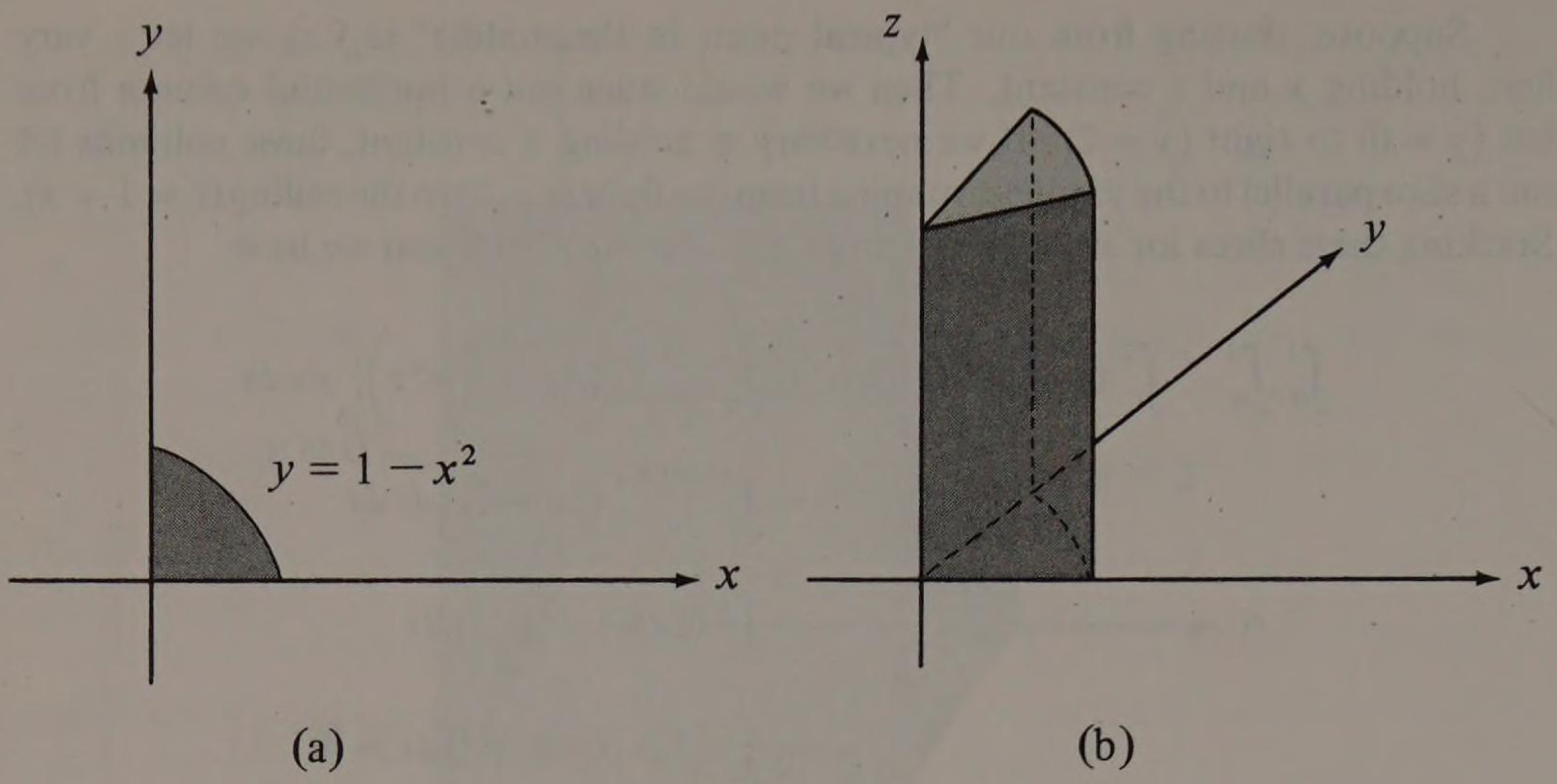

FIGURE 4.26

If we hold x fixed and vary y, the columns generate slices from $y = 0$ to $y = 1 - x^2$. Stacking up these slices for $0 \le x \le 1$, we find

$$\begin{aligned}\int_0^1 \int_0^{1-x^2} \int_0^{2+x+y} dz\,dy\,dx &= \int_0^1 \int_0^{1-x^2} (2 + x + y)\,dy\,dx \\ &= \int_0^1 \left(2y + xy + \frac{1}{2}y^2\right)\Bigg|_0^{1-x^2} dx \\ &= \int_0^1 \left(\frac{5}{2} + x - 3x^2 - x^3 + \frac{1}{2}x^4\right) dx \\ &= \frac{37}{20}\end{aligned}$$

Example 4.24 Find the integral of $f(x,y,z) = y$ over the volume of the sphere contained inside $x^2 + y^2 + z^2 = 1$.

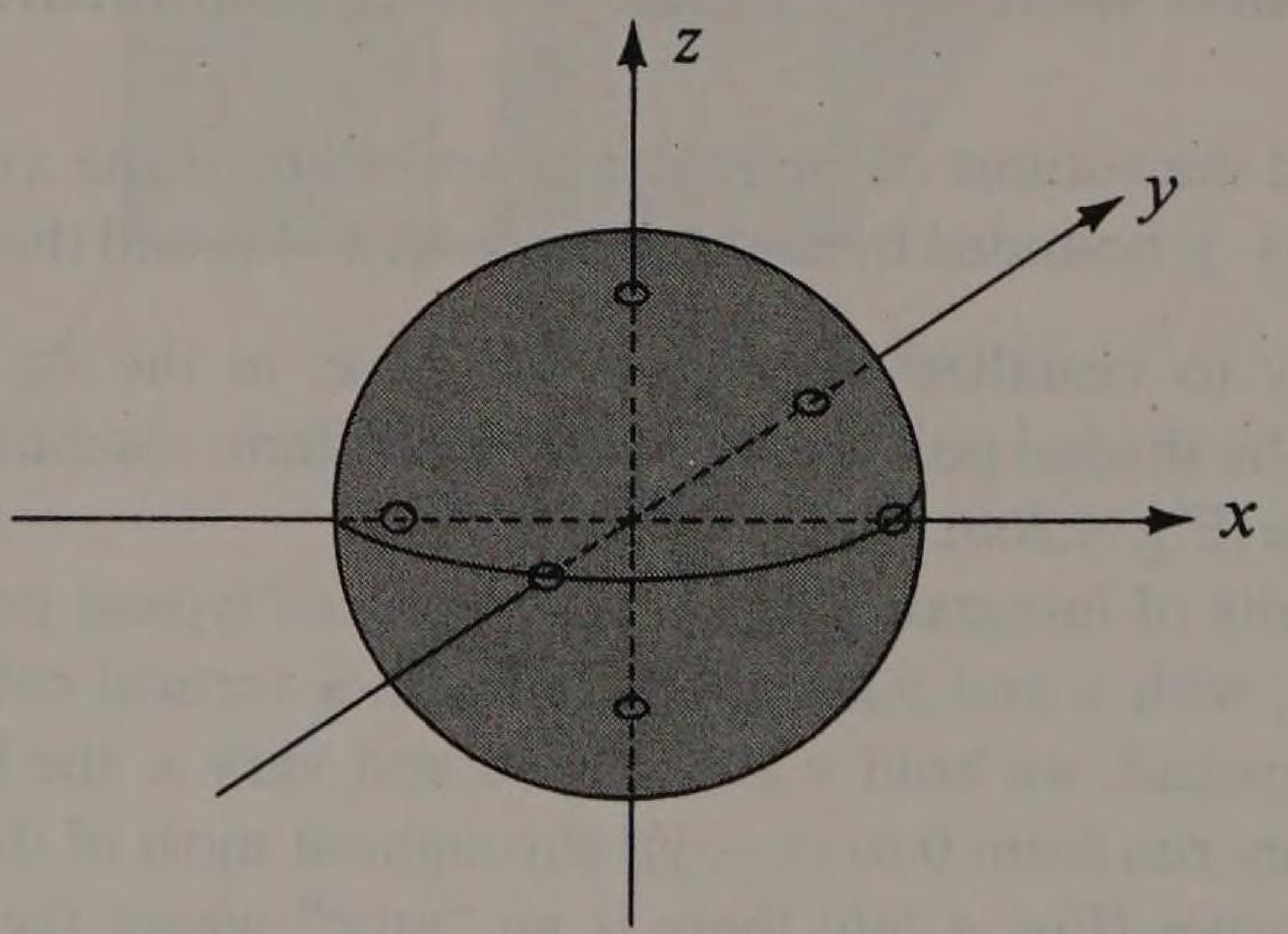

FIGURE 4.27

Solution Obviously, for a sphere we can integrate equally well in any order. From Fig. 4.27, we see that if we fix x and y, z traces out a column between the limits $\pm(1 - x^2 - y^2)^{\frac{1}{2}}$. These columns can be slid in the y direction, for fixed x, between the limits $y = \pm(1 - x^2)^{\frac{1}{2}}$, generating slices that can be stacked from $x = -1$ to $x = +1$. Hence

$$\begin{aligned}\iiint f\,dV &= \int_{-1}^{1}\int_{-(1-x^2)^{\frac{1}{2}}}^{+(1-x^2)^{\frac{1}{2}}}\int_{-(1-x^2-y^2)^{\frac{1}{2}}}^{+(1-x^2-y^2)^{\frac{1}{2}}} y\,dz\,dy\,dx\\ &= 2\int_{-1}^{1}\int_{-(1-x^2)^{\frac{1}{2}}}^{+(1-x^2)^{\frac{1}{2}}} y(1 - x^2 - y^2)^{\frac{1}{2}}\,dy\,dx\\ &= 2\int_{-1}^{1} 0\,dx = 0\end{aligned}$$

(We might have anticipated this answer by symmetry.)

Exercises

1. Compute the volume of the sphere of radius R by iterated integrals.

2. The volume of the region described in Example 4.22 equals 3. Verify this four times by repeating each of the integrations given in Example 4.22, taking $f(x,y,z) = 1$ instead of $f(x,y,z) = x + yz$.

3. Sketch the region whose volume is represented by the triple integral

$$\int_0^2\int_0^3\int_0^{\sqrt{9-y^2}} dx\,dy\,dz$$

4. In this exercise you will be asked to make a simple conjecture on the basis of carrying out the following computations.

(a) Let $\mathbf{F}(x,y,z) = x^2\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Compute

$$\iint_S \mathbf{F}\cdot d\mathbf{S}$$

over the surface of the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$ (Fig. 4.28).

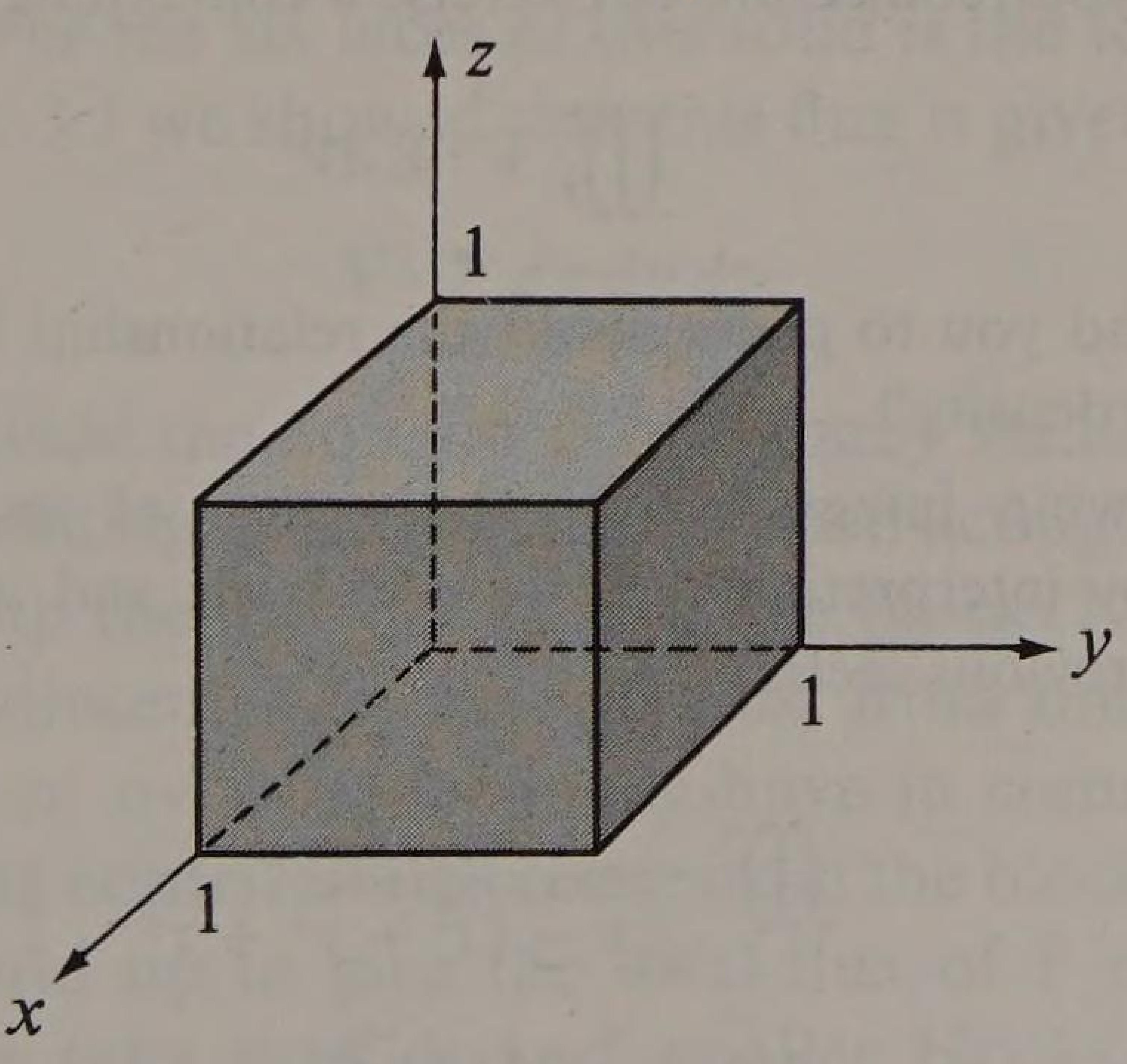

FIGURE 4.28

(b) Let $f(x,y,z) = \nabla \cdot \mathbf{F}$, and compute

$$\iiint_V f(x,y,z)\,dV$$

over the cube. Notice that here limits are no problem; we have simply

$$\int_0^1 \int_0^1 \int_0^1 f(x,y,z)\,dx\,dy\,dz$$

(c) If your answers to (a) and (b) are not equal, check your work until you find the mistake.

(d) Now invent another vector field $\mathbf{F}$ and repeat steps (a) and (b).

(e) What do you conjecture from this?

⑤ Let V be a domain with volume v. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(a) What is

$$\iiint_V \nabla \cdot \mathbf{F}\,dV$$

(b) On the basis of your answer to Exercise 4, what do you conjecture is the value of

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

the surface integral of the normal component of $\mathbf{F}$ over the boundary of V?

⑥ Find the volume of the region bounded by the surface $z = e^{-(x^2+y^2)}$, the cylinder $x^2 + y^2 = 1$, and the plane $z = 0$. (*Hint*: In cylindrical coordinates, $dV = r\,dr\,d\theta\,dz$.)

7. If $\rho(x,y,z)$ denotes the *charge density* (charge per unit volume) in a region of space, then the total charge in this region V is

$$q = \iiint_V \rho(x,y,z)\,dV$$

By Gauss's law we have

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} q = \frac{1}{\varepsilon_0} \iiint_V \rho(x,y,z)\,dV$$

Combine this with your conjecture of Exercise 4 concerning

$$\iiint_V \nabla \cdot \mathbf{E}\,dV$$

What might this lead you to guess about the relationship between the divergence of $\mathbf{E}$ and the charge density?

8. Evaluate the following integrals over the region of space within the sphere $x^2 + y^2 + z^2 = 4$, by interpreting them as potentials and using Gauss's law. (See Exercise 12 of the previous section.)

(a) $$\iiint \frac{dx\,dy\,dz}{\sqrt{(x-1)^2 + y^2 + z^2}}$$

(b) $$\iiint \frac{dx\,dy\,dz}{\sqrt{(x-3)^2 + y^2 + z^2}}$$

9. Evaluate

$$\iiint \frac{dx\,dy\,dz}{\sqrt{(x-9)^2+(y-2)^2+z^2}}$$

where the integral extends over the interior of the sphere $x^2 + y^2 + z^2 = 4$, by interpreting the integral as a potential.

4.9 INTRODUCTION TO THE DIVERGENCE THEOREM AND STOKES' THEOREM

With these preliminaries on integration completed, we can now turn to the interesting part of our work. In this section we introduce two theorems of fundamental importance in vector analysis; most of our work so far has been intended as preliminary to these two theorems. They will be stated more precisely in later sections; here we intend to state them in crude form, without giving the precise conditions on continuity, differentiability, etc., and we will give proofs for the theorems that are instructive but quite nonrigorous. In later sections more careful proofs will be given. First we present the divergence theorem.

THEOREM 4.5 *The volume integral of the divergence* of a vector field, *taken throughout a bounded domain D, equals the surface integral of the normal component of the vector field taken over the boundary of D.* In other words, the total divergence within D equals the net flux emerging from D.

Here is a "simplified proof"; a rigorous proof will be given later.

Proof First let us consider a small rectangular parallelepiped bounded by planes of constant x, $x + dx$, y, $y + dy$, z, and $z + dz$. The surface integral of $\mathbf{F} \cdot \mathbf{n}$ over the six faces of this solid is the total flux of $\mathbf{F}$ out of the box. In Sec. 3.3 we showed that this flux is given, in the limit, by

$$\nabla \cdot \mathbf{F}\,dx\,dy\,dz$$

Now let us divide the domain D into many small parallelepipeds, as if they were building blocks used in constructing D. What do we obtain if we sum up the flux out of all these blocks? If two such parallelepipeds are adjacent, the flux outward from one equals the flux inward to the other, over the face they have in common. Hence the only non-cancelling contributions come from the blocks on the surface, and these terms add up to give the total flux of $\mathbf{F}$ out of the "brick structure". As we take smaller and smaller blocks, we expect that $\sum \nabla \cdot \mathbf{F}\,\delta V$ approaches the volume integral, and the flux out of the structure approaches the flux out of D. Hence,

$$\iiint \nabla \cdot \mathbf{F}\, dV = \iint \mathbf{F} \cdot d\mathbf{S}$$

There are some obvious weaknesses in this proof. The passing to the limits must be considered much more carefully, particularly with regard to the surface integral. For example, it is not clear that one is justified in approximating, say, a spherical surface by a collection of little rectangles parallel to the coordinate planes (recall that in Fig. 4.15 the rectangular patch is taken *tangent to the surface*). These heuristic arguments, however, are very valuable in helping one to recall the theorems and conjecture new approaches.

The divergence theorem is sometimes called Gauss's theorem, because of its close relationship to Gauss's law (Sec. 4.7). To see the connection, it is necessary to know that the divergence of electric field intensity is a scalar multiple of the charge density. Hence the volume integral of the divergence over any domain gives a scalar multiple of the total charge q within the domain. It follows from the divergence theorem that the surface integral of the normal component of the electric intensity, over the boundary of a domain, is a scalar multiple of the charge inside. However, Gauss's law is not just a special case of the divergence theorem, since it can be applied to point charges where the concept of charge per unit volume, in the ordinary sense, is meaningless.

Until some years ago, the divergence theorem was called Green's theorem in three dimensions.

Now let us turn to Stokes' theorem, the other fundamental theorem in vector analysis.

THEOREM 4.6 *The surface integral of the normal component of the curl of a vector field, taken over a bounded surface, equals the line integral of the tangential component of the field, taken over the closed curve bounding the surface.*

Here we are considering a closed curve C in space and a surface S that is bounded by the curve. The theorem states that

$$\iint_S (\mathbf{curl\ F}) \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \tag{4.42}$$

where dS refers to the element of *area* and ds refers to *arc length*. We assume that S is a surface oriented by a field of unit normals $\mathbf{n}$, and that the line integral is taken along C in the direction determined positive by the orientation. (See Fig. 4.29*a*.)

The "proof" we give is as follows (a more rigorous proof will be given later).

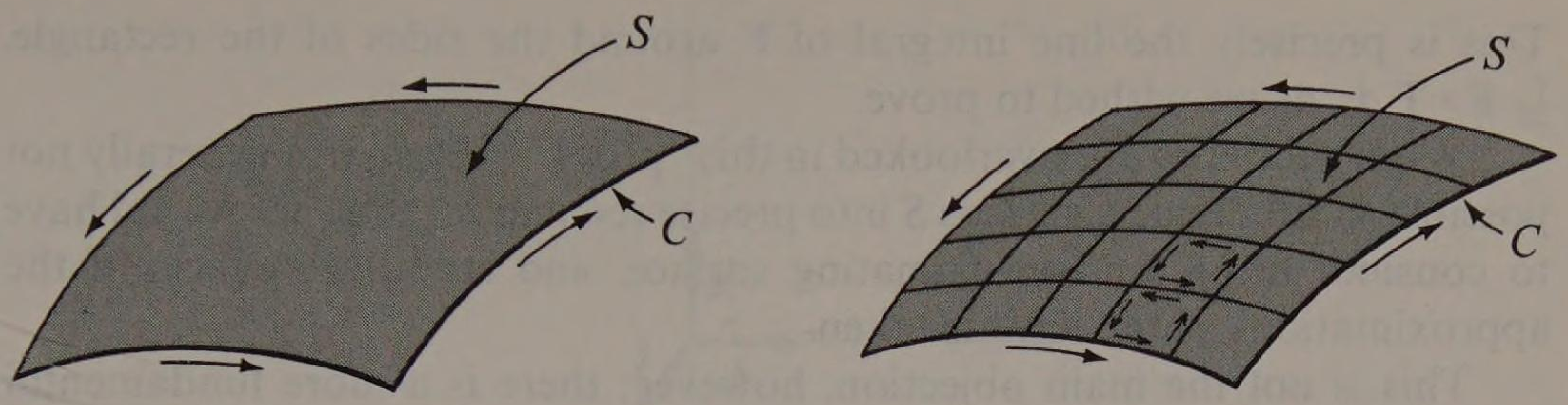

FIGURE 4.29

Proof We shall employ a technique similar to that used in the "proof" of the divergence theorem. Consider the surface divided up into small elements, each approximately rectangular (Fig. 4.29*b*). The flux through S equals the sum of the fluxes through the rectangles. Also, if we add up the line integrals around each rectangle, we obtain cancellations over all the internal boundaries, and the sum equals the line integral around C. Thus, if we can prove Eq. (4.42) for one rectangle, we have it in general.

To prove (4.42) for a small rectangular area, we choose the coordinate axes so that the x and y axes are along the sides of the rectangle and the z axis is in the direction of $\mathbf{n}$. We then have $\mathbf{n} = \mathbf{k}$; hence

$$(\textbf{curl F}) \cdot \mathbf{n} = (\textbf{curl F}) \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Therefore the left side of (4.42) is

$$\iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy,$$

with $0 \leq x \leq a, 0 \leq y \leq b$. We split this up into two integrals, choosing the order of integration differently in the two cases:

$$\begin{aligned}
&\int_0^b \int_0^a \frac{\partial F_2}{\partial x}\,dx\,dy - \int_0^a \int_0^b \frac{\partial F_1}{\partial y}\,dy\,dx \\
&\quad = \int_0^b F_2(x,y)\Big|_{x=0}^{x=a} dy - \int_0^a F_1(x,y)\Big|_{y=0}^{y=b} dx \\
&\quad = \int_0^b [F_2(a,y) - F_2(0,y)]\,dy - \int_0^a (F_1(x,b) - F_1(x,0)]\,dx \\
&\quad = \int_0^b F_2(a,y)\,dy + \int_b^0 F_2(0,y)\,dy + \int_a^0 F_1(x,b)\,dx + \int_0^a F_1(x,0)\,dx \\
&\quad = \int_0^a F_1(x,0)\,dx + \int_0^b F_2(a,y)\,dy + \int_a^0 F_1(x,b)\,dx + \int_b^0 F_2(0,y)\,dy
\end{aligned}$$

This is precisely the line integral of **F** around the sides of the rectangle, $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, as we wished to prove.

What subtleties are overlooked in this "proof"? First, it is generally not possible to subdivide a surface S into precise rectangles; thus, we would have to consider using an approximating surface, and study the effects of the approximations as the limit is taken.

This is not the main objection, however; there is a more fundamental one, involved in choosing the coordinate axes so that x and y axes are along two sides of the rectangle. Let us analyze this more closely.

Suppose, for the sake of argument, that the surface S *is* a rectangle. Then the first objection above does not apply, we do not need to chop up S at all. If it happens that S is already in the xy plane, lying along the x and y axes as shown in Fig. 4.30, then there is no objection to the proof given above (provided we assume continuity of the relevant partial derivatives, etc., so that the integrals exist). But suppose S is not in the xy plane. The above procedure amounts to choosing a new set of coordinates x', y', z' so that S is in the plane $z' = 0$ and has sides along the x' and y' axes, and the above argument shows that Eq. (4.42) is valid when we compute everything relative to the coordinates x', y', and z'. *But how do we know that* (4.42) *is valid relative to the original coordinates* x, y, and z? This is a serious objection, because we have defined **curl F** in terms of a fixed set of coordinates, and we have not yet studied what happens when we change to another set of coordinates.

Let us be very explicit about this, because it is conceptually very important. Let us suppose we are given a vector field **F** in terms of coordinates x, y, and z. Suppose now we are given new coordinates x', y', and z', which we can express as functions of the old coordinates x, y, and z. Substituting into $\mathbf{F}(x,y,z)$, we can now write **F** in terms of x', y', and z'. Now let us compute **curl F** in terms of x', y', and z', and afterward change back to x, y, and z, so that we have **curl F** in terms of x, y, and z. The question is, do we get the same thing as we would get computing **curl F** directly, from the very beginning, in terms of x, y, and z?

In other words, does the curl of a vector field depend only on the nature of the field, or does it also depend on our particular choice of coordinate axes?

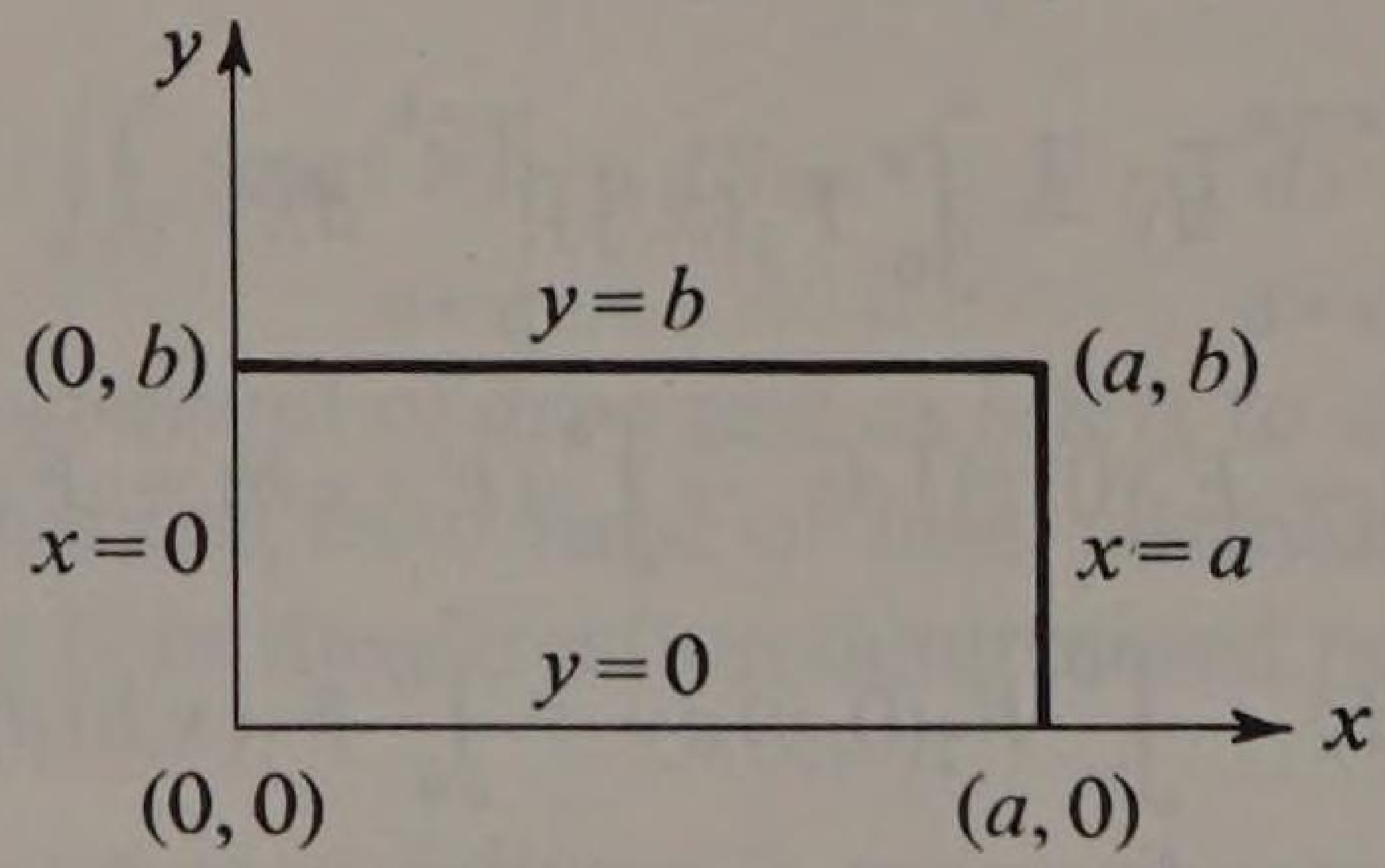

FIGURE 4.30

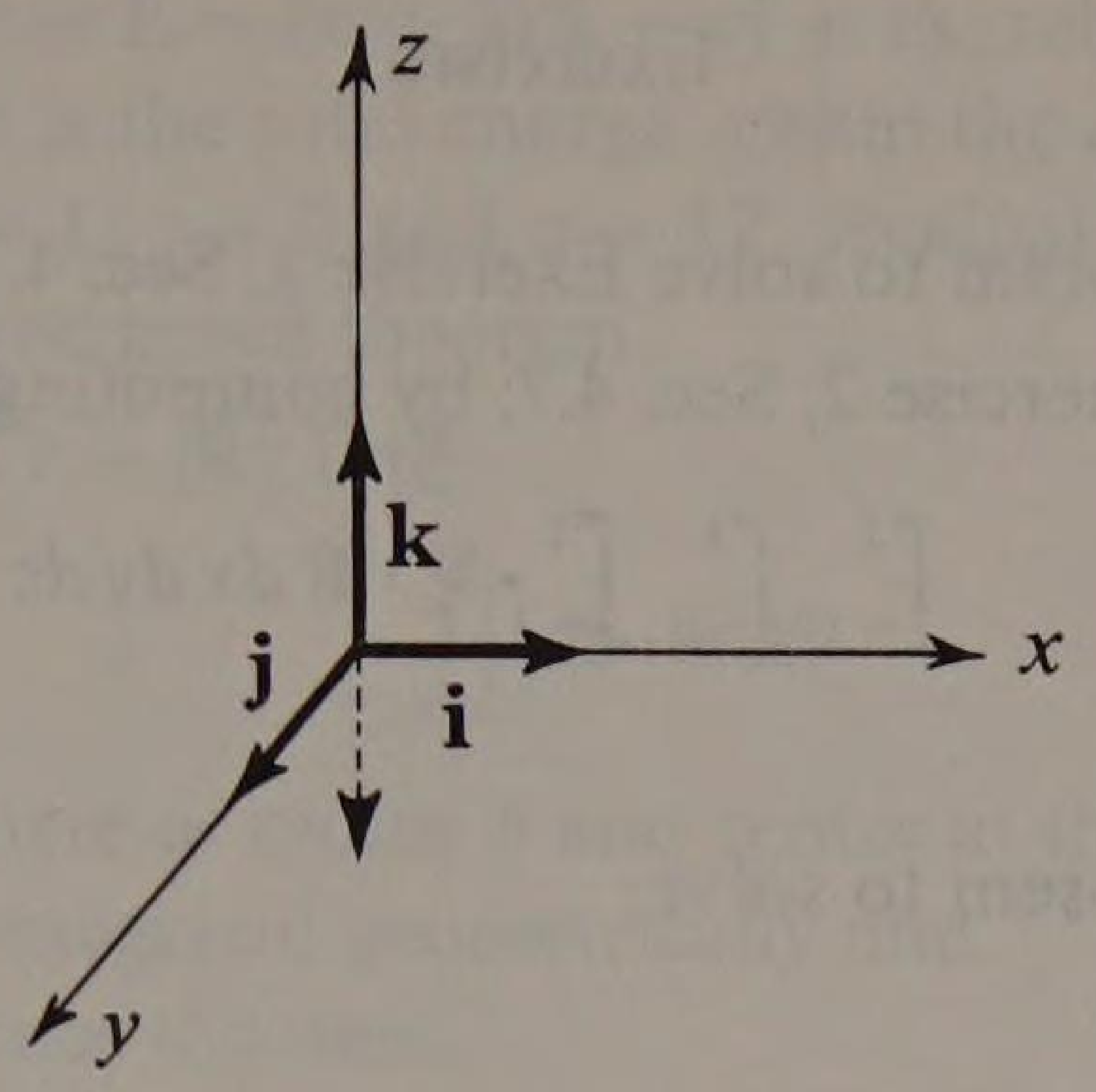

FIGURE 4.31

We shall show in the next chapter that the curl does not depend on the choice of coordinate axes provided (*i*) that we always choose axes that are mutually perpendicular, (*ii*) that we are consistent in the way we mark off distances on these axes (physically, this means that we select some unit of distance, say centimeters, and mark all axes so that distances come out in centimeters), (*iii*) and that we always take a right-handed coordinate system, i.e., one for which $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

In fact, we should expect this to be true; after all, we saw in Sec. 3.4 that the curl has a "coordinate-free" interpretation, as the local angular velocity of a fluid. Similarly, the divergence measures the rate of change of density of a compressible fluid (Sec. 3.3), and the gradient gives the direction and magnitude of the maximum rate of change of a scalar. Therefore it should not be surprising that these three quantities behave in an "invariant" manner under coordinate transformations.

In this connection, it is worth mentioning that some textbooks consistently use left-handed coordinates, as in Fig. 4.31. Then either $\mathbf{i} \times \mathbf{j} = -\mathbf{k}$ or the definition of vector cross products must be modified to give $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, in which case our "right-hand rule" (Sec. 1.12) becomes a "left-hand rule". (This convention has an obvious advantage for a right-handed student who is pressed for time when taking an examination; he need not put his pencil down when applying the left-hand rule, since his left hand is free.)

We shall return later on to this important matter of coordinate transformations.

Summarizing, we have rough statements of the divergence theorem and Stokes' theorem, and have given instructive but incorrect proofs of both theorems. Before proceeding to a more careful analysis the reader is strongly urged to study the following exercises. These are the most important theorems in this book and they must ultimately be thoroughly understood.

The student who has studied attentively up to this point is "over the hump". *The rest of this book is devoted entirely to the deeper study of concepts already introduced.*

Exercises

1. Use the divergence theorem to solve Exercise 1, Sec. 4.7.
2. Do all seven parts of Exercise 2, Sec. 4.7, by computing

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \nabla \cdot \mathbf{F}\, dx\, dy\, dz$$

in each case.

3. Use the divergence theorem to solve:
   (a) Exercise 6, Sec. 4.7;
   (b) Exercise 7, Sec. 4.7.
4. Use Stokes' theorem to solve Exercise 8, Sec. 4.1.
5. Use Stokes' theorem to solve Exercise 10, Sec. 4.1.
6. Verify Stokes' theorem in the following special cases. Let C be the square in the xy plane with equation $|x| + |y| = 1$. Let $\mathbf{F}$ be as follows:
   (a) $\mathbf{F} = x\mathbf{i}$
   (b) $\mathbf{F} = y\mathbf{i}$
   (c) $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$
   (d) $\mathbf{F} = \mathbf{i} + \mathbf{j}$
   (e) $\mathbf{F} = y^3\mathbf{i}$
7. Despite the fact that the surface of Exercise 8, Sec. 4.7, is not closed, the divergence theorem can be used to reduce this to a problem in mental arithmetic. Show how to do this.
8. The moment of inertia about the z axis of a uniform solid is proportional to

$$\iiint_V (x^2 + y^2)\, dx\, dy\, dz$$

Express this as the flux of some vector field through the surface of the body.

9. One can compute the volume of a room by calculating the flux of the vector $\mathbf{R}$ through the walls. Show this.
10. By means of Stokes' theorem, find

$$\int \mathbf{F} \cdot d\mathbf{R}$$

around the ellipse $x^2 + y^2 = 1$, $z = y$, where

$$\mathbf{F} = x\mathbf{i} + (x + y)\mathbf{j} + (x + y + z)\mathbf{k}$$

11. The abstract concept of a gooney sphere is derived from the shape of a gooney egg. A gooney bird is born with a pointed head and a prominent stubby tail; therefore the shape of the egg is roughly ellipsoidal but with pointed ends. Surface integrals over gooney spheres are difficult to compute; tables of gooney functions are needed, but these were tabulated during the war and are still classified top secret. All that is known is that a gooney sphere of minimal diameter $d = 1$ has volume approximately 0.7. (a) Find the surface integral of the normal component of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the surface of a gooney sphere with center at the origin and minimal diameter $d = 2$, making any assumptions you deem reasonable. (b) Would your answer be the same if the gooney sphere had center at $(2,7,-3)$?

12. If electric field intensity is $\mathbf{E} = (x + 1)^2\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, relevant to suitable choices of the units involved, what is the total charge within the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$? Evaluate the left side of Eq. (4.39), (a) directly, (b) by the divergence theorem.

13. If $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{R}|$, find

$$\iint r\mathbf{R} \cdot \mathbf{n}\, dS$$

over the surface of a sphere of radius b and center at the origin,
(a) by interpreting the integrand geometrically and
(b) by using the divergence theorem.

14. Given

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2}$$

find the surface integral of the normal component of $\mathbf{F}$ over the surface of the sphere $x^2 + y^2 + z^2 = 4$. Can you use the divergence theorem?

15. Stokes' theorem provides an interesting interpretation of Theorem 4.3, which identifies irrotational fields with conservative fields in simply connected domains. Show that if $\mathbf{curl\ F} = \mathbf{0}$, then the line integral of $\mathbf{F}$ around any closed curve that bounds an oriented surface in the domain is zero. Where does simple connectedness come into play?

16. Given $\phi(x,y,z) = xyz + 5$, find the surface integral of the normal component of $\mathbf{grad}\ \phi$ over $x^2 + y^2 + z^2 = 9$.

17. (a) Show that, if ϕ is harmonic, $\boldsymbol{\nabla} \cdot (\phi\boldsymbol{\nabla}\phi) = |\boldsymbol{\nabla}\phi|^2$.
(b) Given $\phi = 3x + 2y + 4z$, evaluate

$$\iint \phi \frac{\partial \phi}{\partial n}\, dS$$

over the surface $x^2 + y^2 + z^2 = 4$. Here, $\partial\phi/\partial n$ represents the normal derivative of ϕ, that is, $\mathbf{n} \cdot \boldsymbol{\nabla}\phi$.

18. Let $\mathbf{F} = \phi\boldsymbol{\nabla}\phi$. Find the surface integral of the normal component of $\mathbf{F}$ over the surface of a sphere of radius 3 and center at the origin,
(a) if $\phi = x + y + z$;
(b) if $\phi = x^2 + y^2 + z^2$.

4.10 THE DIVERGENCE THEOREM

As previously promised, we shall now delve into a more careful, detailed analysis of the divergence theorem. To fix ideas for the moment, let us consider a vector field

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$$

defined throughout a region, with components F_1, F_2, F_3 having continuous partial derivatives in this region. Let S denote the surface of a *sphere*, located within the region, and let D denote the set of points within S. Let $\mathbf{n}$ denote the field of unit vectors normal to S. At each point on S, we take $\mathbf{n}$ to be the *outward* normal, thus orienting S in the conventional way.

Consider the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS \tag{4.43}$$

Written out in terms of its components, this becomes

$$\iint_S (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \cdot \mathbf{n}\, dS \tag{4.44}$$

which equals

$$\iint_S F_1(\mathbf{n} \cdot \mathbf{i})\, dS + \iint_S F_2(\mathbf{n} \cdot \mathbf{j})\, dS + \iint_S F_3(\mathbf{n} \cdot \mathbf{k})\, dS \tag{4.45}$$

Let us concentrate on only one of these integrals, the one in the middle. Consider the sphere to be cut up into filaments, each parallel to the y axis. A typical filament is shown in Fig. 4.32. It has cross-sectional area $dx\, dz$ and cuts out two portions from S, having areas $\delta S'$ and $\delta S''$. The contribution to the middle integral of the two portions is approximately

$$F_2'(\mathbf{n}' \cdot \mathbf{j})\, \delta S' + F_2''(\mathbf{n}'' \cdot \mathbf{j})\, \delta S''$$

where F_2' and F_2'' are, respectively, values of F_2 at points on the two portions. By the area cosine principle (Sec. 4.6), $(\mathbf{n}'' \cdot \mathbf{j})\, \delta S''$ and $(\mathbf{n}' \cdot \mathbf{j})\, \delta S'$ are approximately equal to $dx\, dz$ and $-(dx\, dz)$ respectively, since the scalar product of two unit vectors equals the cosine of the angle between them. Therefore, the contribution from these two portions is $(F_2'' - F_2')\, dx\, dz$.

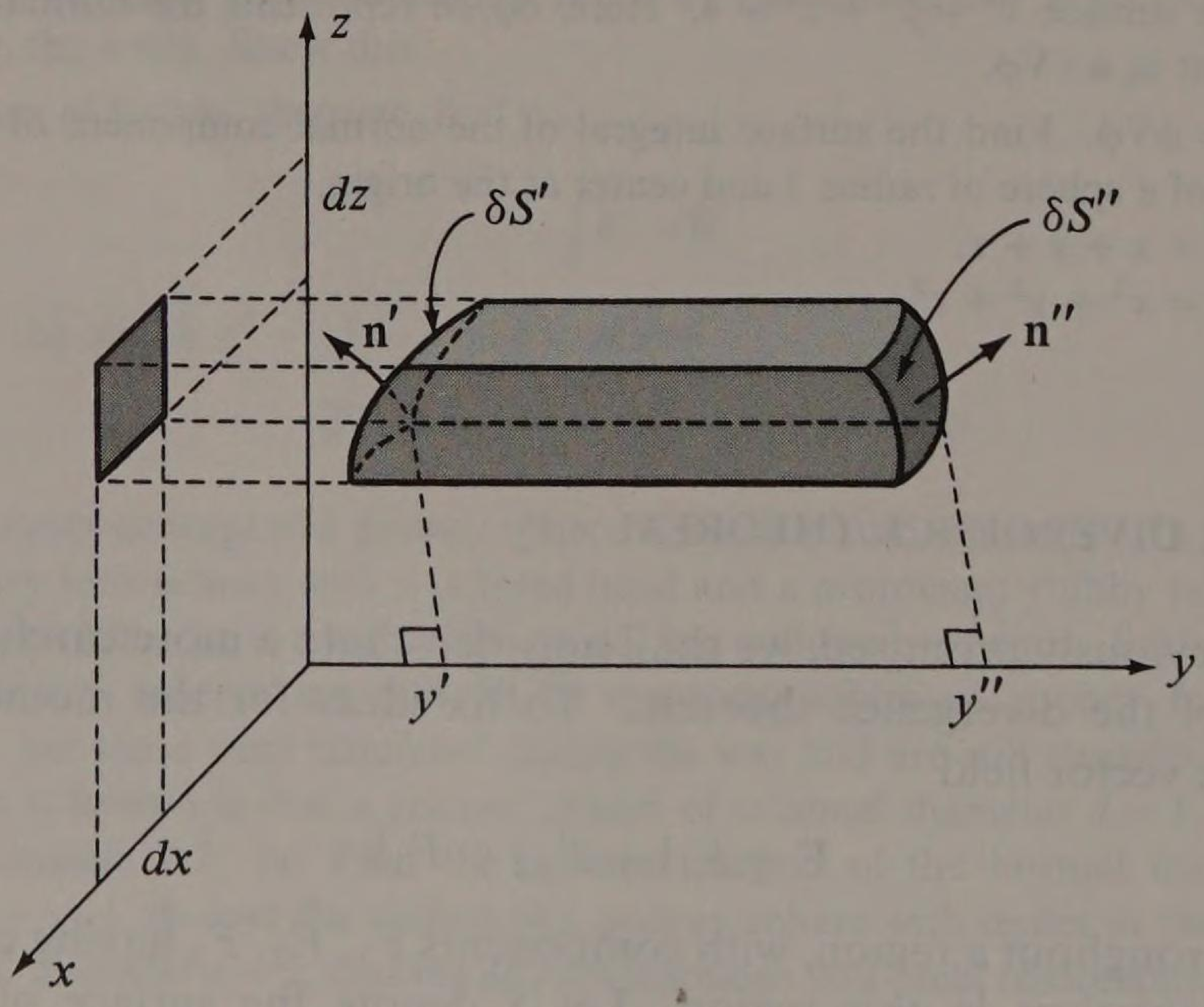

FIGURE 4.32

Since, by the fundamental theorem of calculus, we have

$$\int_{y'}^{y''} \frac{\partial F_2}{\partial y}\, dy = F_2'' - F_2'$$

it follows from the above discussion that the middle integral in (4.45) can be written

$$\iint_S \left(\int_{y'}^{y''} \frac{\partial F_2}{\partial y}\, dy \right) dx\, dz$$

where the middle integral is taken with y varying from y' to y'' within the sphere, and the double integral is taken over the projection of S on the xz plane. However, this equals the volume integral over D. Hence we can write

$$\iint_S F_2 \mathbf{j} \cdot \mathbf{n}\, dS = \iiint_D \frac{\partial F_2}{\partial y}\, dx\, dy\, dz$$

So the surface integral of $F_2(\mathbf{n} \cdot \mathbf{j})$ over S equals the volume integral of $\partial F_2/\partial y$ throughout the domain enclosed by S.

Similarly, we can show that the surface integral of $F_1(\mathbf{n} \cdot \mathbf{i})$ over S equals the volume integral of $\partial F_1/\partial x$ throughout D; and $F_3(\mathbf{n} \cdot \mathbf{k})$ and $\partial F_3/\partial z$ are similarly related. Therefore, (4.45) becomes

$$\iiint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz$$

This is simply the volume integral, throughout D, of the divergence of the vector field $\mathbf{F}$. Since (4.43) equals (4.45), we obtain

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \operatorname{div} \mathbf{F}\, dV \tag{4.46}$$

In this way we prove the divergence theorem for a sphere. However, notice that the same proof applies when S is the surface of an ellipsoid, a cube, a right circular cylinder, or even a potato-shaped region of a fairly arbitrary nature. Here is a more precise statement of what we have shown.

THEOREM 4.5 (*The Divergence Theorem*) *Let D be any domain with the property that each straight line through any interior point of the domain cuts the boundary in exactly two points, and such that the boundary S is a piecewise-smooth, closed, oriented surface with unit normal directed outward from the domain. Let $\mathbf{F}$ be a vector field, $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, continuous throughout a region containing D and its boundary, and such that the partial derivatives of F_1, F_2, and F_3 are also continuous in this region. Then*

$$\iiint_D \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

In proving this theorem, we made strong use of the idea of cutting up the sphere by filaments parallel to a coordinate axis. We assumed in Fig. 4.32 that any such filament cuts out two portions from the surface. Thus the

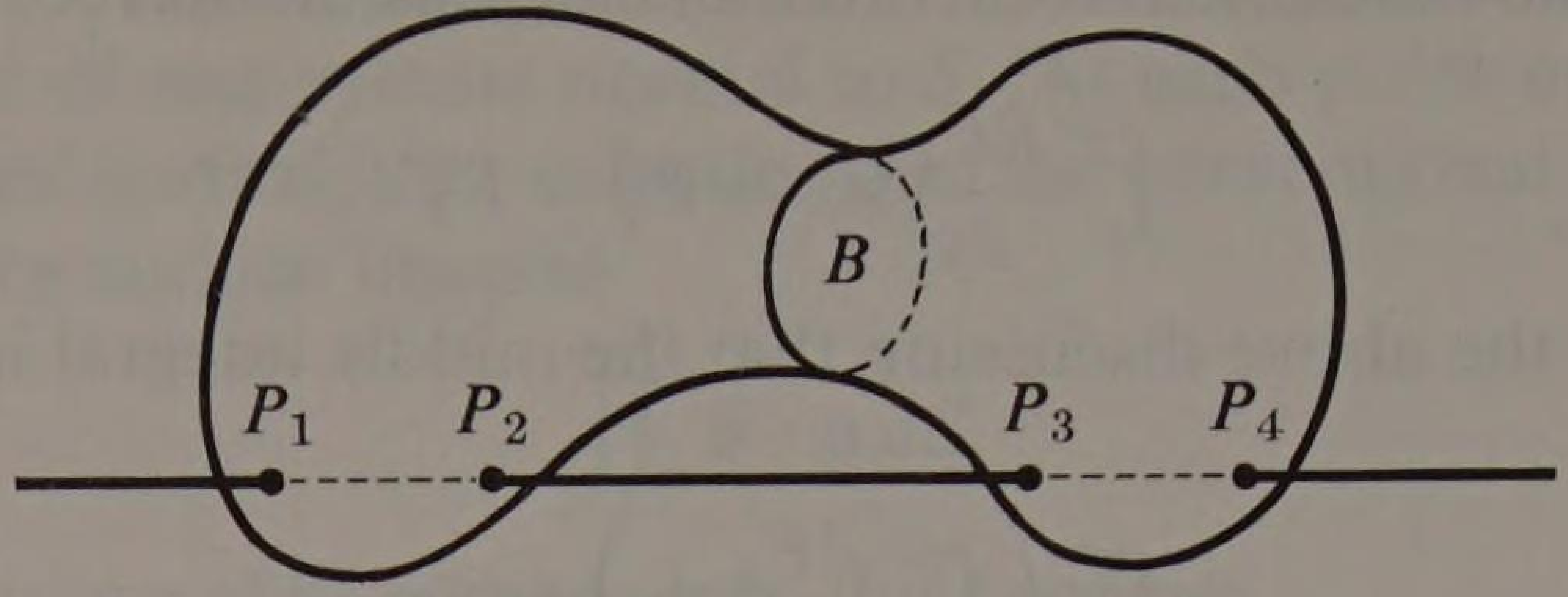

FIGURE 4.33

proof does not apply without modification to a domain such as the dumbbell-shaped one in Fig. 4.33. Here, such a filament can cut out four portions from the surface. However, it is easy to see that the theorem still applies to such a domain, since the dumbbell can be cut in the middle and the theorem applied separately to the two parts. The volume integral over the whole domain equals the sum of the two separate volume integrals, and the corresponding surface integrals add up to give the surface integral over the dumbbell (there will be two contributions from the common boundary B, but they will cancel each other, since $\mathbf{n}$ will have opposite directions in the two integrals).

Let us now investigate one interesting consequence of the divergence theorem. Let us suppose that the domain D is a very small one surrounding a point P. If it is sufficiently small, div $\mathbf{F}$ will be approximately constant, and the volume integral of div $\mathbf{F}$ over the volume V will be approximately equal to the product (div $\mathbf{F}$)V. More precisely, we have

$$\lim_{V\to 0} \frac{\iiint \operatorname{div} \mathbf{F}\, dV}{V} = \operatorname{div} \mathbf{F}$$

By the divergence theorem, we can replace the volume integral of div $\mathbf{F}$ by the surface integral of $\mathbf{F}$ over the boundary enclosing the volume, from which it follows that

$$\operatorname{div} \mathbf{F} = \lim_{V\to 0} \frac{\iint_S \mathbf{F}\cdot\mathbf{n}\, dS}{V}$$

Recall that this was our original motivation for the definition of divergence in Sec. 3.3, but then we were restricted to rectangular parallelepipeds with sides parallel to the axes. The divergence theorem frees us from this restriction; it implies that the divergence of $\mathbf{F}$ at any point P gives the outward flux per unit volume at P, regardless of the shape of the volume. The divergence is, indeed, a "coordinate-free" concept.

Exercises

Note: Computational exercises on the divergence theorem were given at the end of Sec. 4.9. The following exercises are relatively more theoretical. Throughout these exercises, D and $\mathbf{F}$ have the properties stated in the divergence theorem.

1. At what point in the proof of the divergence theorem did we make use of the requirement that the partial derivative $\partial F_2/\partial y$ be a continuous function of y?
2. In the proof, we required that the three partial derivatives be continuous, i.e., that each of them be continuous in all three variables. Why, for example, should we care whether or not the partial derivative $\partial F_2/\partial y$ is a continuous function of x?
3. Show, by a diagram similar to that of Fig. 4.32, that the volume integral of a function, taken over D, can be obtained by first integrating with respect to z and then integrating over the projection of S on the xy plane.
4. Outline a proof of the divergence theorem, taking Exercise 3 as the starting point. Start with

$$\iiint_D \operatorname{div} \mathbf{F}\, dV$$

integrating first with respect to z. Your proof will differ only slightly from that given in this section, i.e., you will integrate first with respect to z rather than y. By using the definition of surface integral you can avoid completely any use of such words as "approximately"; for simplicity, assume that S is a smooth surface.

5. Where, in your "proof" (Exercise 4) did you make unconscious use of the fact that the points on S with normals parallel to the xy plane have a projection on the xy plane of zero area? [*Hint*: Look again at the definition of the area of a surface (Sec. 4.6). What is $\cos \gamma$ for such points?]
6. What is the flux output per unit volume at (3,1,−2) if $\mathbf{F} = x^3\mathbf{i} + yx\mathbf{j} - x^3\mathbf{k}$?
7. What is the flux output from an ellipsoid of volume v if $\mathbf{F} = 3x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$?
8. If $\mathbf{F} = 3x^2\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, would the flux output from an ellipsoid depend on the location of the ellipsoid as well as on its volume?
9. (a) Describe the oriented surface enclosing the region

$$1 \leq x^2 + y^2 + z^2 \leq 4$$

assuming the usual convention concerning the orientation of a closed surface. (In Sec. 4.6 it was mentioned that if a surface encloses a region of space, the unit normal points away from the enclosed region; in this problem, the surface has two disconnected parts.)

(b) How would you compute the surface integral of the normal component of a vector field $\mathbf{F}$ over this surface?

(c) If $\operatorname{div} \mathbf{F} = 0$ except perhaps at the origin, what can you say about

$$\iint \mathbf{F} \cdot \mathbf{n}\, dS$$

over the two parts comprising this surface, taking $\mathbf{n}$ to be the unit normal outward from the origin in each case?

(d) Would your answer to (c) be any different if the region were that between the sphere $x^2 + y^2 + z^2 = 1$ and the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

(e) Compute the surface integral of the normal component of

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

over the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

10. (a) Using the divergence theorem, prove that

$$\iiint_D \nabla^2\phi \, dV = \iint_S \frac{\partial \phi}{\partial n} \, dS$$

where $\partial\phi/\partial n$, at any point on S, denotes the rate of change of the scalar field in the direction of the outward normal to S at that point. (*Hint*: Let $\mathbf{F} = \mathbf{grad}\,\phi$).

(b) Apply the divergence theorem to $\mathbf{F} = \phi\nabla\psi$ to obtain the *first Green formula*:

$$\iiint_D \phi\nabla^2\psi \, dV = \iint_S \phi\frac{\partial \psi}{\partial n} \, dS - \iiint_D \nabla\phi \cdot \nabla\psi \, dV$$

(c) Interchange the role of ϕ and ψ to derive the *second Green formula*:

$$\iiint_D (\phi\nabla^2\psi - \psi\nabla^2\phi) dV = \iint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS$$

11. Let ϕ be a scalar field, and define the *lumpiness* of ϕ at any point to be the scalar

$$-\lim_{V\to 0} \frac{\iint_S \frac{\partial\phi}{\partial n} \, dS}{V}$$

with notation as used in this section.

(a) Explain in your own words why the word "lumpiness" is appropriate. (*Hint*: Think of ϕ as the density of a fluid or, if you prefer, as the concentration of salt at each point in a brine solution.)

(b) How is lumpiness related to the laplacian? (See Exercise 10.)

(c) What can you say about the lumpiness of a harmonic function? (See Sec. 3.6.)

12. Let $\phi(x,y,z)$ be the temperature at (x,y,z). If ϕ represents a steady-state temperature distribution, show that ϕ is a harmonic function. [*Hint*: This can be done directly, using the fact (Sec. 4.7) that $\mathbf{Q} = -k\,\mathbf{grad}\,\phi$ gives the rate of heat flow per unit area, by drawing a small parallelepiped. However, it is intended here that you make use of Exercise 10 and the ideas of Exercise 11.]

13. Suppose that ϕ represents a temperature distribution that is not steady-state, so that ϕ is a function of both position and time. Find the relationship between the laplacian of ϕ and the time rate of change of ϕ at each point. (Let k denote the coefficient of thermal conductivity, let c denote specific heat capacity, and let ρ denote the mass density.)

14. Let S be a sphere of radius b and center at a point P and let ϕ be a continuous function. Consider the integral

$$\iint_S \phi\nabla\left(\frac{1}{r}\right) \cdot \mathbf{n} \, dS$$

where $\mathbf{n}$ is the unit outward normal, and $r = |\mathbf{R}|$ where $\mathbf{R}$ extends from the center of the sphere to a variable point on the surface. What is the limit of this integral as b tends to zero? (You cannot use the divergence theorem since $1/r$ is not defined at $r = 0$. Observe that

$$\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{R}}{r^3} \qquad \text{and} \qquad \mathbf{n} = \frac{\mathbf{R}}{r}$$

whence

$$\nabla\left(\frac{1}{r}\right) \cdot \mathbf{n} = -\frac{1}{r^2} = -\frac{1}{b^2}$$

for points on the surface.)

15. Let $\mathbf{F}$ be a vector field, defined and continuously differentiable everywhere except at a point P, and having zero divergence (except at P). Let S' be a closed surface (say, an ellipsoid) enclosing P, and let S denote the surface of a small sphere with center at P completely within S'. Compare

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS \qquad \text{and} \qquad \iint_{S'} \mathbf{F} \cdot \mathbf{n}\, dS'$$

where $\mathbf{n}$ denotes the *outward* unit normal in each case.

16. Let ϕ be a twice continuously differentiable function in a region D bounded by a suitably smooth surface S, and let r denote distance measured from a fixed point P inside D. Derive the *third Green formula*:

$$\phi(P) = -\frac{1}{4\pi} \iiint_D \frac{\nabla^2 \phi}{r}\, dV + \frac{1}{4\pi} \iint_S \left[\frac{\nabla \phi}{r} - \phi \nabla\left(\frac{1}{r}\right)\right] \cdot d\mathbf{S}$$

(*Hint*: Exclude a small sphere around P, set $\psi = 1/r$ in the second Green formula, and apply the theory in Exercises 14 and 15 to handle the sphere).

17. Evaluate

$$\iint_S \left[\frac{1}{r}\frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] dS$$

over the surface of the sphere $(x-3)^2 + y^2 + z^2 = 25$, where $r^2 = x^2 + y^2 + z^2$ and $\phi = xyz + 5$. By using the formula given in Exercise 16, you should be able to write the answer down at once.

18. Evaluate

$$\iint_S \left[\phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n}\right] dS$$

(a) over the surface of the ellipsoid $\dfrac{x^2}{9} + \dfrac{y^2}{16} + \dfrac{z^2}{25} = 1$ where $r^2 = x^2 + (y-1)^2 + z^2$ and $\phi = x^2 + y^2 - 2z^2 + 4$;

(b) over the surface of the cylindrical pillbox bounded by $x^2 + y^2 = 25$ and $z = \pm 10$, where $r^2 = (x-2)^2 + (y-1)^2 + (z-3)^2$ and $\phi = x^2 - z^2 + 5$.

4.11 GREEN'S THEOREM

This section is relatively elementary and is intended to provide some preparation for the rigorous proof of Stokes' theorem.

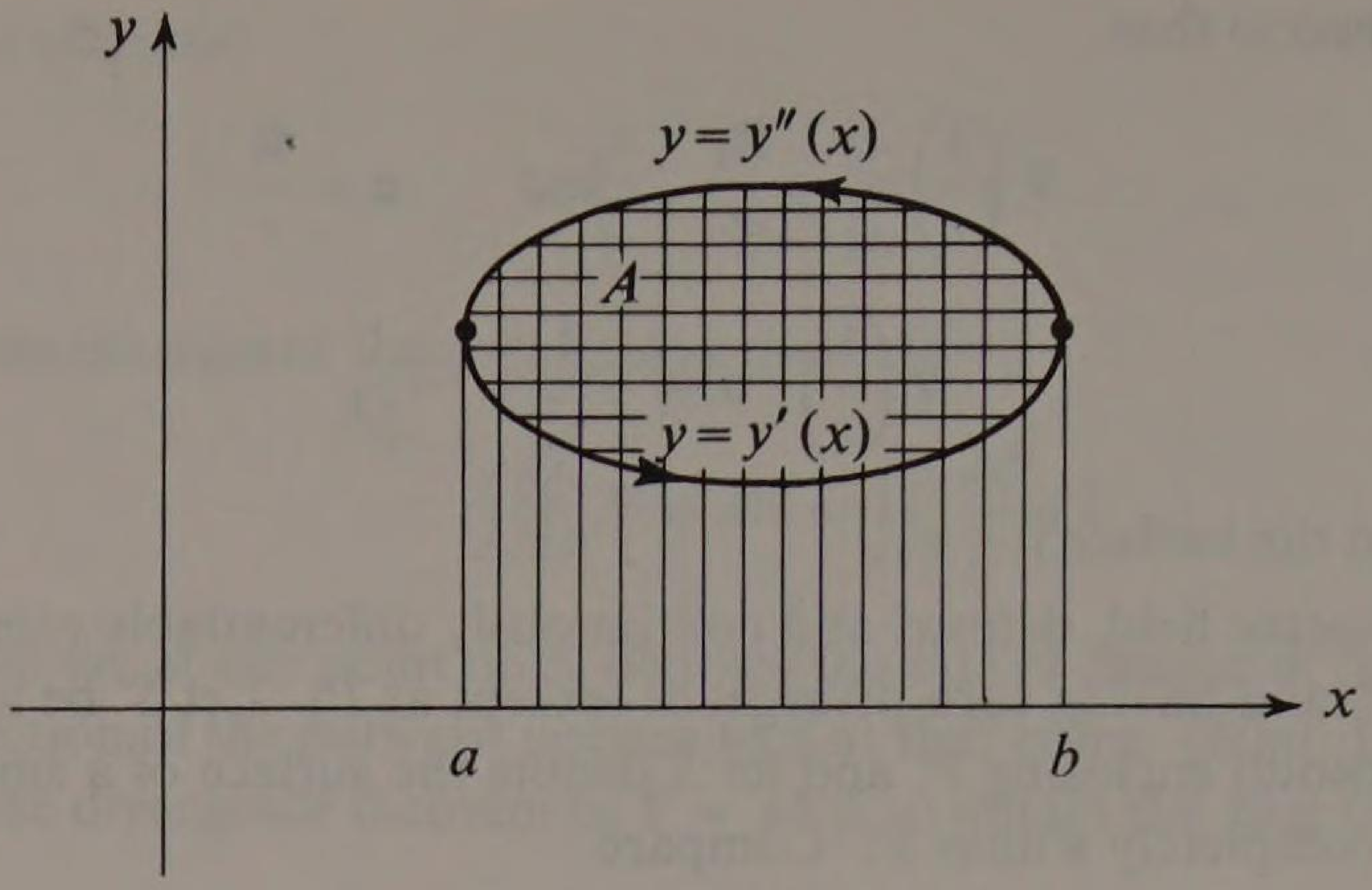

FIGURE 4.34

We shall work entirely in the xy plane. Let C denote a closed smooth arc in the plane (Fig. 4.34). Consider the line integral of the vector field $\mathbf{F} = y\mathbf{i}$ around C.

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \int_C y\,dx$$

(Since it is conventional to orient closed curves in the xy plane so that $\mathbf{k}$ is the positive normal to the plane, we traverse C in a counterclockwise direction.) Then the line integral can be expressed as the sum of two ordinary integrals,

$$\int_C y\,dx = \int_a^b y'(x)\,dx + \int_b^a y''(x)\,dx \tag{4.47}$$

where the first integral is along the bottom portion of the curve and the second is along the top portion; the notation should be self-evident from the figure. (Note that the primes here do *not* denote derivatives.)

The first integral gives the area beneath the lower curve and above the x axis. The second integral equals

$$-\int_a^b y''(x)\,dx$$

and gives the negative of the area beneath the upper curve and above the x axis. Therefore the sum of the two integrals is $-A$, the negative of the area within C,

$$\int_C y\,dx = -A \tag{4.48}$$

Here we assumed C to be in the upper half-plane, but the reader can easily verify that (4.48) also holds if C intersects the x axis or if C is beneath the x axis.

A similar argument shows that

$$\int_C x\,dy = A \tag{4.49}$$

Here we obtain A rather than $-A$, and is easily seen from Fig. 4.35.

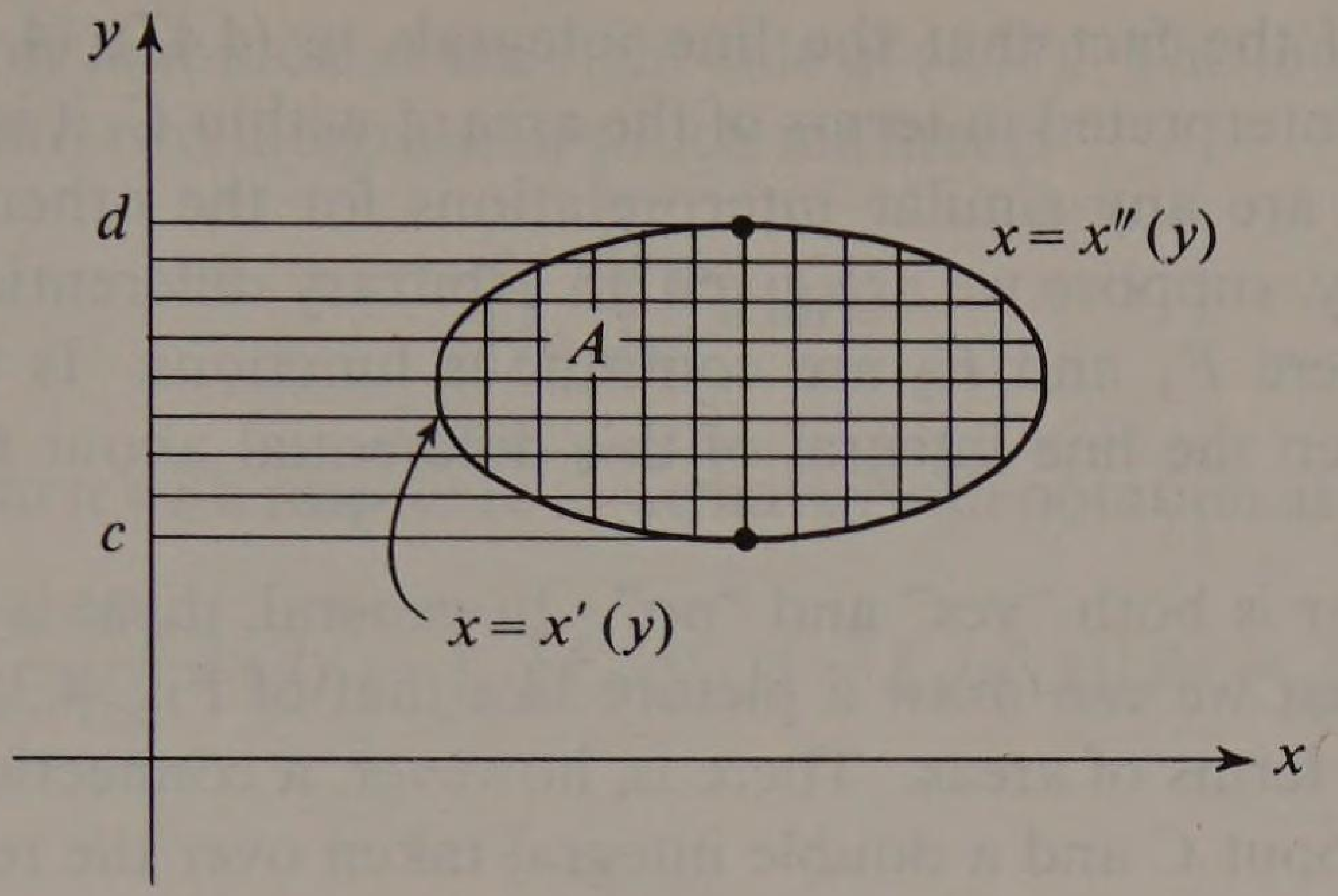

FIGURE 4.35

Now let us consider various other simple integrals about C. For instance, it is easy to verify that if the vector field is taken to be $\mathbf{F} = x\mathbf{i}$, then the line integral is

$$\int_C x\,dx = 0 \tag{4.50}$$

Indeed, $x\mathbf{i}$ is the gradient of the function $x^2/2$ so that the line integral of $x\,dx$ gives the change in $x^2/2$ as we move from initial point to final point, but for any closed curve these points coincide and hence the line integral is zero. In similar fashion, since $y\mathbf{j}$ is a gradient also, we have

$$\int_C y\,dy = 0 \tag{4.51}$$

Also, we have

$$\int_C dx = 0 \tag{4.52}$$

$$\int_C dy = 0 \tag{4.53}$$

It is entertaining, though not particularly instructive, to combine these line integrals in various ways. For instance, if x_0 is a constant, we derive

$$\int_C (x - x_0)\,dy = A \tag{4.54}$$

by using Eqs. (4.48) and (4.53). Similarly,

$$\int_C (x - x_0)\,dx = 0 \tag{4.55}$$

by (4.49) and (4.52).

Somewhat more interesting is

$$\int_C \frac{1}{2}(x\,dy - y\,dx) = A \tag{4.56}$$

which we obtain by combining (4.47) and (4.48).

In view of the fact that the line integrals in (4.47), (4.48), (4.54), and (4.56) may be interpreted in terms of the area A within C, it is natural to ask whether there are any similar interpretations for the other line integrals. More generally, suppose we are given an arbitrary differential $F_1(x, y)\,dx + F_2(x, y)\,dy$, where F_1 and F_2 are continuous functions. Is there any connection between the line integral of this differential about C and the area within?

The answer is both "yes" and "no". In general, there is no connection in the sense that we can draw a picture like that of Fig. 4.35 and interpret the integral in terms of areas. There is, however, a connection between the line integral about C and a double integral taken over the region within C. We will show that

greens

$$\int_C F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \tag{4.57}$$

where D is the domain within C (having area A). In the special case that the integrand in the double integral on the right side of (4.57) is identically equal to one, as in Eq. (4.56), the right side of (4.57) gives precisely A. If the integrand is zero, we get zero for the integral. In general, however, our result may not be related to A in any elementary manner and may be difficult to compute even with the help of (4.57).

The reader may recognize (4.57) as a special case of Stokes' theorem, discussed briefly in Sec. 4.9. To see this, let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ and $\mathbf{n} = \mathbf{k}$. The integral on the left is the line integral of the tangential component of $\mathbf{F}$ about C, and that on the right is the surface integral of the normal component of **curl F** over the surface enclosed by C.

This special case of Stokes' theorem is sometimes called *Green's theorem.* (Several other theorems are also called Green's theorem, incidentally.) The precise statement of the theorem is as follows:

THEOREM 4.7 *Let F_1 and F_2 be continuous functions of x and y for which the partial derivatives $\partial F_2/\partial x$ and $\partial F_1/\partial y$ exist and are continuous throughout a domain D in the xy plane. We require that D be bounded by a regular closed curve C, oriented by choosing* $\mathbf{k}$ *as the unit normal to the plane. We also require that any line passing through an interior point and parallel to either coordinate axis cuts the boundary in exactly two points. Then (4.57) is valid. More generally, (4.57) is valid for regions in the plane that can be decomposed into finitely many domains having these properties.*

The proof of the theorem is similar to that of the divergence theorem and goes as follows.

Proof Let us first look at the right side of (4.57). The integral can be broken up into two integrals, of which the first is

$$\iint_D \frac{\partial F_2}{\partial x}\,dx\,dy$$

Integrating first with respect to x, we have (with notation as in Fig. 4.35)

$$\int_c^d \int_{x'(y)}^{x''(y)} \frac{\partial F_2}{\partial x}\,dx\,dy = \int_c^d [F_2(x'',y) - F_2(x',y)]\,dy = \int_C F_2\,dy$$

Similarly,

$$\begin{aligned} -\iint \frac{\partial F_1}{\partial y}\,dx\,dy &= -\int_a^b \int_{y'(x)}^{y''(x)} \frac{\partial F_1}{\partial y}\,dy\,dx \\ &= \int_a^b [F_1(y') - F_1(y'')]\,dx = \int_C F_1\,dx \end{aligned}$$

Adding these two gives the desired result. If D is a region that can be decomposed into finitely many domains having the stated properties, we simply sum the integrals involved over all the domains. The double integral then extends over all the parts, and the line integral extends over the entire boundary. If the boundaries of two parts have arcs in common, these arcs may be neglected, since the integrals will cancel (as in Fig. 4.29).

Exercises

1. Use Green's theorem to derive Eq. (4.48).
2. Use Green's theorem to derive (4.49).
3. Use Green's theorem to derive (4.50).
4. Use Green's theorem to derive (4.56).
5. Let $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$ and $d\mathbf{R} = dx\,\mathbf{i} + dy\,\mathbf{j}$.
   (a) Compute the magnitude of the vector cross product $\mathbf{R} \times (\mathbf{R} + d\mathbf{R})$.
   (b) Thus give a direct geometrical interpretation of the integrand of (4.56). [*Hint*: Consider the triangle with vertices (0,0), (x, y), and $(x + dx, y + dy)$.]
   (c) Using Fig. 4.36, give an alternative derivation of (4.56).
6. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, and let C be an oriented closed curve enclosing an area A. What is

   $$\int_C \mathbf{F} \cdot \mathbf{T}\,ds$$

   (As usual, $\mathbf{T}$ denotes the unit tangent to C in the positive direction.)
7. Let C denote the circle $x^2 + y^2 = 9$, and let $\mathbf{F} = y\mathbf{i} - 3x\mathbf{j}$. What is the line integral of the tangential component of $\mathbf{F}$ around C, taken in the usual counter-clockwise direction?

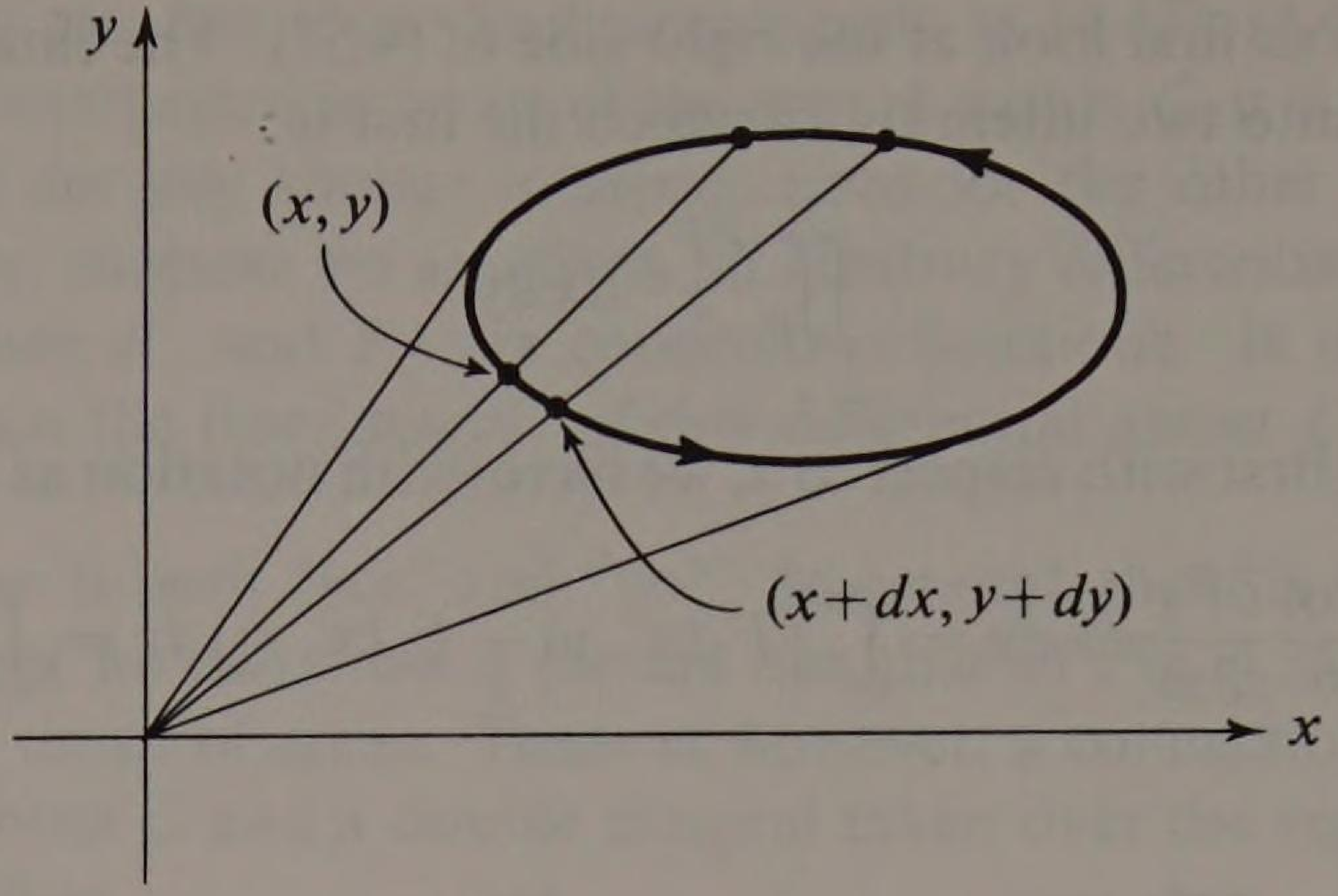

FIGURE 4.36

8. Let C denote the ellipse $(x^2/4) + (y^2/9) = 1$, and let

$$\mathbf{F} = (3y^2 - y)\mathbf{i} + (x^2 + 2)\mathbf{j}$$

(a) What is the area enclosed by C? (Don't integrate, for heaven's sake; we have already derived the area of an ellipse by using the area cosine principle.)

(b) Find the line integral of the tangential component of **F** around C, in the counterclockwise direction. [*Hint*: By Green's theorem, this resolves itself to a double integral, but no computation is necessary if you observe that the symmetry enables you to ignore certain terms. Just multiply the area by the average value of $(\partial F_2/\partial x) - (\partial F_1/\partial y)$.]

9. Compute

$$\int_C (4y^3\,dx - 2x^2\,dy)$$

around the square bounded by the lines $x = \pm 1$ and $y = \pm 1$,

(a) directly, by performing the line integration;

(b) by using Green's theorem.

(c) By symmetry, it is obvious that one of the terms in the integrand of the above line integral can be ignored. Which term?

10. Let $\mathbf{F} = 4z\mathbf{i} - 3x\mathbf{k}$. Compute the line integral of the tangential component of **F** about the circle $(x - 5)^2 + (z - 7)^2 = 4$ in the xz plane. Orient the plane by taking **j** to be unit normal. [*Careful*: If you just replace y by z in Eq. (4.57) you will get the wrong orientation.]

11. In (4.57), the functions F_1 and F_2 are fairly arbitrary functions of x and y (we only require that certain partial derivatives be continuous). It therefore appears that we can interchange F_1 and F_2 and also x and y to obtain the formula

$$\int_C (F_2\,dy + F_1\,dx) = \iint_D \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\right) dy\,dx$$

The left side of this equation is the same as the left side of (4.57), but the right side has the opposite sign. It follows that this expression is incorrect. Give a clue, *in only one word*, to explain this paradox.

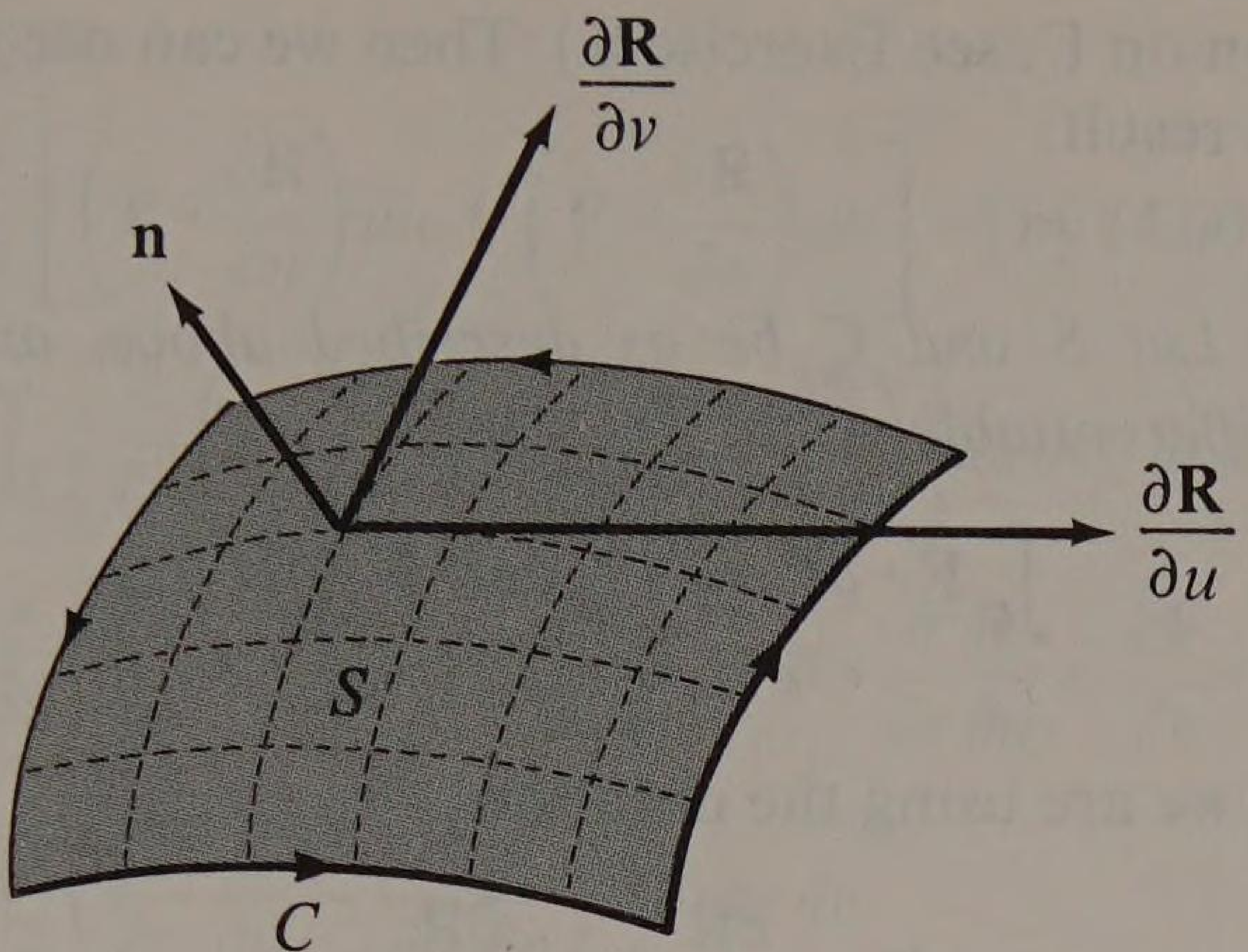

FIGURE 4.37

4.12 STOKES' THEOREM

We are now in a position to give a rigorous proof of Stokes' theorem, which reduces certain surface integrals to line integrals. We are given a smooth oriented surface S in space, bounded by a piecewise smooth, closed curve C whose orientation is consistent with that of S (Fig. 4.37). We assume that the surface can be parametrized by $\mathbf{R} = \mathbf{R}(u,v)$ in such a way that the coordinates x, y, and z are twice continuously differentiable functions of u and v (so that the mixed partials are equal in either order), with $(\partial\mathbf{R}/\partial u \times \partial\mathbf{R}/\partial v)$ pointing in the direction of the normal.

The set of values (u,v) in the uv plane that correspond to points on S will be denoted Σ (Fig. 4.38). We assume that distinct points of Σ correspond to distinct point on S (the mapping $\mathbf{R}(u,v)$ is one-to-one) and that the region Σ and its boundary Γ satisfy the hypothesis of Green's theorem. (Notice that these assumptions imply that the positive orientation on C corresponds to the

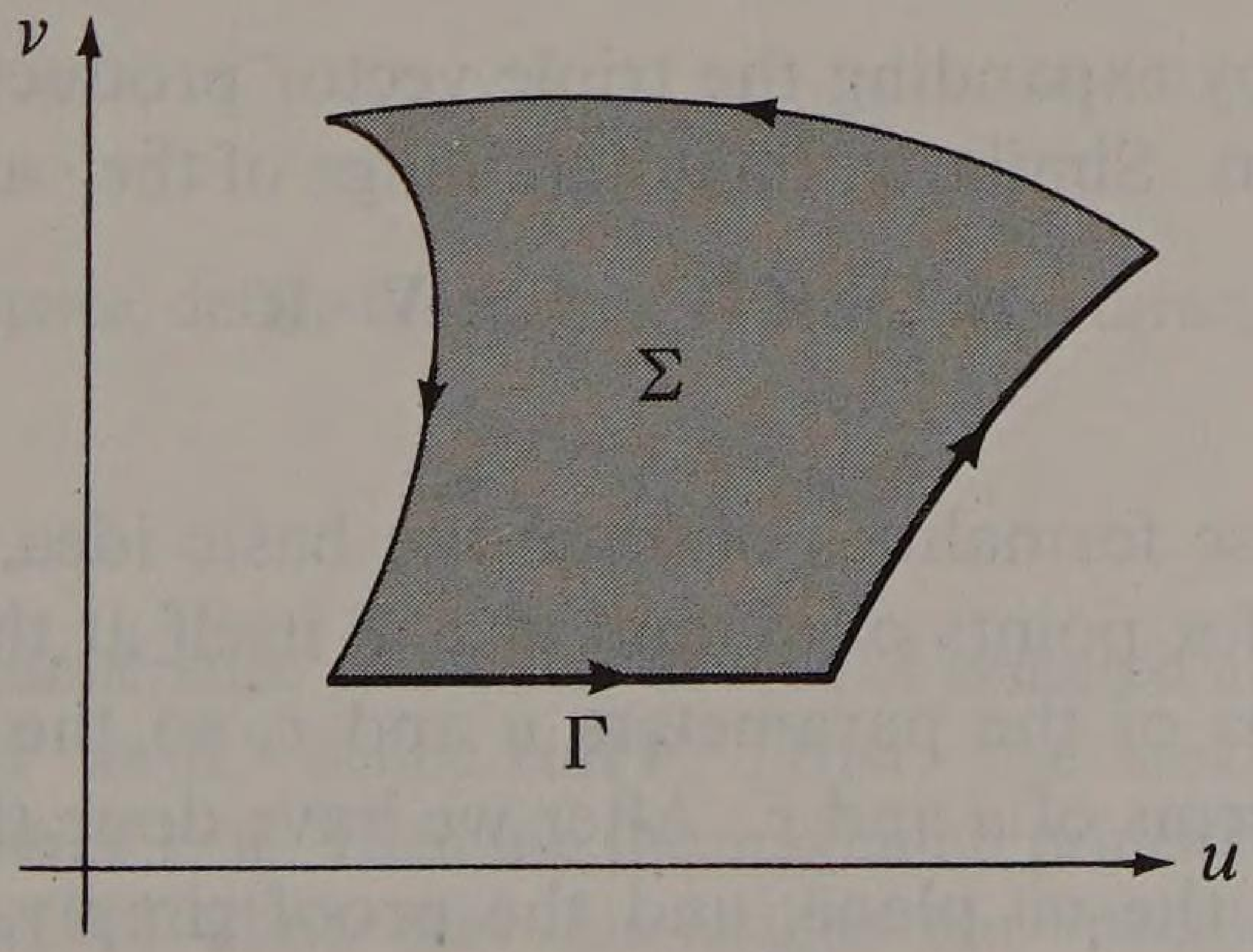

FIGURE 4.38

correct orientation on Γ; see Exercise 8.) Then we can use this parametrization to derive the result:

THEOREM 4.6 *Let S and C be as described above, and let* **F** *be a continuously differentiable vector field; then,*

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \iint_S \mathbf{curl}\ \mathbf{F} \cdot d\mathbf{S} \tag{4.58}$$

Observe that we are using the notation

$$d\mathbf{S} = \frac{\partial \mathbf{R}}{\partial u}\,du \times \frac{\partial \mathbf{R}}{\partial v}\,dv \tag{4.59}$$

introduced in Sec. 4.6. In the derivation we will use the identities

$$d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial u}\,du + \frac{\partial \mathbf{R}}{\partial v}\,dv \tag{4.60}$$

and

$$\frac{\partial}{\partial u} = \frac{\partial \mathbf{R}}{\partial u} \cdot \boldsymbol{\nabla} \tag{4.61}$$

$$\frac{\partial}{\partial v} = \frac{\partial \mathbf{R}}{\partial v} \cdot \boldsymbol{\nabla} \tag{4.62}$$

Before going through the derivation, the reader may wish to review Sec. 4.6, and also the operator convention (first paragraph of Sec. 3.7). Thus, to derive Eq. (4.61), we simply use the chain rule in operator form,

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z} = \frac{\partial \mathbf{R}}{\partial u} \cdot \boldsymbol{\nabla} \tag{4.63}$$

and (4.62) is derived similarly. We will also use

$$\left(\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\right) \times \boldsymbol{\nabla} = \frac{\partial \mathbf{R}}{\partial v}\left(\frac{\partial \mathbf{R}}{\partial u} \cdot \boldsymbol{\nabla}\right) - \frac{\partial \mathbf{R}}{\partial u}\left(\frac{\partial \mathbf{R}}{\partial v} \cdot \boldsymbol{\nabla}\right) \tag{4.64}$$

which is obtained by expanding the triple vector product and by using the operator convention. Similarly, the interchange of the · and × in

$$\mathbf{A} \cdot \boldsymbol{\nabla} \times \mathbf{B} = \mathbf{A} \times \boldsymbol{\nabla} \cdot \mathbf{B} \tag{4.65}$$

is easily verified.

Do not let these formalities obscure the basic idea, which is that the position vector **R**, for points on *S*, and also **F** itself at these points, can be written as functions of the parameters *u* and *v*, so the integrals in (4.58) can be written in terms of *u* and *v*. After we have done this, it appears that we are working in the *uv* plane, and the proof simply amounts to some inspired juggling of vector identities, aided by Green's theorem.

Proof We write

$$
\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_\Gamma \left[\left(\mathbf{F}\cdot\frac{\partial \mathbf{R}}{\partial u}\right)du + \left(\mathbf{F}\cdot\frac{\partial \mathbf{R}}{\partial v}\right)dv\right] \quad [\text{by (4.60)}]\\
&= \iint_\Sigma \left[\frac{\partial}{\partial u}\left(\mathbf{F}\cdot\frac{\partial \mathbf{R}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\mathbf{F}\cdot\frac{\partial \mathbf{R}}{\partial u}\right)\right] du\,dv \quad (\text{Green's theorem})\\
&= \iint_\Sigma \left[\frac{\partial \mathbf{F}}{\partial u}\cdot\frac{\partial \mathbf{R}}{\partial v} + \mathbf{F}\cdot\frac{\partial^2 \mathbf{R}}{\partial u\,\partial v} - \mathbf{F}\cdot\frac{\partial^2 \mathbf{R}}{\partial v\,\partial u} - \frac{\partial \mathbf{F}}{\partial v}\cdot\frac{\partial \mathbf{R}}{\partial u}\right] du\,dv\\
&= \iint_\Sigma \left(\frac{\partial \mathbf{R}}{\partial v}\frac{\partial}{\partial u} - \frac{\partial \mathbf{R}}{\partial u}\frac{\partial}{\partial v}\right)\cdot\mathbf{F}\,du\,dv\\
&= \iint_\Sigma \left[\frac{\partial \mathbf{R}}{\partial v}\left(\frac{\partial \mathbf{R}}{\partial u}\cdot\nabla\right) - \frac{\partial \mathbf{R}}{\partial u}\left(\frac{\partial \mathbf{R}}{\partial v}\cdot\nabla\right)\right]\cdot\mathbf{F}\,du\,dv\\
&\qquad\qquad [\text{by (4.61) and (4.62)}]\\
&= \iint_\Sigma \left(\frac{\partial \mathbf{R}}{\partial u}\times\frac{\partial \mathbf{R}}{\partial v}\right)\times\nabla\cdot\mathbf{F}\,du\,dv \quad [\text{by (4.64)}]\\
&= \iint_\Sigma \left(\frac{\partial \mathbf{R}}{\partial u}\times\frac{\partial \mathbf{R}}{\partial v}\right)\cdot\nabla\times\mathbf{F}\,du\,dv \quad [\text{by (4.65)}]\\
&= \iint_\Sigma (\mathbf{curl\ F})\cdot\left(\frac{\partial \mathbf{R}}{\partial u}\times\frac{\partial \mathbf{R}}{\partial v}\right)du\,dv\\
&= \iint_S \mathbf{curl\ F}\cdot d\mathbf{S} \quad [\text{by (4.59)}]]
\end{aligned}
$$

which completes the derivation.

This proof avoids the pitfalls of the argument in Sec. 4.9 only by completely devoiding itself of any physical content. That is the value, and the liability, of relying on parametrizations.

Exercises

1. At any point P in space, define the "swirl" of $\mathbf{F}$ at P in a direction $\mathbf{n}$ to be

$$\lim_{A\to 0}\frac{1}{A}\int_C \mathbf{F}\cdot d\mathbf{R} \tag{4.66}$$

where C is the circumference of a circle of area A centered at P with unit normal $\mathbf{n}$. Using the word "swirl," define **curl F**. [*Hint*: Use Stokes' theorem to show that (4.66) equals (**curl F**) · **n**. Then use the maximum principle of Sec. 1.9 to define the direction of **curl F**.] Show that this justifies our "paddlewheel" definition in Sec. 3.4. It also provides a coordinate-free description of curl.

2. Be a bit fanciful, and imagine that S is the surface of a laundry bag with a draw-string forming the boundary C. Then Stokes' theorem states that the surface integral of the normal component of **curl F** over the laundry bag equals the line integral of its tangential component around the drawstring. Now suppose that we close the bag by pulling the drawstring; the effective length of the drawstring becomes zero and the line integral is therefore zero. S has become a closed surface.

(a) What is the surface integral of the normal component of **curl F** over a closed surface?

We now apply the divergence theorem, which says that the volume integral of the divergence of a vector field through the interior of a closed laundry bag equals the surface integral of the normal component of the field over its surface. Let the vector field be **curl F**.

(b) What is the volume integral of the divergence of **curl F** over a domain?

If the laundry bag is very, very small, the divergence of **curl F** will be approximately constant throughout, and the volume integral of div (**curl F**) will be approximately div (**curl F**), at a point within the laundry bag, times the volume the bag encloses.

(c) What is div (**curl F**) at any point P?

(d) To which of the identities of Sec. 3.7 is this related?

3. This is very similar to Exercise 2, but the point of view is somewhat different. Let S be the surface of a sphere, and let us imagine the sphere divided into two parts, an upper hemisphere and a lower hemisphere, by a plane parallel to the xy plane passing through its center. (Draw a diagram.) Let **F** be a vector field, and consider the surface integral of the normal component of **curl F** over the upper hemisphere. Relate this mentally to the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{R}$$

where C is the equator, oriented relative to the outward normal of the upper hemisphere (i.e., the positive direction is west to east). Now do the same thing for the lower hemisphere: the surface integral of (**curl F**) · **n** over the lower hemisphere equals the line integral over the equator with, however, an east-to-west direction of integration. Add the two.

(a) What is the surface integral of the normal component of **curl F** over a sphere?

(b) What is the volume integral of div (**curl F**) through the interior of a sphere?

(c) Let the sphere shrink to a point; what does this say about div (**curl F**) at a point?

4. Suppose that $\mathbf{F} = \mathbf{grad}\ \phi$, so that the line integral of the tangential component of **F** along any curve is equal to the difference in the values of ϕ at the endpoints of the curve. In particular, if C is a closed curve,

$$\int_C \mathbf{F} \cdot d\mathbf{R} = 0$$

Let S be a surface with boundary C.

(a) What is the surface integral of the normal component of **curl** (**grad** ϕ) over a surface S?

If S is a very small element of surface, bounded by a closed curve C, **curl** (**grad** ϕ) will be approximately constant on S, and the surface integral of the normal component of **curl** (**grad** ϕ) will be approximately **n** · **curl** (**grad** ϕ) times the area of the surface.

(b) For any unit vector **n**, and any point in space, what is **n** · **curl** (**grad** ϕ) at this point?

(c) Since this result is independent of the direction of **n**, what can you say about **curl (grad** ϕ**)**?

(d) To which of the identities of Sec. 3.7 is this related?

5. Let **J** denote electric current density (a vector in the direction of the current, with magnitude in units of current/area) and **B** denote the magnetic field intensity. One of Maxwell's laws of electromagnetism states that, in the absence of a time-varying electric field,

$$\mathbf{curl\ B} = \mu_0 \mathbf{J} \tag{4.67}$$

where μ_0 is a constant. Use Stokes' theorem to derive

$$\int_C \mathbf{B} \cdot d\mathbf{R} = \mu_0 i \tag{4.68}$$

In words: the line integral of the tangential component of magnetic field intensity, around a closed loop, is proportional to the current i passing across any surface bounded by the loop.

6. Given the vector field $\mathbf{F} = 3y\mathbf{i} + (5 - 2x)\mathbf{j} + (z^2 - 2)\mathbf{k}$, find (a) div **F**, (b) **curl F**, (c) the surface integral of the normal component of **curl F** over the open hemispherical surface $x^2 + y^2 + z^2 = 4$ above the xy plane. [*Hint*: By a double application of Stokes' theorem, part (c) can be reduced to a triviality.]

7. Given that $\mathbf{curl\ F} = 2y\mathbf{i} - 2z\mathbf{j} + 3\mathbf{k}$, find the surface integral of the normal component of **curl F** (*not* **F**) over (a) the open hemispherical surface $x^2 + y^2 + z^2 = 9$, $z > 0$, and (b) the sphere $x^2 + y^2 + z^2 = 9$. (In both parts, you should be able to write the answer down by inspection.)

8. Show why the positive orientations on C and Γ, in the proof of Stokes' theorem, correspond. (*Hint*: Reread the beginning of Sec. 4.6.)

4.13 OPTIONAL READING: TRANSPORT THEOREMS

In some physics and engineering applications it is necessary to compute the time derivative of a surface or volume integral, when the surface or volume of integration is in motion. For instance, in an electric generator a loop of wire is moved through a magnetic field in such a manner that the flux of the field through the surface bounded by the loop is changing. According to Faraday's law, an electromotive force is set up in the loop, proportional to the rate of change of this flux integral.

Similarly, one may wish to note the rate of change of some quantity, like charge or stored energy, associated with a specific portion of a moving fluid. If this quantity is given as a volume integral of some density function, then the volume of interest is being transported downstream with the fluid as the time derivative is taken.

These two problems are related. First we treat the moving surface problem, then we use the answer in analyzing the transported volume problem.

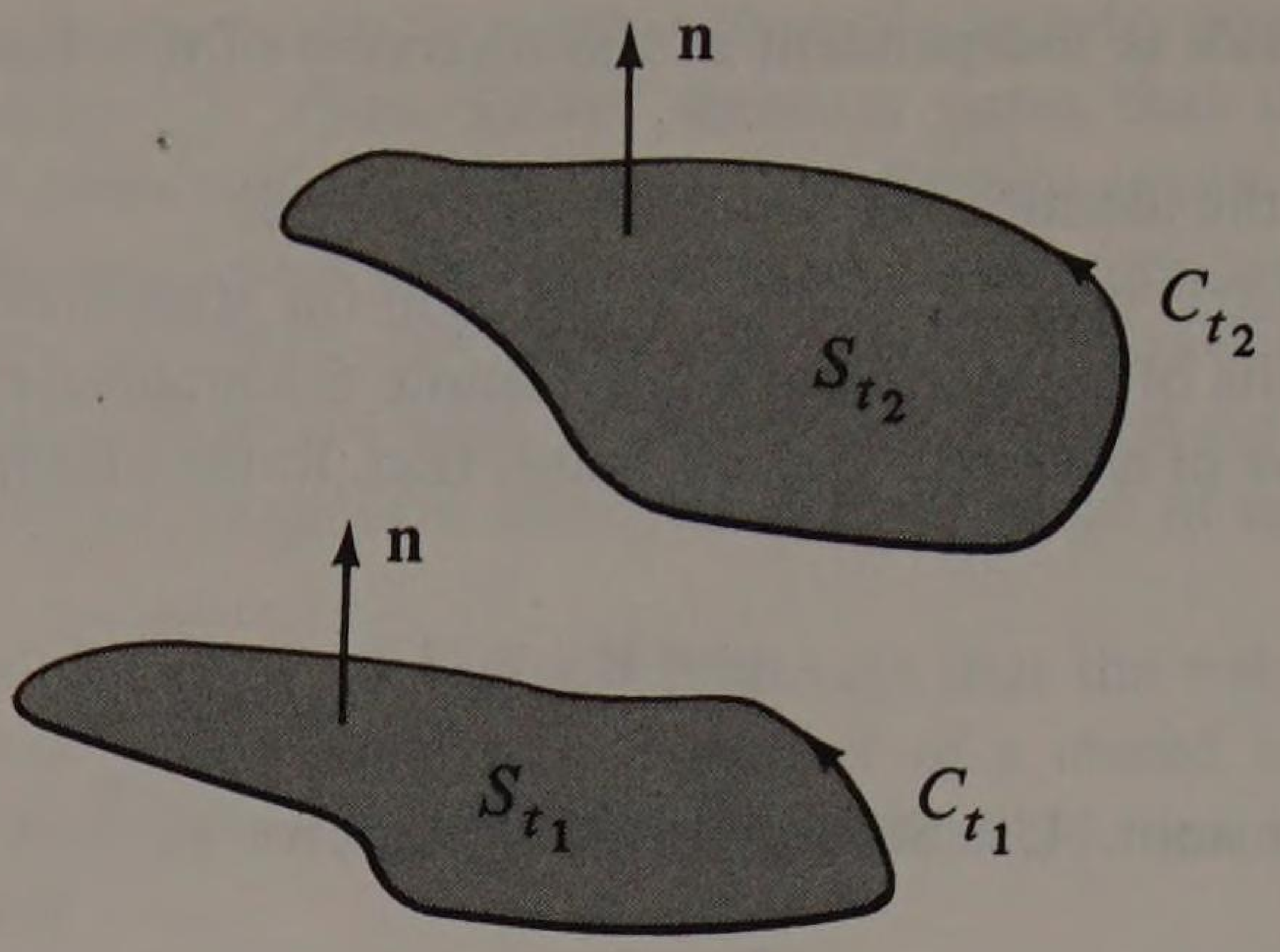

FIGURE 4.39

The situation is as follows. We have a vector field **F** that changes with time, $\mathbf{F} = \mathbf{F}(\mathbf{R},t)$, and an oriented surface S that, together with its properly oriented boundary curve C, moves through space; we use the notation S_t and C_t to designate the surface and curve at time t. Let $\Phi(t)$ be the flux of $\mathbf{F}(\mathbf{R},t)$ through S_t, at time t:

$$\Phi(t) = \iint_{S_t} \mathbf{F}(\mathbf{R},t) \cdot d\mathbf{S}$$

Notice that $\Phi(t)$ changes due to two effects; namely, the changing field **F** and the motion of the surface S_t. The problem is to compute $d\Phi/dt$.

We shall give two derivations, first a heuristic argument based on Fig. 4.39, and then a more rigorous argument using parametrizations.

Figure 4.39 shows the location of the surface at times t_1 and t_2, together with its boundary and the orientations. In Fig. 4.40, the displacements of corresponding points on S_{t_1} and S_{t_2} are shown. If we can associate a velocity field $\mathbf{v} = \mathbf{v}(\mathbf{R},t)$ on S that describes the pointwise motion of the surface, then for $t_2 - t_1 \equiv dt$ sufficiently small, the point **R** on S_{t_1} is carried to the point $\mathbf{R} + \mathbf{v}(\mathbf{R},t_1)\,dt$ on S_{t_2}. (The reader who is uneasy about the vagueness of these notions will feel more comfortable with the second derivation.)

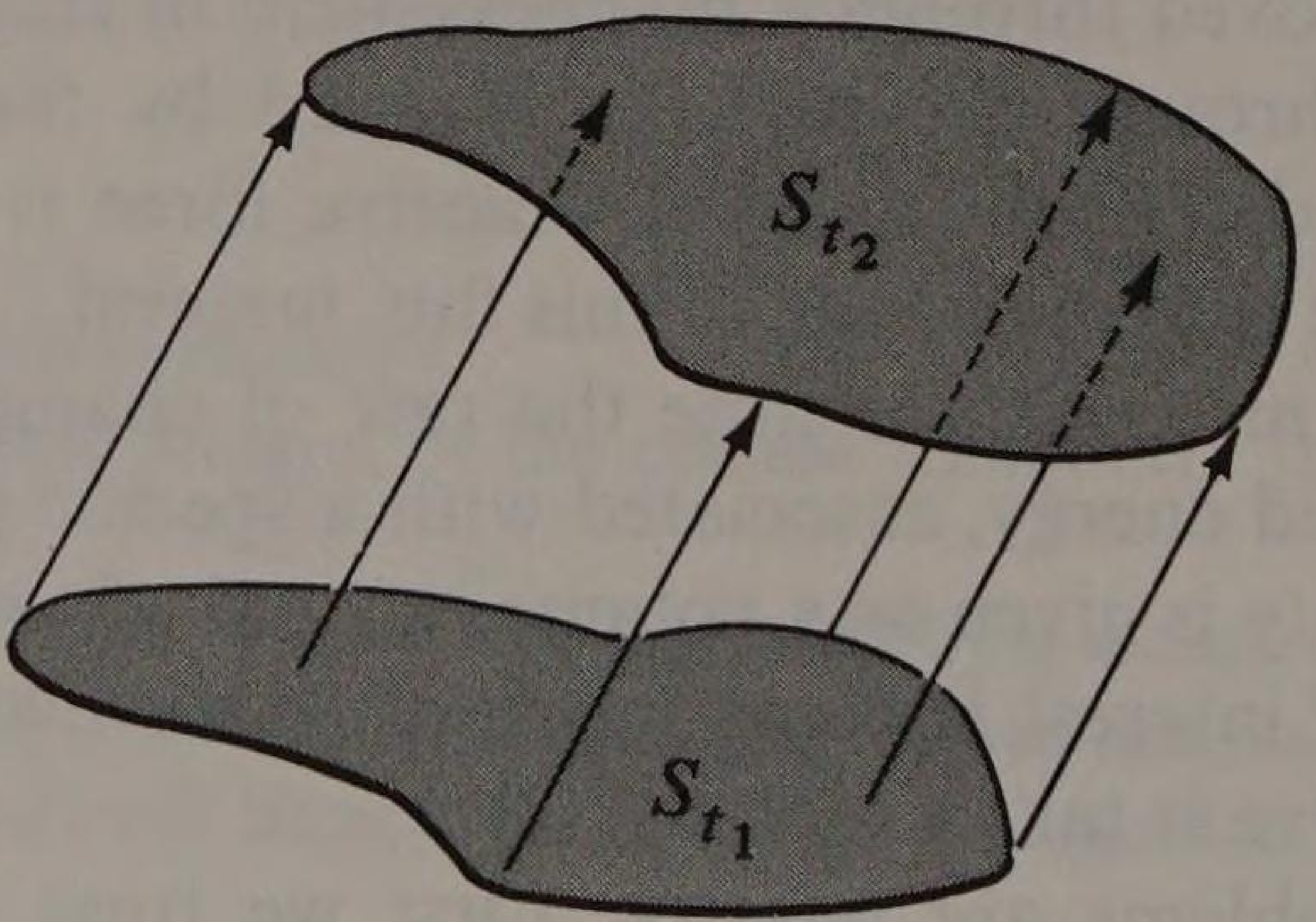

FIGURE 4.40

To compute $d\Phi/dt$, we must evaluate

$$\frac{d\Phi}{dt} = \lim_{t_2 \to t_1} \frac{1}{t_2 - t_1}\left[\iint_{S_{t_2}} \mathbf{F}(\mathbf{R},t_2)\cdot d\mathbf{S} - \iint_{S_{t_1}} \mathbf{F}(\mathbf{R},t_1)\cdot d\mathbf{S}\right] \tag{4.69}$$

Visualizing these fluxes in Fig. 4.40 suggests that the divergence theorem, applied to the region D swept out by the surfaces in the intervening times, may be useful. Applying the theorem at time t_1, with due consideration for the distinction between the surface normal and the outward normal, we have

$$\begin{aligned}\iiint_D \nabla\cdot\mathbf{F}(\mathbf{R},t_1)\,dV &= \iint_{S_{t_2}} \mathbf{F}(\mathbf{R},t_1)\cdot d\mathbf{S} - \iint_{S_{t_1}} \mathbf{F}(\mathbf{R},t_1)\cdot d\mathbf{S}\\ &\quad + \iint_{\text{(sides)}} \mathbf{F}(\mathbf{R},t_1)\cdot d\mathbf{S}\end{aligned} \tag{4.70}$$

Expressing

$$\mathbf{F}(\mathbf{R},t_2) \approx \mathbf{F}(\mathbf{R},t_1) + \frac{\partial\mathbf{F}}{\partial t}\,dt$$

and using (4.70) for the S_{t_1} integral we find

$$\begin{aligned}&\iint_{S_{t_2}} \mathbf{F}(\mathbf{R},t_2)\cdot d\mathbf{S} - \iint_{S_{t_1}} \mathbf{F}(\mathbf{R},t_1)\cdot d\mathbf{S}\\ &\quad = \iint_{S_{t_2}} \mathbf{F}(\mathbf{R},t_1)\cdot d\mathbf{S} + dt\iint_{S_{t_2}} \frac{\partial\mathbf{F}}{\partial t}\cdot d\mathbf{S}\\ &\qquad - \iint_{S_{t_2}} \mathbf{F}(\mathbf{R},t_1)\cdot d\mathbf{S} - \iint_{\text{(sides)}} \mathbf{F}(\mathbf{R},t_1)\cdot d\mathbf{S}\\ &\qquad + \iiint_D \nabla\cdot\mathbf{F}(\mathbf{R},t)\,dV\end{aligned} \tag{4.71}$$

Obviously, two of these integrals cancel. On the sides, Fig. 4.41 shows that the surface element $d\mathbf{S}$ equals $d\mathbf{R} \times \mathbf{v}\,dt$, where $d\mathbf{R}$ is taken along C_{t_1}. The element of volume in D has base $|d\mathbf{S}|$ and height $|\mathbf{v}\,dt\cdot\mathbf{n}|$, so that

$$dV = d\mathbf{S}\cdot\mathbf{v}\,dt$$

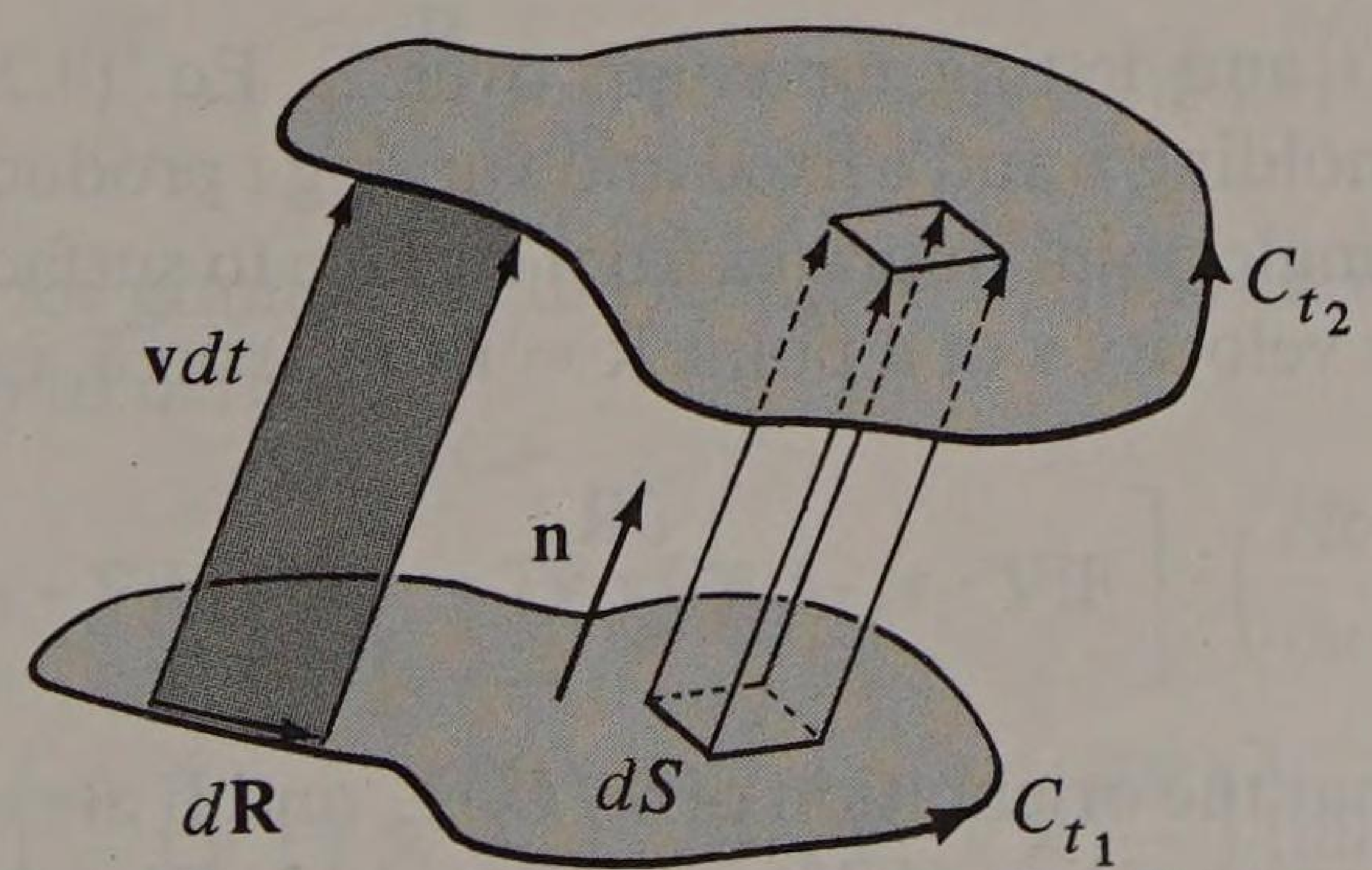

FIGURE 4.41

Consequently, Eq. (4.71) becomes

$$\begin{aligned}\iint_{S_{t_2}} \mathbf{F}(\mathbf{R},t_2)\cdot d\mathbf{S} - \iint_{S_{t_1}} \mathbf{F}(\mathbf{R},t_1)\cdot d\mathbf{S} \\ = dt\iint_{S_{t_2}} \frac{\partial \mathbf{F}}{\partial t}\cdot d\mathbf{S} - dt\oint_{C_{t_1}} \mathbf{F}(\mathbf{R},t_1)\cdot d\mathbf{R}\times\mathbf{v} \\ + dt\iint_{S_{t_1}} (\nabla\cdot\mathbf{F})\mathbf{v}\cdot d\mathbf{S}\end{aligned}$$

Dividing by dt, we arrive at the *flux transport theorem*

$$\begin{aligned}\frac{d\Phi}{dt} = \iint_{S_t}\left[\frac{\partial \mathbf{F}}{\partial t} + (\nabla\cdot\mathbf{F})\mathbf{v}\right]\cdot d\mathbf{S} \\ + \oint_{C_t} \mathbf{F}\times\mathbf{v}\cdot d\mathbf{R}\end{aligned} \tag{4.72}$$

One of the biggest flaws of the previous argument is the vagueness of the velocity field concept. It presupposes some way of identifying, in a one-to-one fashion, the points on the surface S_{t_2} with corresponding points on the surface S_{t_1}, so that $\mathbf{v}(\mathbf{R})\,dt$ describes the displacements. However, if S_t is only a mathematical surface, without physical substance, this correspondence between points is rather arbitrary; hence so, also, is the velocity field.

A more logical way to proceed would be to specify a parametrization of the surface at some fixed time, say, at $t = 0$:

$$\mathbf{R} = \mathbf{R}_0(u,v) \tag{4.73}$$

Here u,v range over a region Σ, bounded by the curve Γ in the uv plane (recall Sec. 4.12). As time progresses, each point originally on S_0 traces out a curve, and we write

$$\mathbf{R} = \mathbf{R}(\mathbf{R}_0,t) \tag{4.74}$$

to describe the location at time t of the point originating from $\mathbf{R}_0$. Substituting Eq. (4.73), we rewrite (4.74) as

$$\mathbf{R} = \mathbf{R}(u,v,t) \tag{4.75}$$

Thus, fixing t and letting u,v roam over Σ, Eq. (4.75) traces out the surface S_t; while holding u and v fixed and varying t produces a function that describes how a single point migrates from surface to surface. In this context it is clear that the velocity $\mathbf{v}$ of a point $\mathbf{R} = \mathbf{R}(u,v,t)$ on S_t is given by

$$\mathbf{v} = \frac{\partial \mathbf{R}}{\partial t}$$

We assume that the orientation of S_t, C_t, Σ, and Γ are all consistent with the parametrization equation (4.75), as in Sec. 4.12. The rigorous derivation of (4.72) then proceeds as follows: we have

$$\Phi = \iint_{S_t} \mathbf{F} \cdot d\mathbf{S} = \iint_{\Sigma} \mathbf{F}(\mathbf{R}(u,v,t),t) \cdot \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\, du\, dv$$

Thus, since Σ is fixed,

$$\begin{aligned} \frac{d\Phi}{dt} &= \iint_{\Sigma} \frac{d\mathbf{F}}{dt} \cdot \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\, du\, dv \\ &\quad + \iint_{\Sigma} \mathbf{F} \cdot \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\right) du\, dv \end{aligned} \tag{4.76}$$

For the first term we have, by the chain rule,

$$\begin{aligned} \frac{d\mathbf{F}[\mathbf{R}(u,v,t),t]}{dt} &= \frac{\partial \mathbf{F}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \mathbf{F}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial \mathbf{F}}{\partial t} \\ &= (\mathbf{v} \cdot \nabla)\mathbf{F} + \frac{\partial \mathbf{F}}{\partial t} \end{aligned} \tag{4.77}$$

The second term requires considerable labor, but nothing more profound than an inspired juggling of identities. Observe that

$$\begin{aligned} \mathbf{F} \cdot \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\right) &= \mathbf{F} \cdot \left[\frac{\partial}{\partial u}\left(\frac{\partial \mathbf{R}}{\partial t} \times \frac{\partial \mathbf{R}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{\partial \mathbf{R}}{\partial t} \times \frac{\partial \mathbf{R}}{\partial u}\right)\right] \\ &= \mathbf{F} \cdot \frac{\partial}{\partial u}\left(\mathbf{v} \times \frac{\partial \mathbf{R}}{\partial v}\right) - \mathbf{F} \cdot \frac{\partial}{\partial v}\left(\mathbf{v} \times \frac{\partial \mathbf{R}}{\partial u}\right) \end{aligned}$$

and, moreover, that this equals

$$\begin{aligned} &\frac{\partial}{\partial u}\left(\mathbf{F} \cdot \mathbf{v} \times \frac{\partial \mathbf{R}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\mathbf{F} \cdot \mathbf{v} \times \frac{\partial \mathbf{R}}{\partial u}\right) - \frac{\partial \mathbf{F}}{\partial u} \cdot \left(\mathbf{v} \times \frac{\partial \mathbf{R}}{\partial v}\right) \\ &\quad + \frac{\partial \mathbf{F}}{\partial v} \cdot \left(\mathbf{v} \times \frac{\partial \mathbf{R}}{\partial u}\right) \end{aligned} \tag{4.78}$$

In the next paragraph we will show that

$$\frac{\partial \mathbf{F}}{\partial v} \cdot \mathbf{v} \times \frac{\partial \mathbf{R}}{\partial u} - \frac{\partial \mathbf{F}}{\partial u} \cdot \mathbf{v} \times \frac{\partial \mathbf{R}}{\partial v} = [(\nabla \cdot \mathbf{F})\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{F}] \cdot \left(\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\right) \tag{4.79}$$

Taking Eq. (4.79) for granted for the moment, we insert it into (4.78) and use the result in (4.76) to derive

$$\begin{aligned} \frac{d\Phi}{dt} &= \iint_{\Sigma} \left[(\mathbf{v} \cdot \nabla)\mathbf{F} + \frac{\partial \mathbf{F}}{\partial t} + (\nabla \cdot \mathbf{F})\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{F}\right] \cdot \left(\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\right) du\, dv \\ &\quad + \iint_{\Sigma} \left[\frac{\partial}{\partial u}\left(\mathbf{F} \cdot \mathbf{v} \times \frac{\partial \mathbf{R}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\mathbf{F} \cdot \mathbf{v} \times \frac{\partial \mathbf{R}}{\partial u}\right)\right] du\, dv \end{aligned}$$

In the last term we interchange the dot and cross products and apply Green's theorem (Sec. 4.11):

$$\begin{aligned}\frac{d\Phi}{dt} &= \iint_{\Sigma}\left[(\nabla\cdot\mathbf{F})\mathbf{v}+\frac{\partial\mathbf{F}}{\partial t}\right]\cdot\left(\frac{\partial\mathbf{R}}{\partial u}\times\frac{\partial\mathbf{R}}{\partial v}\right)du\,dv \\ &\quad+\oint_{\Gamma}\left(\mathbf{F}\times\mathbf{v}\cdot\frac{\partial\mathbf{R}}{\partial u}\,du+\mathbf{F}\times\mathbf{v}\cdot\frac{\partial\mathbf{R}}{\partial v}\,dv\right)\end{aligned} \tag{4.80}$$

Identifying $d\mathbf{S}$ and

$$d\mathbf{R}=\frac{\partial\mathbf{R}}{\partial u}\,du+\frac{\partial\mathbf{R}}{\partial v}\,dv$$

we recover Eq. (4.72).

The proof of the identity (4.79) is best handled with tensor notation. First observe that, by the chain rule,

$$\frac{\partial\mathbf{F}}{\partial u}=\left(\frac{\partial\mathbf{R}}{\partial u}\cdot\nabla\right)\mathbf{F}$$

and similarly for v. Using the symbols $x_i{}^u$, $x_i{}^v$ for the ith components of $\partial\mathbf{R}/\partial u$ and $\partial\mathbf{R}/\partial v$, respectively, we write the left-hand side of (4.79) as

$$\begin{aligned}&\varepsilon_{ijk}(x_\ell{}^v\partial_\ell F_i)v_jx_k{}^u-\varepsilon_{ijk}(x_\ell{}^u\partial_\ell F_i)v_jx_k{}^v \\ &\quad=\varepsilon_{ijk}(x_\ell{}^vx_k{}^u-x_\ell{}^ux_k{}^v)v_j\partial_\ell F_i \\ &\quad=\varepsilon_{ijk}(\delta_{\ell s}\delta_{kt}-\delta_{\ell t}\delta_{ks})x_s{}^vx_t{}^uv_j\partial_\ell F_i \\ &\quad=\varepsilon_{ijk}\varepsilon_{\ell kp}\varepsilon_{stp}x_s{}^vx_t{}^uv_j\partial_\ell F_i && \text{by (1.39)} \\ &\quad=\varepsilon_{ijk}\varepsilon_{p\ell k}\varepsilon_{stp}x_s{}^vx_t{}^uv_j\partial_\ell F_i && \text{by (1.36)} \\ &\quad=(\delta_{ip}\delta_{j\ell}-\delta_{i\ell}\delta_{jp})\varepsilon_{stp}x_s{}^vx_t{}^uv_j\partial_\ell F_i && \text{by (1.39)} \\ &\quad=\varepsilon_{sti}x_s{}^vx_t{}^uv_\ell\partial_\ell F_i-\varepsilon_{stj}x_s{}^vx_t{}^uv_j\partial_i F_i \\ &\quad=-\varepsilon_{its}v_\ell(\partial_\ell F_i)x_t{}^ux_s{}^v+\varepsilon_{jts}(\partial_iF_i)v_jx_t{}^ux_s{}^v \\ &\quad=-[(\mathbf{v}\cdot\nabla)\mathbf{F}]\cdot\frac{\partial\mathbf{R}}{\partial u}\times\frac{\partial\mathbf{R}}{\partial v}+(\nabla\cdot\mathbf{F})\mathbf{v}\cdot\frac{\partial\mathbf{R}}{\partial u}\times\frac{\partial\mathbf{R}}{\partial v}\end{aligned}$$

The identity is proved.

Finally, we add the remark that if the velocity field $\mathbf{v}$, defined on the surface S_t, can be extended as a continuously differentiable vector field throughout some region containing S_t, then Stokes' theorem can be employed to recast (4.72) as

$$\frac{d\Phi}{dt}=\iint_{S_t}\left[\frac{\partial\mathbf{F}}{\partial t}+(\nabla\cdot\mathbf{F})\mathbf{v}+\nabla\times(\mathbf{F}\times\mathbf{v})\right]\cdot d\mathbf{S} \tag{4.81}$$

This would be the case if, for instance, the surface were being transported inside a moving volume of fluid.

Now we turn to the transport theorem for volume integrals. Let $\rho(\mathbf{R},t)$ be a continuously differentiable vector field and let V_t denote the volume of integration at time t. The points inside V_t move with velocity $\mathbf{v}(\mathbf{R},t)$, generating the motion of the volume. Our task is to compute

$$\frac{d}{dt}\iiint_{V_t} \rho(\mathbf{R},t)\,dV \tag{4.82}$$

The answer can be obtained heuristically as follows. If the volume is divided into small rectangular parallelepipeds, each of volume ΔV (as in Sec. 3.3), then the integral in (4.82) is approximated by

$$\sum \rho(\mathbf{R},t)\,\Delta V$$

For the derivative we have

$$\sum \frac{d\rho(\mathbf{R},t)}{dt}\Delta V + \sum \rho(\mathbf{R},t)\frac{d\,\Delta V}{dt}$$

As in Eq. (4.77), we have

$$\frac{d\rho(\mathbf{R},t)}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho$$

while Exercise 13 of Sec. 3.3 states that

$$\frac{1}{\Delta V}\cdot\frac{d\,\Delta V}{dt} = \nabla\cdot\mathbf{v}$$

Putting all this together, we arrive at

$$\begin{aligned}\frac{d}{dt}\iiint_{V_t} \rho\,dV &= \iiint_{V_t}\left(\frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v}\right)dV \\ &= \iiint_{V_t}\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right]dV\end{aligned}$$

Applying the divergence theorem produces

$$\frac{d}{dt}\iiint_{V_t} \rho\,dV = \iiint_{V_t}\frac{\partial \rho}{\partial t}\,dV + \iint_{S_t}\rho\mathbf{v}\cdot d\mathbf{S} \tag{4.83}$$

which is known as *Reynold's transport theorem.*

A more rigorous proof of (4.83) is based on the observation that any continuous scalar field ρ can be written as the divergence of some vector field $\mathbf{F}$; for instance, it is shown in Appendix D that $\mathbf{F}$ defined by

$$\mathbf{F}(\mathbf{R},t) = \frac{1}{4\pi}\iiint \frac{\rho(\mathbf{R}',t)}{|\mathbf{R}-\mathbf{R}'|^3}(\mathbf{R}-\mathbf{R}')\,dV'$$

satisfies

$$\nabla\cdot\mathbf{F}(\mathbf{R},t) = \rho(\mathbf{R},t)$$

With this in hand we use the divergence theorem to express

$$\iiint_{V_t} \rho\, dV = \iint_{S_t} \mathbf{F} \cdot d\mathbf{S}$$

and apply Eq. (4.72) to the (closed) surface S_t to derive

$$\frac{d}{dt}\iiint_{V_t} \rho\, dV = \iint_{S_t} \frac{\partial \mathbf{F}}{\partial t} \cdot d\mathbf{S} + \iint_{S_t} (\nabla \cdot \mathbf{F})\mathbf{v} \cdot d\mathbf{S}$$

One more application of the divergence theorem does it:

$$\begin{aligned}\frac{d}{dt}\iiint_{V_t} \rho\, dV &= \iiint_{V_t} \nabla \cdot \frac{\partial \mathbf{F}}{\partial t}\, dV + \iint_{S_t} (\nabla \cdot \mathbf{F})\mathbf{v} \cdot d\mathbf{S} \\ &= \iiint_{V_t} \frac{\partial \rho}{\partial t}\, dV + \iint_{S_t} \rho\mathbf{v} \cdot d\mathbf{S}\end{aligned}$$

in agreement with (4.83).

Exercises

1. Let S_t be a uniformly expanding hemisphere described by

$$x^2 + y^2 + z^2 = (vt)^2 \qquad z \geq 0$$

and let $\mathbf{F}$ be the vector field

$$\mathbf{F}(\mathbf{R},t) = \mathbf{R}t$$

Verify the flux transport theorem in this case.

2. Verify the flux transport theorem when S_t is the square with corners $(0,0,t)$, $(0,1,t)$, $(1,0,t)$, $(1,1,t)$, and $\mathbf{F}(\mathbf{R},t) = xz\mathbf{k}$.
3. Suppose the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ is rotated about the x axis at a constant angular velocity. Verify the flux transport theorem with the uniform vector field $\mathbf{F}(\mathbf{R},t) = \mathbf{k}$.
4. Verify Reynold's transport theorem for the expanding sphere V_t:

$$x^2 + y^2 + z^2 \leq (vt)^2$$

and

$$\rho(\mathbf{R},t) = |\mathbf{R}|^2 t$$

5. Verify Reynold's theorem for a unit cube with edges parallel to the axes, sliding in the x-direction at constant velocity and with $\rho(\mathbf{R},t) = xy$.
6. Prove *Euler's expansion formula*:

$$\frac{d}{dt}\iiint_{V_t} dV = \frac{d(\text{volume})}{dt} = \iiint_{V_t} \nabla \cdot \mathbf{v}\, dV = \iint_{S_t} \mathbf{v} \cdot d\mathbf{S}$$

Relate this to Exercise 13, Sec. 3.3.

7. Use the continuity equation in Sec. 3.3 together with Reynold's theorem to prove that the mass of a specific portion of a moving fluid remains constant during the flow.

SUPPLEMENTARY PROBLEMS

1. Let C be the curve given by

$$\mathbf{R}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + e^t\mathbf{k} \qquad 0 \le t \le \frac{\pi}{2}$$

and let
$$\mathbf{F} = -\frac{2x}{x^2+y^2}\mathbf{i} + \frac{2y}{x^2+y^2}\mathbf{j} + 2z\mathbf{k}$$

Express $\int_C \mathbf{F} \cdot d\mathbf{R}$ in terms of t and evaluate the resulting integral.

2. Evaluate the following line integrals over the straight-line segment C joining the point (2,1,4) to the point (3,3,4):

(a) $\int_C 3xy\,dx + 3\,dy + yz\,dz$

(b) $\int_C e^{xyz}(yz\,dx + xz\,dy + xy\,dz)$

3. Evaluate

$$\oint [(y + yz\cos xyz)\,dx + (x^2 + xz\cos xyz)\,dy + (z + xy\cos xyz)\,dz]$$

along the ellipse $x = 2\cos\theta$, $y = 3\sin\theta$, $z = 1$, $0 \le \theta \le 2\pi$.

4. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$, where C is the intersection of the plane $x + y + z = 1$ with the cylinder $x^2 + y^2 = 1$ and $\mathbf{F} = (x + y)\mathbf{i} + (y + z)\mathbf{j} + (z + x)\mathbf{k}$. Orient C clockwise as viewed from above.

5. Evaluate

$$\oint_C (\sin x + y^2)\,dx + (x - e^{-y})\,dy$$

when C is the boundary of the semicircular region $x^2 + y^2 \le 4$, $y \ge 0$.

6. Let $\mathbf{F} = (x^2/y)\mathbf{i} + y\mathbf{j} + \mathbf{k}$.

(a) Find the equation for the flow line for $\mathbf{F}$ that passes through the point (1,1,0).

(b) Show that this flow line passes also through the point (e,e,1).

(c) Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{R}$$

where C is the path, along the given flow line, from (1,1,0) to (e,e,1).

7. Let $\mathbf{F}(x,y) = (x^2 + y^2)(\mathbf{i} + \mathbf{j})$, and let C be a directed straight-line segment of unit length, with one end point at the origin (0,0). Find the direction of C such that the line integral

$$I = \int_C \mathbf{F} \cdot d\mathbf{R}$$

is:

(a) a maximum (give the direction of C and the value of I),

(b) a minimum (give the direction of C and the value of I), and

(c) zero (give the direction of C).

8. Let

$$\mathbf{F}(x,y) = \frac{x-y}{x^2+y^2}\mathbf{i} + \frac{x+y}{x^2+y^2}\mathbf{j}$$

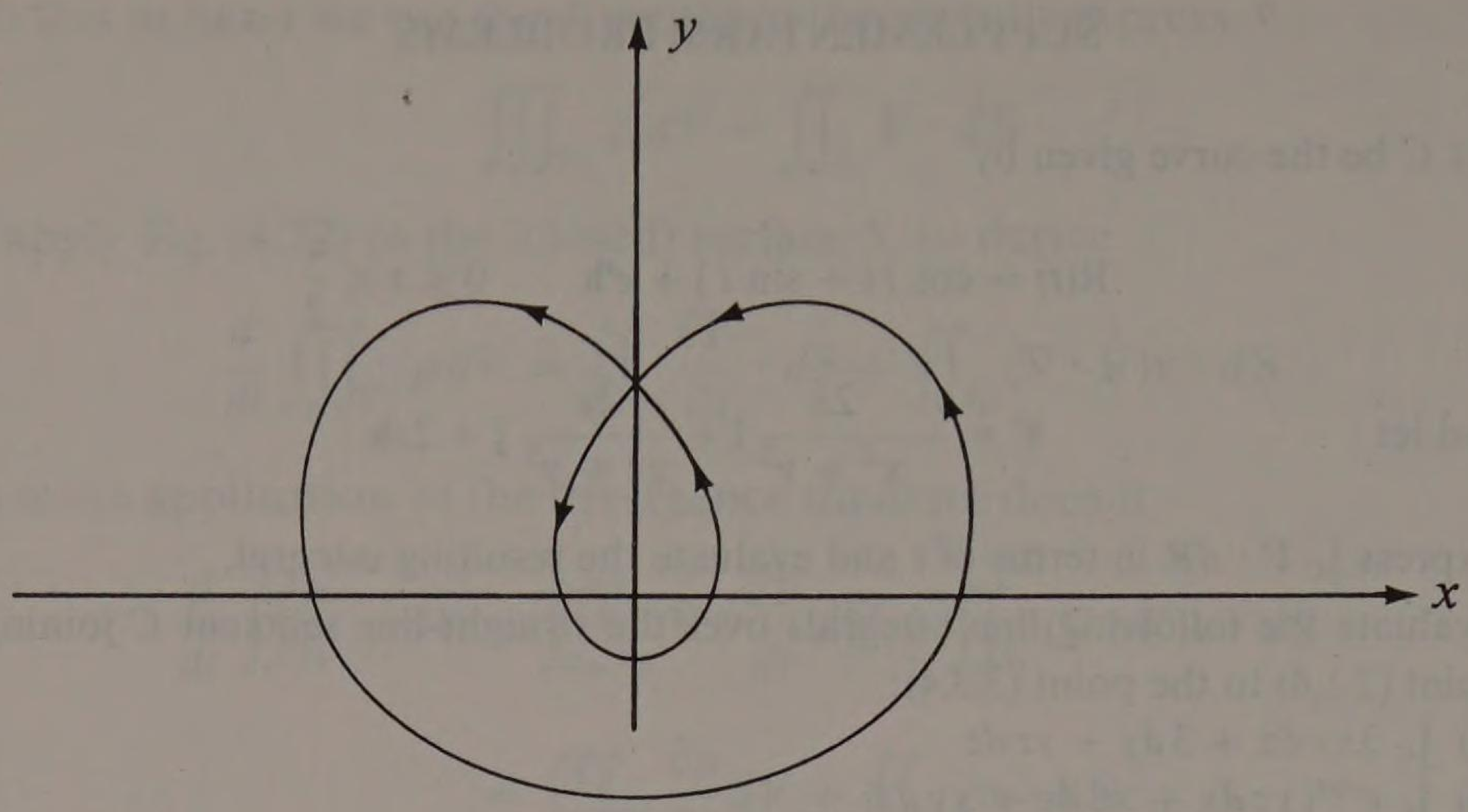

FIGURE 4.42

(a) Show that

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

(b) Is **F** conservative? Justify your answer.
(c) Evaluate $\int \mathbf{F} \cdot d\mathbf{R}$ along the curve shown in Fig. 4.42.

9. Let

$$\mathbf{F} = (6x - 2e^{2x}y^2)\mathbf{i} - 2ye^{2x}\mathbf{j} + \cos z\,\mathbf{k}$$

(a) Determine whether **F** is conservative or not. Explain.
(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$ along the path parametrized by

$$\mathbf{R}(t) = t\mathbf{i} + (t-1)(t-2)\mathbf{j} + \frac{\pi}{2}t^3\mathbf{k} \qquad 0 \le t \le 1$$

(c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$ along

$$\mathbf{R}(t) = \frac{1}{2}(t-1)\mathbf{i} + t(3-t)\mathbf{j} + \frac{\pi}{4}(t-1)\mathbf{k} \qquad 1 \le t \le 3$$

10. Let

$$\mathbf{F} = [(1+x)e^{x+y}]\mathbf{i} + [xe^{x+y} + 2y]\mathbf{j} - 2z\mathbf{k}$$
$$\mathbf{G} = [(1+x)e^{x+y}]\mathbf{i} + [xe^{x+y} + 2z]\mathbf{j} - 2y\mathbf{k}$$

(a) Show that **F** is conservative by finding a potential ϕ for **F**.
(b) Evaluate $\int_C \mathbf{G} \cdot d\mathbf{R}$ when C is the path given by

$$x = (1-t)e^t \qquad y = t \qquad z = 2t \qquad 0 \le t \le 1$$

(*Hint*: Take advantage of the similarity between **F** and **G**.)

11. (a) Show that the field

$$\mathbf{F} = xe^y\mathbf{i} + ye^z\mathbf{j} + ze^x\mathbf{k}$$

is not conservative.

(b) Find a potential ϕ for the field $\mathbf{G}$ when

$$\begin{aligned}\mathbf{G} = {} & [xe^{-x^2} + (xyz + y)e^{zx} + y^2ze^{xy} + ze^{yz}]\mathbf{i} \\ & + [ye^{-y^2} + (yzx + z)e^{xy} + z^2xe^{yz} + xe^{zx}]\mathbf{j} \\ & + [ze^{-z^2} + (zxy + x)e^{yz} + x^2ye^{zx} + ye^{xy}]\mathbf{k}\end{aligned}$$

12. (a) Find a potential ϕ for the field

$$\mathbf{F} = (2xyz + z^2 - 2y^2 + 1)\mathbf{i} + (x^2z - 4xy)\mathbf{j} + (x^2y + 2xz - 2)\mathbf{k}$$

(b) The field

$$\mathbf{G} = \frac{x}{(x^2 + z^2)^2}\mathbf{i} + \frac{z}{(x^2 + z^2)^2}\mathbf{k}$$

satisfies the condition that $\nabla \times \mathbf{G} = \mathbf{0}$ at all points except on the y-axis. Is $\mathbf{G}$ conservative?

13. Find the elements of surface area $d\mathbf{S}$ and dS, in terms of du and dv, for the surface S given parametrically by $x = u^2$, $y = \sqrt{2}uv$, and $z = v^2$.

14. Let $\mathbf{F} = y\mathbf{i} + (x + 2)\mathbf{j} + x^3 \sin(yz)\mathbf{k}$, and let S be the portion of the cylinder $x^2 + y^2 = 1$ that lies in the first octant and below $z = 1$. Calculate

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

15. Find $d\mathbf{S}$ and dS, in terms of $d\phi$ and $d\theta$, for the surface with parametric equations

$$\begin{aligned} x &= (1 + \cos\theta)\cos\phi \\ y &= (1 + \cos\theta)\sin\phi \\ z &= \sin\theta \end{aligned}$$

16. Let $\mathbf{E} = -\mathbf{grad}(|\mathbf{R}|^{-1})$ where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(a) Show that $\mathbf{E} = \mathbf{R}/|\mathbf{R}|^3$.

(b) Find $\int_C \mathbf{E} \cdot d\mathbf{R}$ when C is the line segment joining the points (0,1,0) and (0,0,1).

(c) Compute $\iint_{S_1} \mathbf{E} \cdot d\mathbf{S}$ when S_1 is the sphere $x^2 + y^2 + z^2 = 9$.

(d) Evaluate $\iint_{S_2} \mathbf{E} \cdot d\mathbf{S}$ when S_2 is a cube with edges one unit long, centered at the origin.

(e) Give, if possible, an example of a sphere S with positive radius such that

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = 0$$

17. Let S be the portion of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 0$, and let

$$\mathbf{F} = (y - z)\mathbf{i} - (x + z)\mathbf{j} + (x + y)\mathbf{k}$$

Find $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

18. Find

$$\iiint_V \operatorname{div} \mathbf{F}\, dV$$

when $\mathbf{F} = (x^2 + xy)\mathbf{i} + (y^2 + yz)\mathbf{j} + (z^2 + zx)\mathbf{k}$, and V is the cube centered at the origin and with faces on the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$.

19. Evaluate

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

where $\mathbf{F} = 2y\mathbf{i} + (x - 2x^3 z)\mathbf{j} + xy^3\mathbf{k}$ and where S is the curved surface of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

20. Use Stokes' theorem to evaluate

$$\int_C [x \sin y\,\mathbf{i} - y \sin x\,\mathbf{j} + (x + y)z^2\mathbf{k}] \cdot d\mathbf{R}$$

along the path consisting of straight-line segments successively joining the points $P_0 = (0,0,0)$ to $P_1 = (\pi/2,0,0)$ to $P_2 = (\pi/2,0,1)$ to $P_3 = (0,0,1)$ to $P_4 = (0,\pi/2,1)$ to $P_5 = (0,\pi/2,0)$, and back to $(0,0,0)$.

21. Use the divergence theorem to evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

when $\mathbf{F} = y^2 x\mathbf{i} + x^2 y\mathbf{j} + z^2\mathbf{k}$, and when S is the complete surface of the region bounded by the cylinder $x^2 + y^2 = 4$ and by the planes $z = 0$ and $z = 2$.

22. Let $\mathbf{F} = (x - yz)\mathbf{i} + (y + xz)\mathbf{j} + (z + 2xy)\mathbf{k}$, and let S_1 be the portion of the cylinder $x^2 + y^2 = 2$ that lies inside the sphere $x^2 + y^2 + z^2 = 4$. Let S_2 be the portion of the surface of the sphere $x^2 + y^2 + z^2 = 4$ that lies outside cylinder $x^2 + y^2 = 2$. Let V be the volume bounded by S_1 and S_2.

(a) Draw a diagram illustrating S_1, S_2, and V.

(b) Compute

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS_1$$

with $\mathbf{n}_1$ pointing inward.

(c) Compute

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV$$

(d) Compute

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS_2$$

with $\mathbf{n}_2$ pointing outward.

23. Let $\mathbf{F} = xyz\mathbf{i} + (y^2 + 1)\mathbf{j} + z^3\mathbf{k}$, and let S be the surface of the unit cube $0 \leq x, y, z \leq 1$. Evaluate the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

using

(a) the divergence theorem,

(b) Stokes' theorem, and

(c) direct computation.

24. Let $\mathbf{F}$ be the field $\mathbf{F} = ye^x\mathbf{i} + (x + e^x)\mathbf{j} + z^2\mathbf{k}$ and let C be the curve given by

$$\mathbf{R}(t) = (1 + \cos t)\mathbf{i} + (1 + \sin t)\mathbf{j} + (1 - \sin t - \cos t)\mathbf{k}$$

for $0 \le t \le 2\pi$. Find

$$\int_C \mathbf{F} \cdot d\mathbf{R}$$

(*Hint*: Use Stokes' theorem, observing that C is contained in a certain plane and that the projection of C on the xy plane is a circle.)

25. If $\mathbf{F} = xz\mathbf{i} - y\mathbf{j} + x^2y\mathbf{k}$, use Stokes' theorem to evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{R}$$

where C is the closed path consisting of the edges of the triangle with vertices at the points $P_1 = (1,0,0)$, $P_2 = (0,0,1)$, $P_3 = (0,0,0)$ transversed from P_1 to P_2 to P_3, and back to P_1.

26. Indicate which of the following statements are TRUE, and which are FALSE. You may assume that all functions have continuous derivatives for all orders at all points:

(a) The divergence of $\nabla \times \mathbf{F}$ is zero, for every $\mathbf{F}$.

(b) In a simply connected region, $\int_C \mathbf{F} \cdot d\mathbf{R}$ depends only on the endpoints of C.

(c) If $\nabla f = \mathbf{0}$, then f is a constant function.

(d) If $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a constant vector field.

(e) If div $\mathbf{F} = 0$, then $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$ for every closed surface S.

(f) If $\int_C \mathbf{F} \cdot d\mathbf{R} = 0$ for every closed contour C, then $\nabla \times \mathbf{F} = \mathbf{0}$.

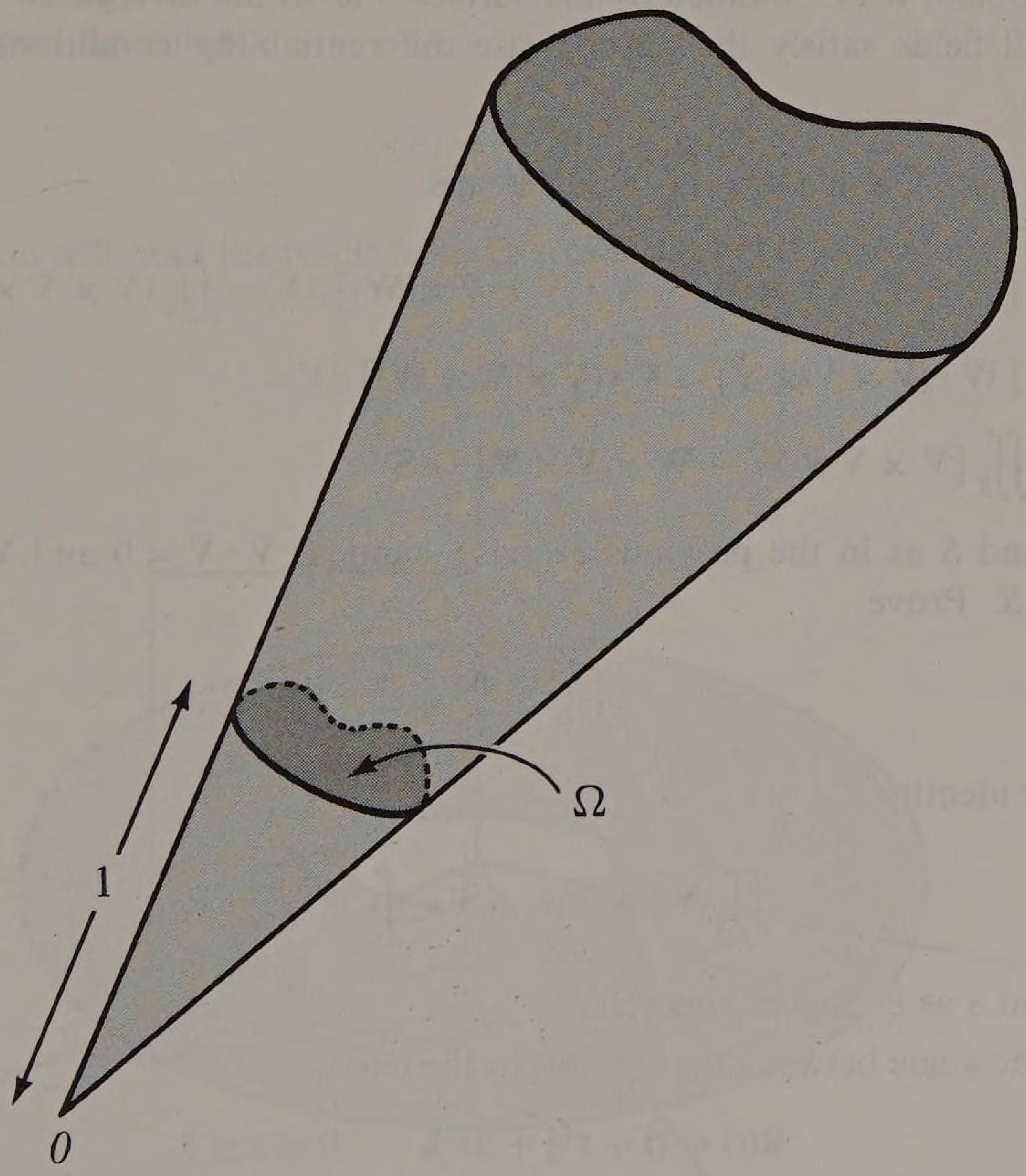

FIGURE 4.43

27. Consider a cone with vertex at the origin, as in Fig. 4.43. The *solid angle* Ω at the vertex is defined to be the surface area that this cone cuts out of the unit sphere centered at the origin.

(a) If the cone is perfectly flat, i.e., a plane, what is Ω?

(b) What is Ω for the corner of a cube?

(c) What is Ω for the 45° cone: $z = (x^2 + y^2)^{\frac{1}{2}}$?

(d) What is the total solid angle around a point?

(e) Suppose the surface S, bounded by the simple closed curve C, has the property that every ray from the origin intersects S at most once. Then the *solid angle Ω subtended at the origin by S* is the solid angle at the vertex of the cone generated by the rays through C. Show that, if S is properly oriented,

$$\Omega = \iint_S \frac{\mathbf{R} \cdot d\mathbf{S}}{|\mathbf{R}|^3} \qquad (\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

28. Use the results of the previous exercise to check Gauss' law, Eq. (4.39), for a point charge at the origin. The expression for the electric field appears in Example 4.19.

29. The flux of a solenoidal field through a surface depends only on the curve bounding the surface. Explain this.

30. Show that any level curve $\mathbf{R}(t)$ for the function $f(x,y,z)$ satisfies

$$\frac{d\mathbf{R}}{dt} \cdot \nabla f = 0$$

31. Let the domain D be bounded by the surface S as in the divergence theorem, and assume all fields satisfy the appropriate differentiability conditions. Prove the identities:

(a) $\iiint_D \nabla\phi \cdot \nabla \times \mathbf{F}\, dV = \iint_S \mathbf{F} \times \nabla\phi \cdot d\mathbf{S}$

(b) $\iiint_D [(\nabla \times \mathbf{V}) \cdot (\nabla \times \mathbf{W}) - \mathbf{V} \cdot (\nabla \times \nabla \times \mathbf{W})]\, dV = \iint_S (\mathbf{V} \times \nabla \times \mathbf{W}) \cdot d\mathbf{S}$

(c) $\iiint_D [\mathbf{W} \cdot (\nabla \times \nabla \times \mathbf{V}) - \mathbf{V} \cdot (\nabla \times \nabla \times \mathbf{W})]\, dV =$
$\iint_S [\mathbf{V} \times \nabla \times \mathbf{W} - \mathbf{W} \times \nabla \times \mathbf{V}] \cdot d\mathbf{S}$

32. With D and S as in the previous exercise, suppose $\nabla \cdot \mathbf{V} = 0$ and $\mathbf{W} = \nabla\phi$ with $\phi = 0$ on S. Prove

$$\iiint_D \mathbf{V} \cdot \mathbf{W}\, dV = 0$$

33. Prove the identity

$$\iint_S \nabla\phi \times \nabla\psi \cdot d\mathbf{S} = \oint_C \phi\nabla\psi \cdot d\mathbf{R}$$

with C and S as in Stokes' theorem.

34. What is the angle between the tangent to the curve

$$\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t^2\mathbf{k} \qquad 0 \le t \le 3$$

and the normal to the surface $z = 16 - x^2 - y$ at their point of intersection?

35. Use Green's theorem to find the area inside the loop of *Descartes' folium*

$$x = \frac{t}{1+t^3}$$

$$y = \frac{t^2}{1+t^3} \qquad 0 \le t < \infty$$

36. A function $f(x,y,z)$ is said to be *homogeneous of degree k* if $f(tx,ty,tz) = t^k f(x,y,z)$. Suppose the components F_1, F_2, F_3 of the vector field $\mathbf{F}(x,y,z)$ are each homogeneous of degree k, and $\mathbf{curl\ F} = \mathbf{0}$. Prove

$$\mathbf{F} = \nabla\left(\frac{xF_1 + yF_2 + zF_3}{k+1}\right)$$

37. A *torus* (doughnut) is shown in Fig. 4.44. Its major radius is A and its minor radius is a. Derive the parametrization $\mathbf{R}(u,v)$ in terms of the *toroidal angle u* and the *poloidal angle v*

$$x = A \cos u + a \cos u \cos v$$

$$y = A \sin u + a \sin u \cos v$$

$$z = a \sin v$$

Show that the area of the torus is $4\pi^2 Aa$.

38. Show that if ϕ is harmonic and S is a sphere of radius R centered at P, then the third Green formula (Exercise 16, Section 4.10) reduces to the *mean value theorem for harmonic functions*

$$\phi(P) = \frac{1}{4\pi R^2} \iint_S \phi \, dS$$

[*Hint*: You will need the result from Exercise 10(a) of that section.]

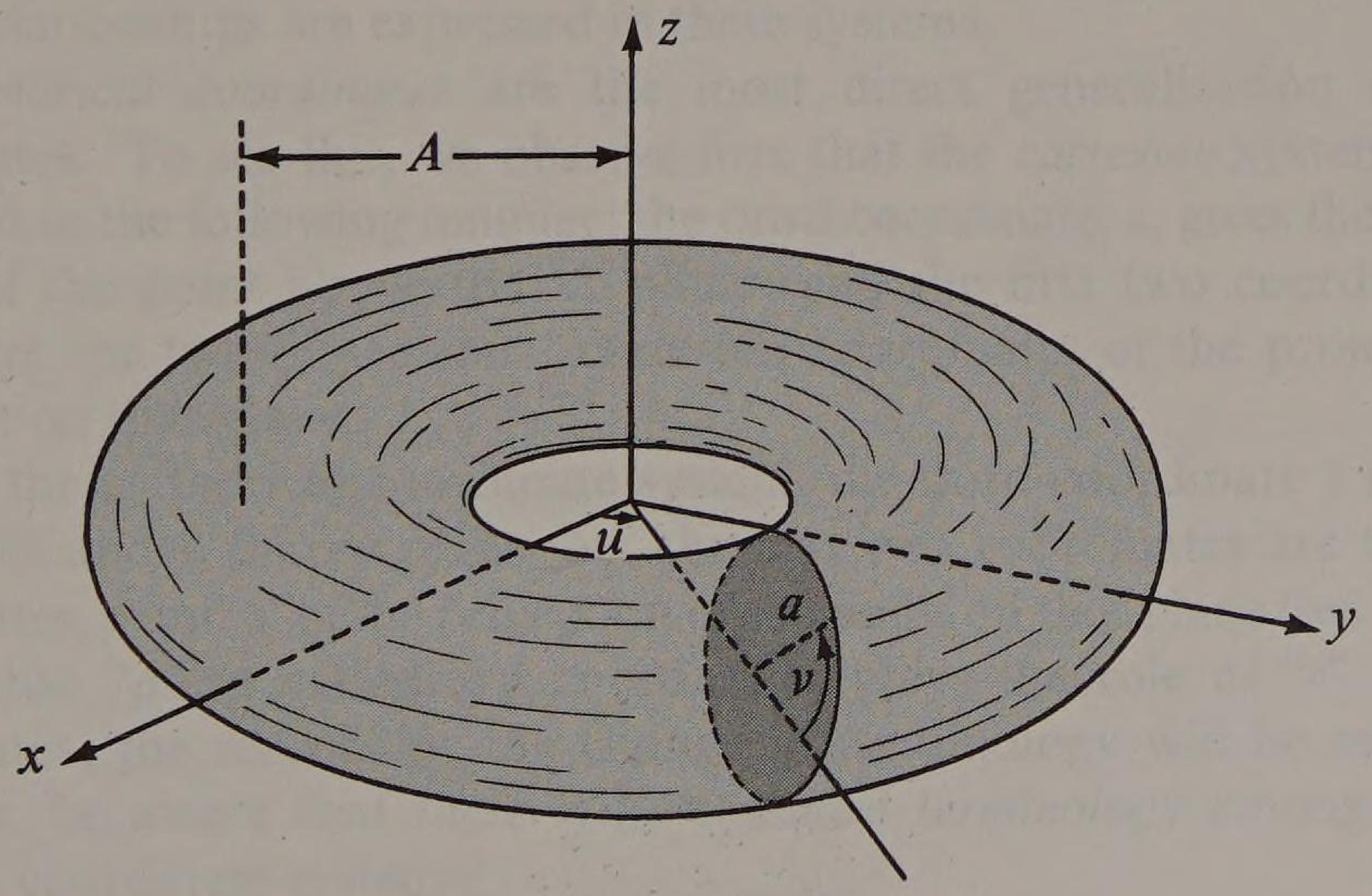

FIGURE 4.44

39. What is the value of the surface integral in the third Green formula if ϕ is harmonic and P lies *outside* the closed surface S?

40. Suppose that the vector fields $\mathbf{V}$ and $\mathbf{W}$ have identical divergences and curls in the region D, and that they have the same normal component on the bounding surface S. Prove $\mathbf{V} = \mathbf{W}$. (*Hint*: Consider the properties of $\mathbf{U} = \mathbf{V} - \mathbf{W}$.)

CHAPTER 5

Generalized Orthogonal Coordinates

5.1 CYLINDRICAL AND SPHERICAL COORDINATES

Recall (Sec. 2.4) that many two-dimensional problems can be expressed more conveniently in polar coordinates than in cartesian coordinates. This is the case, for instance, if some type of circular symmetry is present. Analogous situations arise in three dimensions, of course, and therefore one is led to construct generalizations of the polar coordinate system. The two generalizations that have proved most useful are *cylindrical coordinates* and *spherical coordinates*. In this section we shall study how the various vector relationships are expressed in these systems.

Cylindrical coordinates are the most direct generalization of polar coordinates. To see this, we observe first that the *cartesian* system can be described in the following manner: the third coordinate, z, gives the (signed) height of the point above the xy plane; and the first two coordinates, x and y, are the two-dimensional cartesian coordinates of the projection of the point on that plane.

For the cylindrical coordinate system, the third coordinate z is, again, the height above the xy plane, but the first two coordinates are the *polar* coordinates, ρ and θ, of the projection of the point on that plane (see Fig. 5.1). Notice that "ρ" in cylindrical coordinates plays the role of "r" in polar coordinates; the reason for the change in terminology will be seen later. However, be aware that *there is no standard terminology* among authors for these coordinate systems!

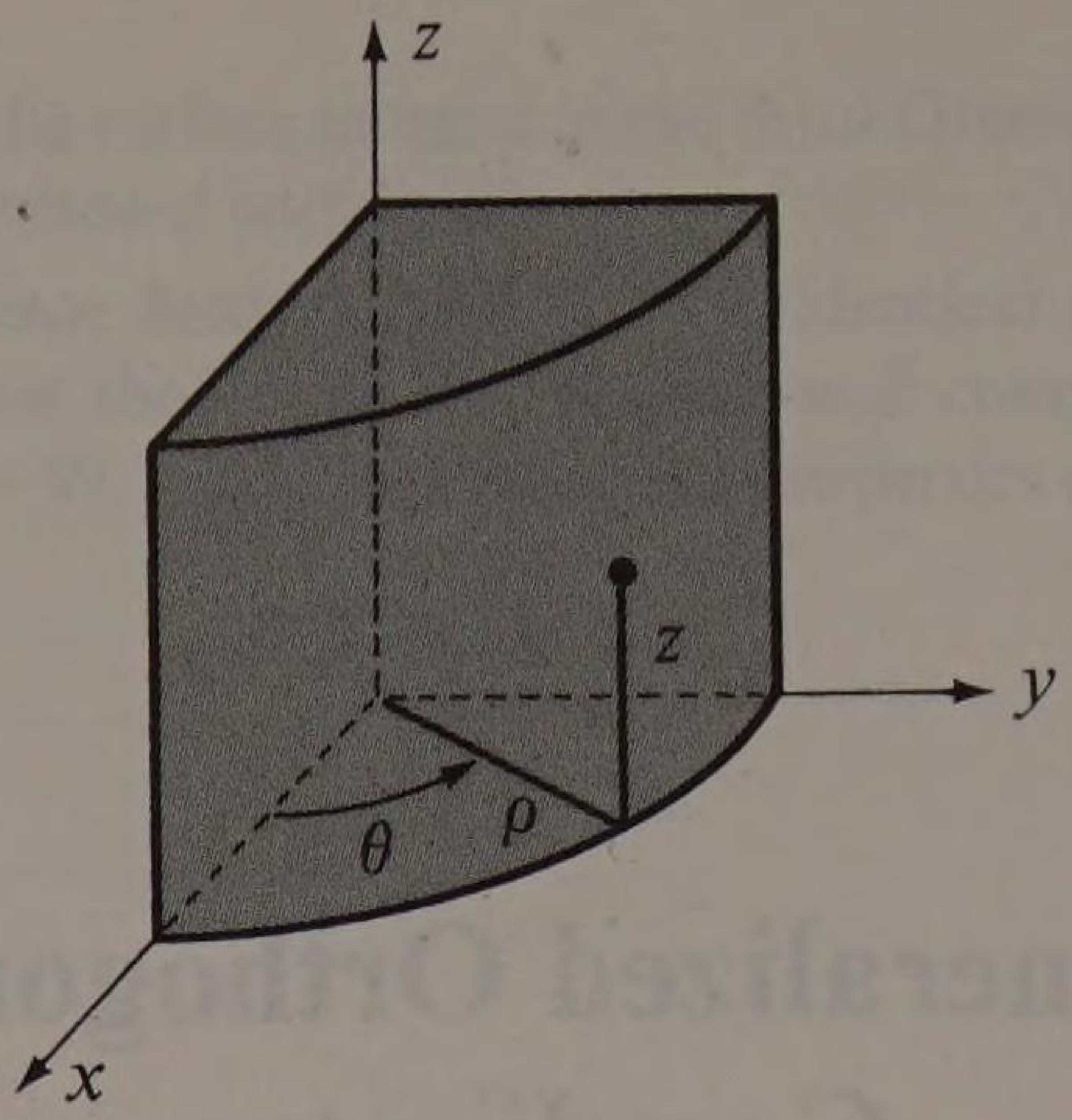

FIGURE 5.1

$$
\begin{aligned}
x &= \rho \cos\theta & \rho &= (x^2 + y^2)^{\frac{1}{2}} \\
y &= \rho \sin\theta & \theta &= \sin^{-1}\frac{y}{\sqrt{x^2+y^2}} = \cos^{-1}\frac{x}{\sqrt{x^2+y^2}} \\
z &= z & z &= z
\end{aligned}
\tag{5.1}
$$

The extra equation for θ serves to remind us that we use the value θ appropriate to the quadrant of (x, y), not necessarily the principal value.

The angle θ in cylindrical coordinates is not defined on the z axis, when $\rho = 0$, but otherwise the equations of (5.1) specify a one-to-one correspondence between the two systems.

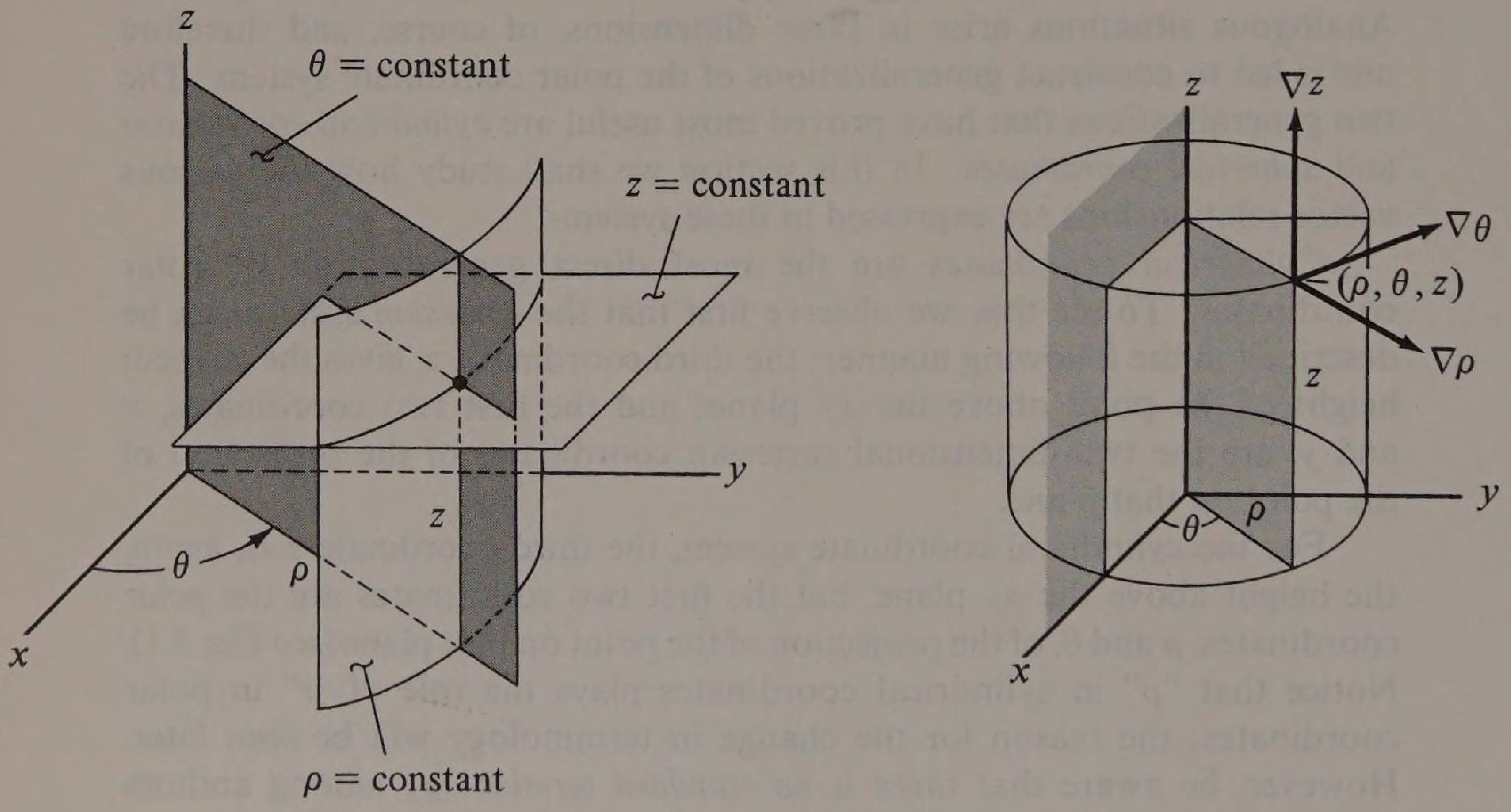
z
θ = constant
z = constant
y
z
θ
ρ
x
ρ = constant
z
∇z
∇θ
(ρ, θ, z)
∇ρ
z
y
θ
ρ
x

FIGURE 5.2

The nomenclature "cylindrical" comes from the fact that the surfaces $\rho = constant$ are cylinders. The surfaces $\theta = constant$ are "half-planes" extending out from the z axis, and, of course, $z = constant$ defines a family of horizontal planes. Normals to these surfaces are given by **grad** ρ, **grad** θ, and **grad** z, respectively. From Fig. 5.2, we can see that **grad** ρ points away from the z axis, **grad** θ points counterclockwise in the horizontal plane, and **grad** z points upward. In the order **grad** ρ, **grad** θ, and **grad** z, these vectors form a right-handed orthogonal system.

Any two surfaces $\rho = constant$ and $\theta = constant$ intersect in a vertical line, which is a curve along which only z varies. It is called a *coordinate curve* for z. Coordinate curves for ρ are horizontal rays extending from the z axis. Coordinate curves for θ are horizontal circles. Notice that **grad** z, **grad** ρ, and **grad** θ are everywhere tangent to their respective coordinate curves.

These features make it convenient to introduce unit vectors in the directions of **grad** z, **grad** ρ, and **grad** θ. Accordingly, we define

$$\begin{aligned} \mathbf{e}_z &= \frac{\mathbf{grad}\ z}{|\mathbf{grad}\ z|} \\ \mathbf{e}_\rho &= \frac{\mathbf{grad}\ \rho}{|\mathbf{grad}\ \rho|} \\ \mathbf{e}_\theta &= \frac{\mathbf{grad}\ \theta}{|\mathbf{grad}\ \theta|} \end{aligned} \tag{5.2}$$

The reader should convince himself that $\mathbf{e}_z$ is the same as $\mathbf{k}$, and $\mathbf{e}_\rho$ and $\mathbf{e}_\theta$ are the three-dimensional analogs of $\mathbf{u}_r$ and $\mathbf{u}_\theta$ in Sec. 2.4. In fact, recalling that $|\mathbf{grad}\, f| = df/ds$ when s measures distance in the direction of $\mathbf{grad}\, f$, one can simplify these equations. Along the coordinate curves of z, $ds = |dz|$. Hence, $|\mathbf{grad}\ z| = dz/dz = 1$. Along the coordinate curves of ρ, $ds = |d\rho|$. Hence also, $|\mathbf{grad}\ \rho| = d\rho/d\rho = 1$. But along coordinate curves of θ, $ds = \rho|d\theta|$. Therefore, $|\mathbf{grad}\ \theta| = d\theta/\rho d\theta = 1/\rho$. This results in

$$\begin{aligned} \mathbf{e}_z &= \mathbf{grad}\ z \\ \mathbf{e}_\rho &= \mathbf{grad}\ \rho \\ \mathbf{e}_\theta &= \rho\ \mathbf{grad}\ \theta \end{aligned} \tag{5.3}$$

The reader should observe that the position vector of a point can be expressed in cylindrical coordinates as

$$\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \rho\mathbf{e}_\rho + z\mathbf{e}_z \tag{5.4}$$

To compute arc length in cylindrical coordinates, observe that in Fig. 5.3 the displacement $d\mathbf{R}$ can be expressed as the sum of three orthogonal displacements

$$d\mathbf{R} = \mathbf{e}_\rho\, d\rho + \mathbf{e}_\theta \rho\, d\theta + \mathbf{e}_z\, dz \tag{5.5}$$

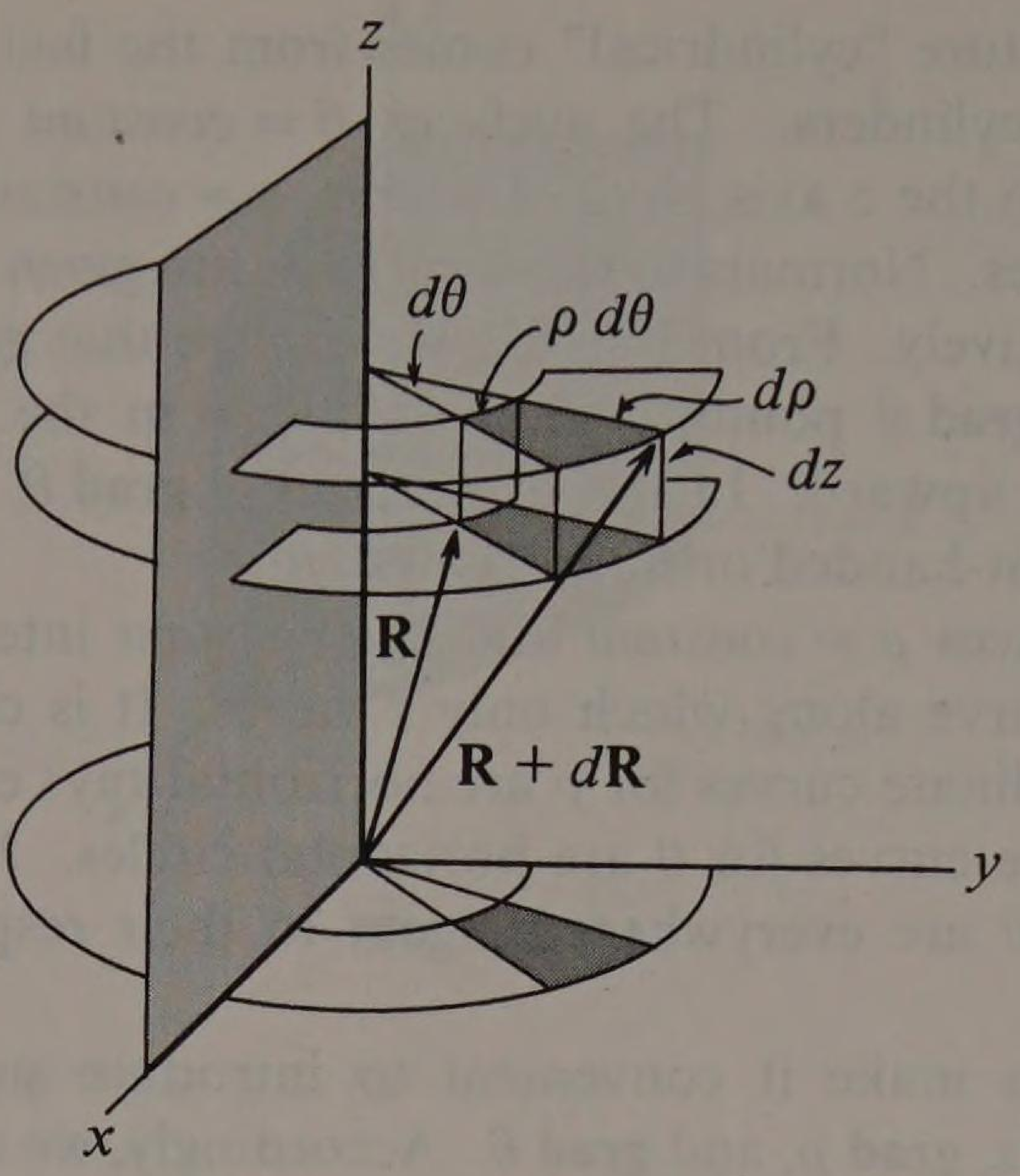

FIGURE 5.3

Hence, *the element of arc length in cylindrical coordinates is given by*

$$ds = |d\mathbf{R}| = (d\rho^2 + \rho^2\, d\theta^2 + dz^2)^{\frac{1}{2}} \tag{5.6}$$

Example 5.1 Find the arc length along the helix

$$x = \sin t \qquad y = \cos t \qquad z = t$$

for $0 \le t \le 4\pi$.

Solution Transforming to cylindrical coordinates we find

$$\rho = 1 \qquad \theta = \frac{\pi}{2} - t \qquad z = t$$

Hence

$$\begin{aligned} s &= \int_0^{4\pi} \left[\left(\frac{d\rho}{dt}\right)^2 + \rho^2 \left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right]^{\frac{1}{2}} dt \\ &= \int_0^{4\pi} [1 + 1]^{\frac{1}{2}}\, dt = 4\sqrt{2}\pi \end{aligned}$$

From Fig. 5.3 it is easy to see that *the element of volume in cylindrical coordinates is given by*

$$dV = d\rho\, \rho\, d\theta\, dz = \rho\, d\rho\, d\theta\, dz \tag{5.7}$$

Example 5.2 Find the volume integral of the function $f(x,y,z) = x^2 + y^2$ over the volume contained between two cylinders, $\rho = 1$ and $\rho = 2$, for $0 \le z \le 2$.

Solution

$$\iiint (x^2 + y^2)\,dV = \int_0^2 \int_0^{2\pi} \int_1^2 \rho^3\,d\rho\,d\theta\,dz = 2(2\pi)\left(\frac{2^4}{4} - \frac{1^4}{4}\right) = 15\pi$$

Now consider a scalar field f expressed in cylindrical coordinates, $f = f(\rho,\theta,z)$. To express **grad** f in these coordinates, we observe that since $\mathbf{e}_\rho$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ are mutually orthogonal unit vectors, $\nabla f = (\mathbf{e}_\rho \cdot \nabla f)\mathbf{e}_\rho + (\mathbf{e}_\theta \cdot \nabla f)\mathbf{e}_\theta + (\mathbf{e}_z \cdot \nabla f)\mathbf{e}_z$. Each of these coefficients gives the rate of change of f with respect to distance, df/ds, in the corresponding direction. Applying the expression (5.6) for ds, we find

$$\mathbf{e}_\rho \cdot \nabla f = \left.\frac{\partial f}{\partial s}\right|_{\theta,z\text{ constant}} = \frac{\partial f}{\partial \rho}$$

$$\mathbf{e}_\theta \cdot \nabla f = \left.\frac{\partial f}{\partial s}\right|_{\rho,z} = \frac{1}{\rho}\frac{\partial f}{\partial \theta}$$

$$\mathbf{e}_z \cdot \nabla f = \left.\frac{\partial f}{\partial s}\right|_{\rho,\theta} = \frac{\partial f}{\partial z}$$

Therefore *the expression for* **grad** f *in cylindrical coordinates is*

$$\mathbf{grad}\ f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z \tag{5.8}$$

Example 5.3 Compute **grad** f in cylindrical coordinates if f is given in cartesian coordinates by $f(x,y,z) = z/(x^2 + y^2)$.

Solution First we express f in cylindrical coordinates then apply Eq. (5.8). We have $f'(\rho,\theta,z) = f(x(\rho,\theta,z), y(\rho,\theta,z), z(\rho,\theta,z)) = z/\rho^2$. Hence,

$$\mathbf{grad}\ f = -\frac{2z}{\rho^3}\mathbf{e}_\rho + \frac{1}{\rho^2}\mathbf{e}_z$$

The expressions for the divergence and curl of a vector field can be derived by heuristic reasoning with infinitesimals (as in Sec. 3.3), but one must be extra careful when the coordinate system is not rectilinear.

Let us compute div **F** as flux per unit volume out of the box in Fig. 5.4. We start with the vector field **F** given in cylindrical coordinates.

$$\mathbf{F} = F_\rho(\rho,\theta,z)\mathbf{e}_\rho + F_\theta(\rho,\theta,z)\mathbf{e}_\theta + F_z(\rho,\theta,z)\mathbf{e}_z \tag{5.9}$$

The flux of **F** out of face I equals the outward-normal component of **F** times the surface area, $(-F_\rho)(\rho\,d\theta\,dz)$. A similar expression holds for face IV, but with a different value of ρ. Hence the contribution from faces I and IV will be given, in the limit, by

$$(F_\rho \rho\,d\theta\,dz)_{\text{IV}} - (F_\rho \rho\,d\theta\,dz)_{\text{I}} = \frac{\partial(\rho F_\rho)}{\partial \rho}\,d\rho\,d\theta\,dz \tag{5.10}$$

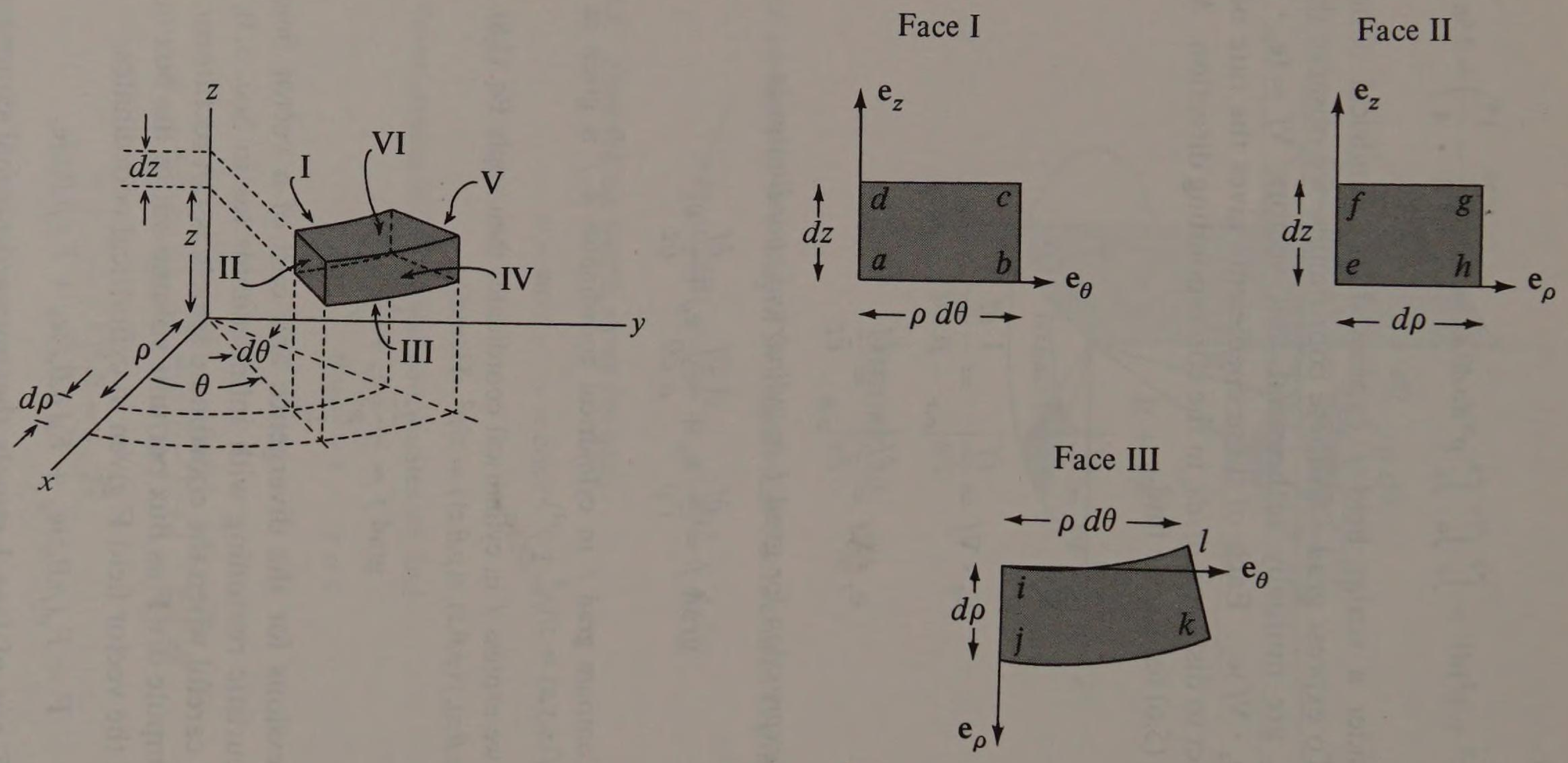
Face I
Face II
Face III
z
dz
z
x
y
ρ
dρ
θ
dθ
I
II
III
IV
V
VI
$\mathbf{e}_z$
$\mathbf{e}_\theta$
$\mathbf{e}_\rho$
dz
$\rho\, d\theta$
$d\rho$
a
b
c
d
e
f
g
h
i
j
k
l

FIGURE 5.4

Notice that we regard the dimensions $d\theta$ and dz as the same for faces I and IV, so they are constants in (5.10).

The flux out of face II is $(-F_\theta)\,d\rho\,dz$, and combining this with face V we obtain a contribution

$$(F_\theta\,d\rho\,dz)_{\mathrm{V}} - (F_\theta\,d\rho\,dz)_{\mathrm{II}} = \frac{\partial F_\theta}{\partial\theta}\,d\theta\,d\rho\,dz \tag{5.11}$$

The flux out of faces III and VI contributes

$$(F_z\rho\,d\theta\,d\rho)_{\mathrm{VI}} - (F_z\rho\,d\theta\,d\rho)_{\mathrm{III}} = \frac{\partial F_z}{\partial z}\,dz\,\rho\,d\theta\,d\rho \tag{5.12}$$

The reader may feel a little queasy about this last expression since face III is not a genuine rectangle, having side ℓi of length $\rho\,d\theta$ and the opposite side kj of length $(\rho + d\rho)\,d\theta$. To play it safe, we may replace ρ in (5.12) by $\tilde{\rho}$, some intermediate value between ρ and $\rho + d\rho$. Adding all the contributions to the flux, we obtain

$$\frac{\partial(\rho F_\rho)}{\partial\rho}\,d\rho\,d\theta\,dz + \frac{\partial F_\theta}{\partial\theta}\,d\theta\,d\rho\,dz + \frac{\partial F_z}{\partial z}\,dz\,\tilde{\rho}\,d\theta\,d\rho$$

Next we divide by the volume element (5.7) and, noticing that in the limit, $\tilde{\rho} \to \rho$ (so our precaution was unnecessary), we find that *the divergence of a vector field is given in cylindrical coordinates by*

$$\operatorname{div} \mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z} \tag{5.13}$$

Example 5.4 Compute the divergence of

$$\mathbf{F}(\rho,\theta,z) = \rho\;\mathbf{e}_\rho + z\sin\theta\,\mathbf{e}_\theta + \rho z\mathbf{e}_z \tag{5.14}$$

Solution Applying (5.13)

$$\nabla\cdot\mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho^2)}{\partial\rho} + \frac{1}{\rho}\frac{\partial(z\sin\theta)}{\partial\theta} + \frac{\partial(\rho z)}{\partial z} = 2 + \frac{z\cos\theta}{\rho} + \rho$$

We compute **curl F** by employing the physical characterization of curl as "swirl" per unit area; see Eq. (4.66). To compute the $\mathbf{e}_\rho$ component, consider the line integral of **F** around the edge of face I in Fig. 5.4. The edge must be oriented in the order *abcda* since $\mathbf{e}_\rho$ points toward the reader.

The contribution to the line integral along *ab* is $F_\theta\rho\,d\theta$; along *cd* it is $(-F_\theta)\rho\,d\theta$, but with a different value of z. Since ρ and $d\theta$ are the same along these edges, we obtain a net contribution of

$$(F_\theta\rho\,d\theta)_{ab} - (F_\theta\rho\,d\theta)_{cd} = -\frac{\partial F_\theta}{\partial z}\,dz\,\rho\,d\theta$$

Similarly, edges *bc* and *da* contribute

$$(F_z\,dz)_{bc} - (F_z\,dz)_{da} = \frac{\partial F_z}{d\theta}\,d\theta\,dz$$

Hence the $\mathbf{e}_\rho$ component of **curl F** is the sum of these divided by the area $\rho\,d\theta\,dz$, or

$$\frac{1}{\rho}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \tag{5.15}$$

To find the $\mathbf{e}_\theta$ component, we integrate around the edge *efghe* of face II (because $\mathbf{e}_\theta$ points into the page). The line integral is

$$\begin{aligned}(F_z\,dz)_{ef} &- (F_z\,dz)_{gh} + (F_\rho\,d\rho)_{fg} - (F_\rho\,d\rho)_{he}\\ &= -\frac{\partial F_z}{\partial \rho}\,d\rho\,dz + \frac{\partial F_\rho}{\partial z}\,dz\,d\rho\end{aligned}$$

and dividing by the area $d\rho\,dz$ we find the $\mathbf{e}_\theta$ component to be

$$\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \tag{5.16}$$

The $\mathbf{e}_z$ component is obtained by integrating around the edge *ijkli* of face III.

$$\begin{aligned}(F_\rho\,d\rho)_{ij} &- (F_\rho\,d\rho)_{kl} + (F_\theta \rho\,d\theta)_{jk} - (F_\theta \rho\,d\theta)_{li}\\ &= -\frac{\partial F_\rho}{\partial \theta}\,d\theta\,d\rho + \frac{\partial(\rho F_\theta)}{\partial \rho}\,d\rho\,d\theta\end{aligned}$$

(keeping in mind that ρ on edge *li* is different from ρ on edge *jk*). Dividing by the area $\tilde{\rho}\,d\theta\,d\rho$, with $\tilde{\rho}$ between ρ and $\rho + d\rho$ as before, and then taking the limit, we find the $\mathbf{e}_z$ component of **curl F** to be

$$\frac{1}{\rho}\left[-\frac{\partial F_\rho}{\partial \theta} + \frac{\partial(\rho F_\theta)}{\partial \rho}\right] \tag{5.17}$$

Combining these components, we see that *the curl of a vector field is given in cylindrical coordinates by*

$$\begin{aligned}\mathbf{curl\ F} &= \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\mathbf{e}_\rho + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\mathbf{e}_\theta\\ &\quad + \frac{1}{\rho}\left(\frac{\partial(\rho F_\theta)}{\partial \rho} - \frac{\partial F_\rho}{\partial \theta}\right)\mathbf{e}_z\end{aligned} \tag{5.18}$$

or, equivalently (Exercise 3),

$$\mathbf{curl\ F} = \frac{1}{\rho}\begin{vmatrix}\mathbf{e}_\rho & \rho\mathbf{e}_\theta & \mathbf{e}_z\\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z}\\ F_\rho & \rho F_\theta & F_z\end{vmatrix} \tag{5.19}$$

Example 5.5 Compute the curl of **F** given in Eq. (5.14).

Solution Applying (5.19).

$$\text{curl } \mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ \rho & \rho z\sin\theta & \rho z \end{vmatrix}$$

$$= (0 - \rho\sin\theta)\frac{\mathbf{e}_\rho}{\rho} + (0 - z)\mathbf{e}_\theta + (z\sin\theta - 0)\frac{\mathbf{e}_z}{\rho}$$

$$= -\sin\theta\,\mathbf{e}_\rho - z\mathbf{e}_\theta + \frac{z\sin\theta}{\rho}\mathbf{e}_z$$

Spherical coordinates are also generalizations of polar coordinates in the plane. The first coordinate, r, is the distance of the point from the origin; hence, it is a three-dimensional generalization of the two-dimensional "r." The second coordinate, ϕ, is the angle between the positive z axis and the position vector **R** (see Fig. 5.5). The third coordinate, θ, is the same angle as in the cylindrical coordinate system.

Surfaces of constant r are, of course, spheres centered at the origin. Surfaces of constant ϕ are cones—right circular cones in fact (see Fig. 5.6). Surfaces of constant θ are half-planes, as in the case of cylindrical coordinates.

The reader should be sure he understands why the angle ϕ is restricted, by definition, to lie between 0 and π radians.

The equations of transformation between spherical and cartesian coordinates are easy to see once we recognize that the cylindrical coordinate

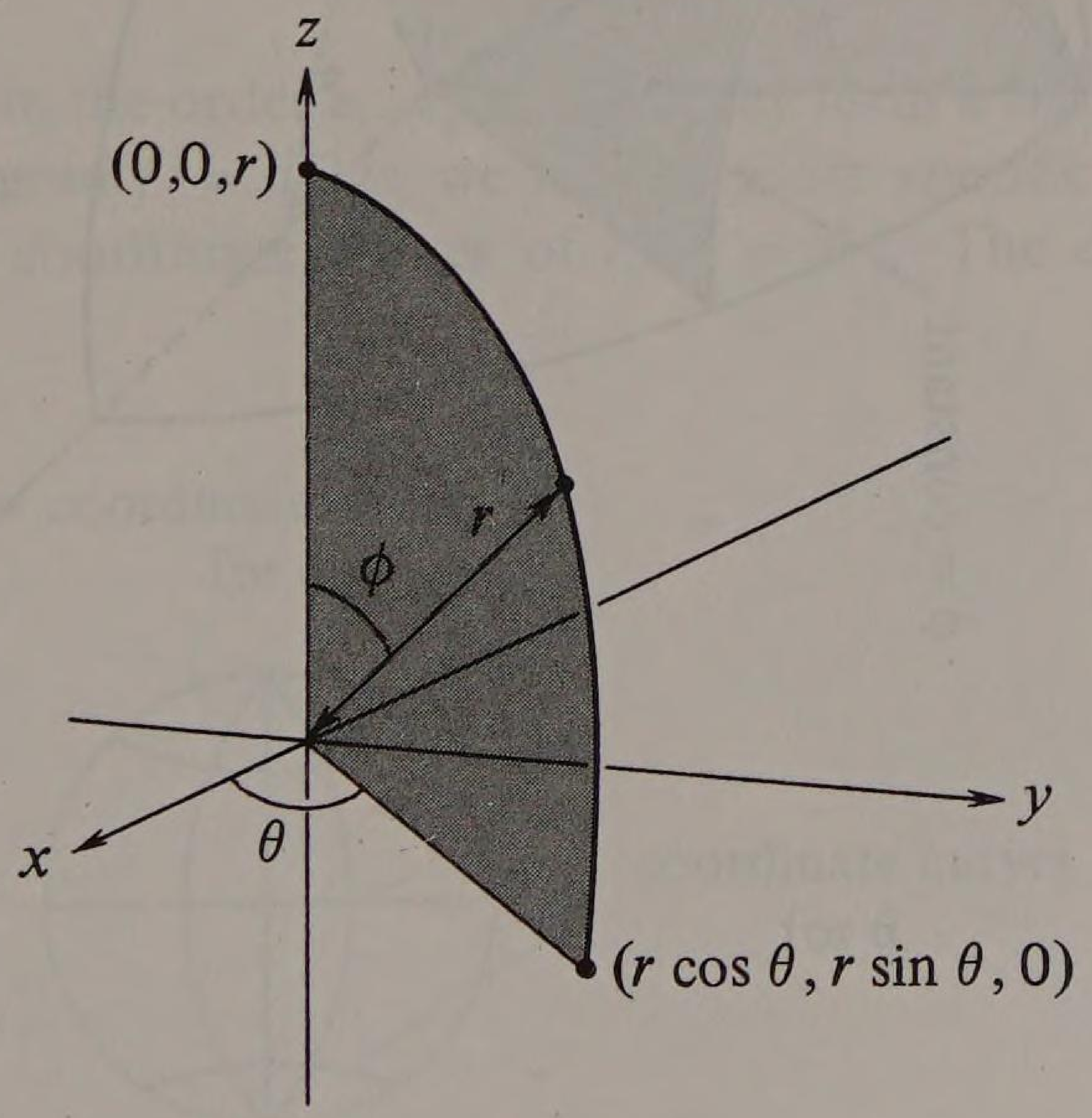

FIGURE 5.5

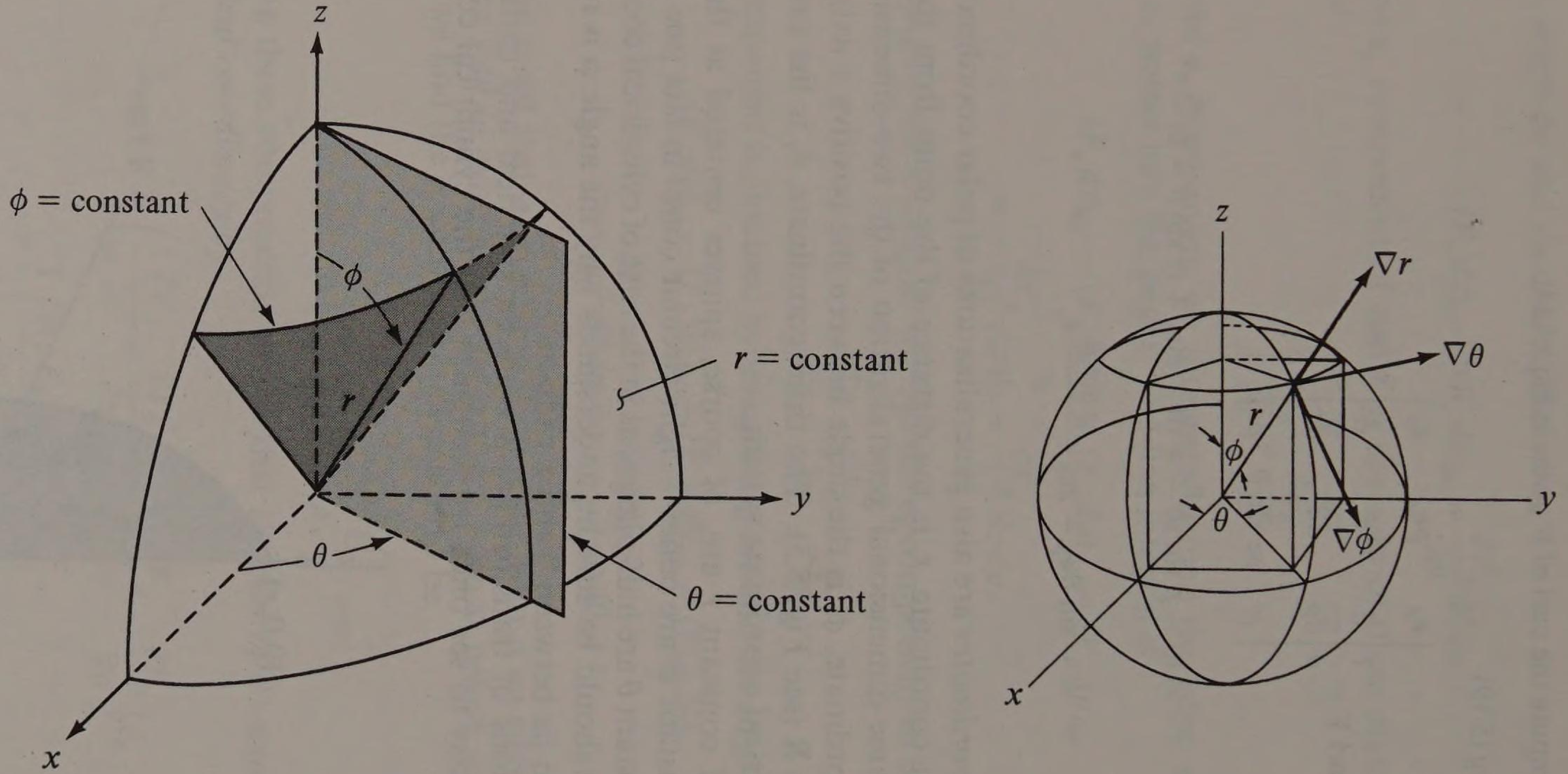
z
ϕ = constant
ϕ
r
r = constant
y
θ
θ = constant
x
z
∇r
$\nabla\theta$
r
ϕ
θ
$\nabla\phi$
y
x

FIGURE 5.6

ρ equals $r \sin \phi$, and z equals $r \cos \phi$. Then with the help of equations (5.1), we find

$$\begin{aligned} x &= r \sin \phi \cos \theta \qquad & r &= (x^2 + y^2 + z^2)^{\frac{1}{2}} \\ y &= r \sin \phi \sin \theta & & \\ z &= r \cos \phi & \phi &= \cos^{-1} \frac{z}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \quad \text{(principle value)} \\ & & \theta &= \sin^{-1} \frac{y}{\sqrt{x^2 + y^2}} = \cos^{-1} \frac{x}{\sqrt{x^2 + y^2}} \end{aligned} \tag{5.20}$$

The coordinate curves (curves where one coordinate varies and the other two are constant) are rays emanating from the origin (for r), vertical semicircles (for ϕ), and horizontal circles (for θ). If we consider the surface of the earth as a sphere, $r =$ constant, the coordinate curves for ϕ are circles of constant longitude and those for θ are circles of constant latitude: $\phi = \pi/2$ defines the equator (see Fig. 5.7).

Staying with this earth analogy for a moment, we can see from the constant surfaces in Fig. 5.6 that **grad** r points along the *local vertical*, **grad** ϕ points due south and **grad** θ points due east. These vectors are also tangent to their respective coordinate curves, and they are mutually orthogonal. Hence, we define a set of unit vectors

$$\mathbf{e}_r = \frac{\mathbf{grad}\, r}{|\mathbf{grad}\, r|} \qquad \mathbf{e}_\phi = \frac{\mathbf{grad}\, \phi}{|\mathbf{grad}\, \phi|} \qquad \mathbf{e}_\theta = \frac{\mathbf{grad}\, \theta}{|\mathbf{grad}\, \theta|} \tag{5.21}$$

and observe that in the order $\mathbf{e}_r$, $\mathbf{e}_\phi$, and $\mathbf{e}_\theta$ they form a right-handed system. Again recalling $|\mathbf{grad}\, f| = df/ds$, we can be more specific in the equations of (5.21). Along coordinate curves of r, $ds = |dr|$. The coordinate curves

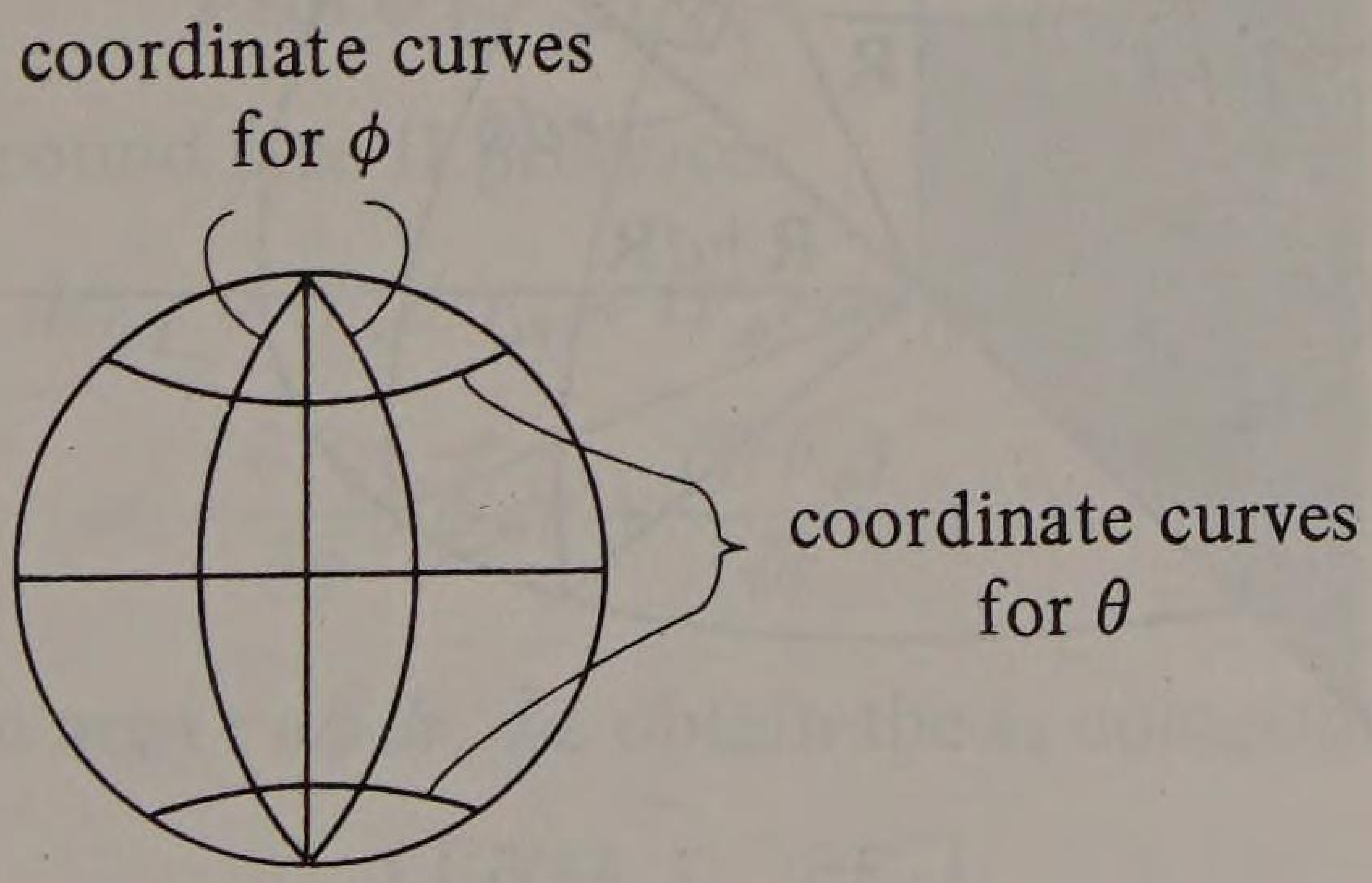

FIGURE 5.7

of ϕ are semicircles of radius r, so $ds = |r\,d\phi|$. The coordinate curves of θ are circles of radius (be careful!) $r \sin \phi$, so $ds = |r \sin \phi\, d\theta|$ (see Fig. 5.8). Therefore,

$$\begin{aligned} \mathbf{e}_r &= \mathbf{grad}\, r \\ \mathbf{e}_\phi &= r\, \mathbf{grad}\, \phi \\ \mathbf{e}_\theta &= r \sin \phi\, \mathbf{grad}\, \theta \end{aligned} \tag{5.22}$$

The position vector in spherical coordinates is simply

$$\mathbf{R} = r\mathbf{e}_r$$

Now we can model the computations, made previously for cylindrical coordinates, to obtain the analogous expressions in spherical coordinates. From Fig. 5.8, we see that the displacement $d\mathbf{R}$ can be expressed

$$d\mathbf{R} = \mathbf{e}_r\, dr + \mathbf{e}_\phi r\, d\phi + \mathbf{e}_\theta r \sin \phi\, d\theta \tag{5.23}$$

Thus *the element of arc length in spherical coordinates is*

$$ds = |d\mathbf{R}| = (dr^2 + r^2\, d\phi^2 + r^2 \sin^2 \phi\, d\theta^2)^{\frac{1}{2}} \tag{5.24}$$

From Fig. 5.8, we see that *the volume element in spherical coordinates is given by*

$$dV = (dr)(r\, d\phi)(r \sin \phi\, d\theta) = r^2 \sin \phi\, dr\, d\phi\, d\theta \tag{5.25}$$

The component of the gradient of $f(r,\phi,\theta)$ in the direction $\mathbf{e}_r$ is the rate of change of f with respect to distance along the r-coordinate curve: and,

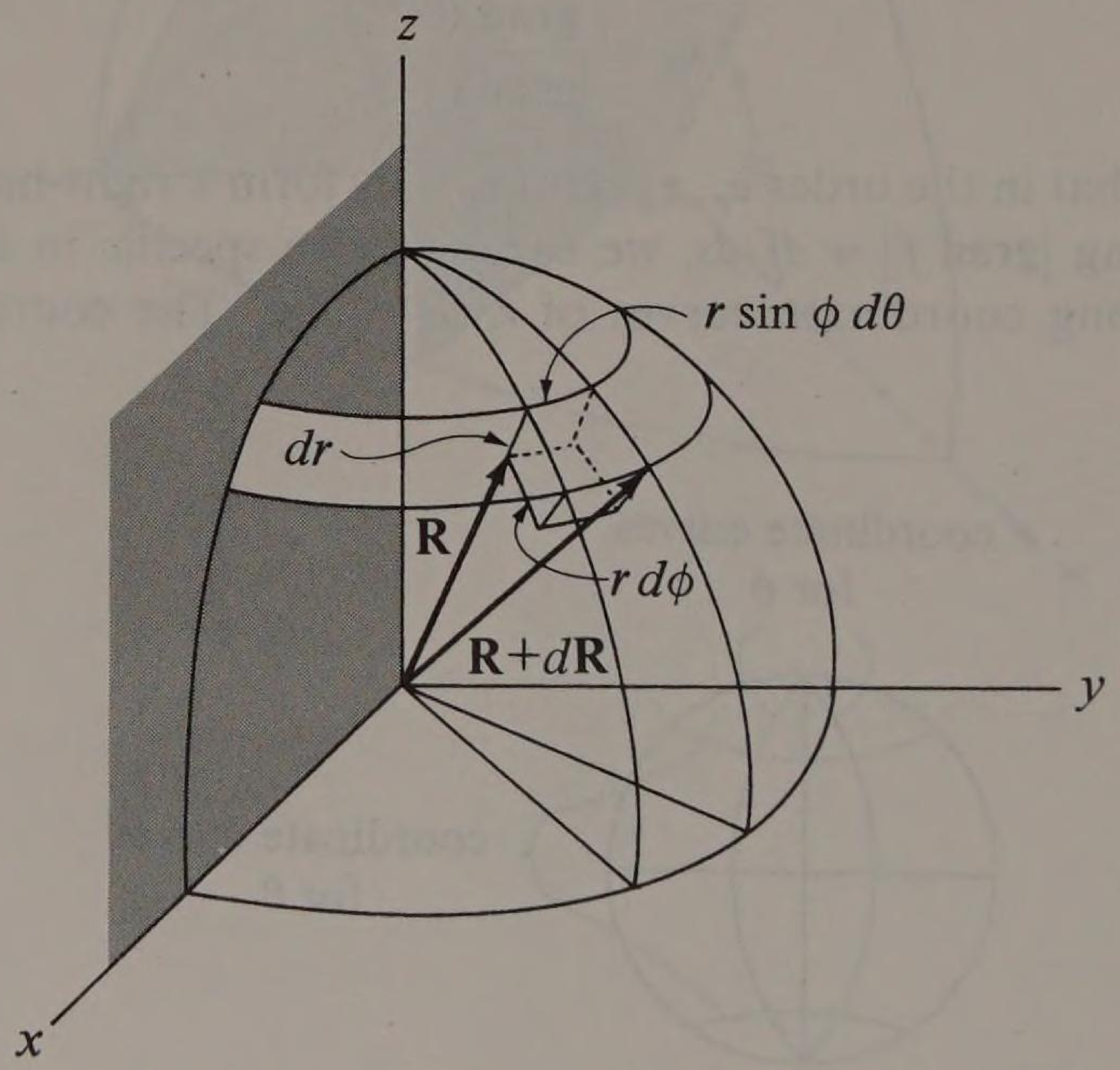

FIGURE 5.8

similarly, for the $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ components. Using (5.24) for distances, we find, analogous to (5.8), that *the expression for the gradient in spherical coordinates is*

$$\textbf{grad}\, f(r,\phi,\theta) = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\,\mathbf{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta \tag{5.26}$$

If $\mathbf{F}$ is a vector field given in spherical coordinates by

$$\mathbf{F}(r,\phi,\theta) = \mathbf{F}_r\mathbf{e}_r + F_\phi\mathbf{e}_\phi + F_\theta\mathbf{e}_\theta$$

we can compute its divergence as before by reasoning on the infinitesimal parallepiped in Fig. 5.9. The total flux out of all the faces can be expressed

$$\begin{aligned}&(F_r r\sin\phi\, d\theta\, r\, d\phi)_{\mathrm{IV}} - (F_r r\sin\phi\, d\theta\, r\, d\phi)_{\mathrm{I}} + (F_\theta r\, d\phi\, dr)_{\mathrm{V}} - (F_\theta r\, d\phi\, dr)_{\mathrm{II}}\\ &\quad + (F_\phi r\sin\phi\, d\theta\, dr)_{\mathrm{VI}} - (F_\phi r\sin\phi\, d\theta\, dr)_{\mathrm{III}}\\ &= \frac{\partial(r^2F_r)}{\partial r}\, dr\sin\phi\, d\theta\, d\phi + \frac{\partial F_\theta}{\partial\theta}\, d\theta\, r\, d\phi\, dr + \frac{\partial(F_\phi\sin\phi)}{\partial\phi}\, d\phi\, r\, d\theta\, dr\end{aligned}$$

(keeping careful track of which variable changes from face to face). Dividing by the volume element (5.25), we find that *the expression for the divergence of a vector field in spherical coordinates is given by*

$$\text{div}\,\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial\theta} + \frac{1}{r\sin\phi}\frac{\partial(F_\phi\sin\phi)}{\partial\phi} \tag{5.27}$$

Analogous reasoning on Fig. 5.9 yields the expression for the curl. The line integral around the properly-oriented edge of face I is

$$\begin{aligned}&(F_\phi r\, d\phi)_{ab} - (F_\phi r\, d\phi)_{cd} + (F_\theta r\sin\phi\, d\theta)_{bc} - (F_\theta r\sin\phi\, d\theta)_{da}\\ &= -\frac{\partial F_\phi}{\partial\theta}\, d\theta\, r\, d\phi + \frac{\partial(F_\theta\sin\phi)}{\partial\phi}\, d\phi\, r\, d\theta\end{aligned}$$

and dividing by the area $r^2\sin\phi\, d\theta\, d\phi$, we obtain the $\mathbf{e}_r$ component of **curl F**,

$$\frac{1}{r\sin\phi}\left[\frac{\partial(F_\theta\sin\phi)}{\partial\phi} - \frac{\partial F_\phi}{\partial\theta}\right] \tag{5.28}$$

The line integral around face II produces

$$\begin{aligned}&(F_r\, dr)_{ef} - (F_r\, dr)_{gh} + (F_\phi r\, d\phi)_{fg} - (F_\phi r\, d\phi)_{he}\\ &= -\frac{\partial F_r}{\partial\phi}\, d\phi\, dr + \frac{\partial(rF_\phi)}{\partial r}\, dr\, d\phi\end{aligned}$$

and dividing by the area $r\, d\phi\, dr$, we obtain the $\mathbf{e}_\theta$ component:

$$\frac{1}{r}\left[\frac{\partial(rF_\phi)}{\partial r} - \frac{\partial F_r}{\partial\phi}\right] \tag{5.29}$$

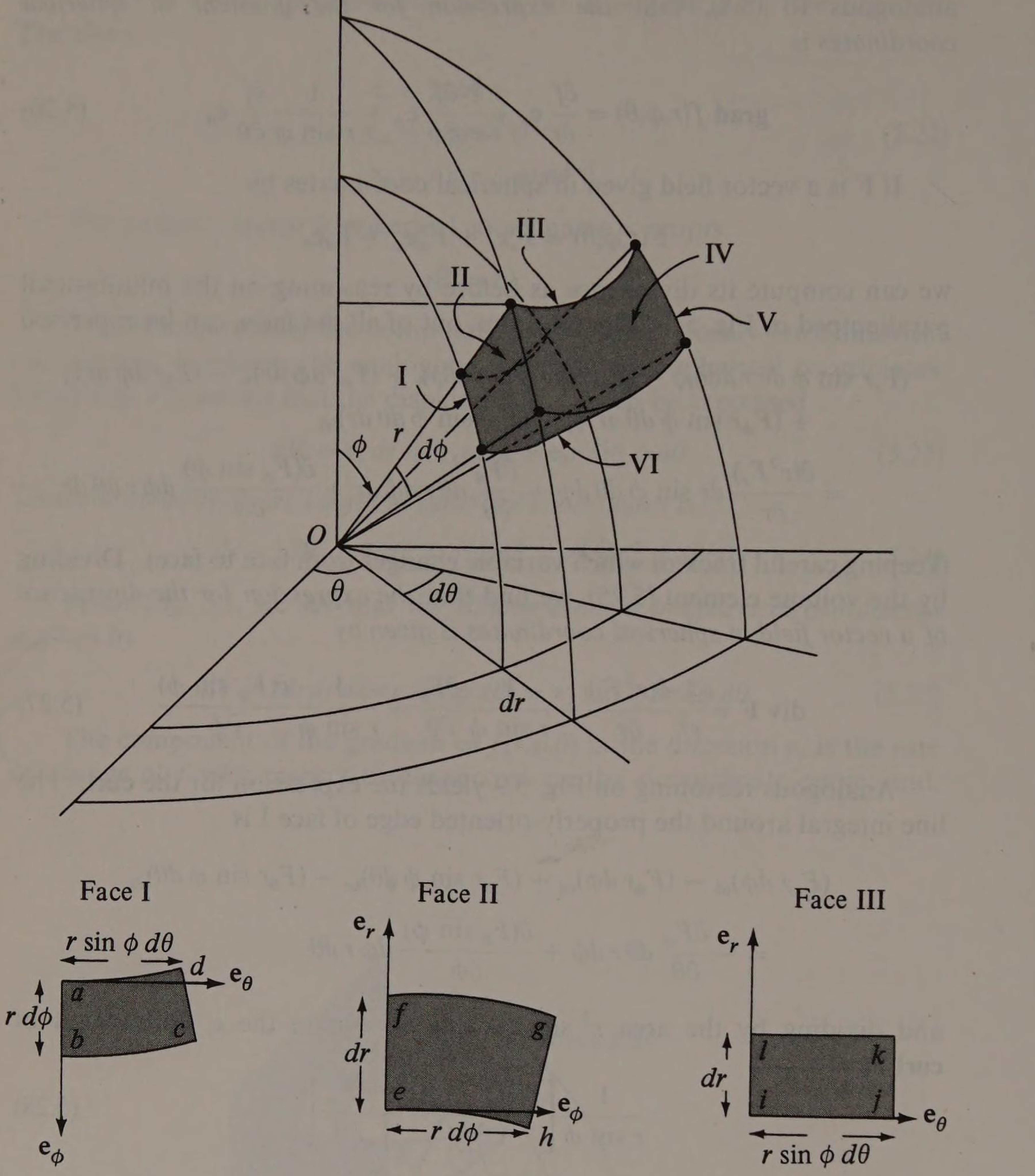
III
II
IV
V
I
VI
r
ϕ
$d\phi$
O
θ
$d\theta$
dr
Face I
$r \sin \phi \, d\theta$
d
$\mathbf{e}_\theta$
a
b
c
$r \, d\phi$
$\mathbf{e}_\phi$
Face II
$\mathbf{e}_r$
f
g
e
dr
$\mathbf{e}_\phi$
$r \, d\phi$
h
Face III
$\mathbf{e}_r$
l
k
i
j
dr
$\mathbf{e}_\theta$
$r \sin \phi \, d\theta$

FIGURE 5.9

The line integral around face III produces

$$(F_\theta r \sin \phi \, d\theta)_{ij} - (F_\theta r \sin \phi \, d\theta)_{kl} + (F_r \, dr)_{jk} - (F_r \, dr)_{li}$$

$$= -\frac{\partial (rF_\theta)}{\partial r} \, dr \sin \phi \, d\theta + \frac{\partial F_r}{\partial \theta} \, d\theta \, dr$$

and dividing by the area $r \sin \phi \, d\theta \, dr$ yields the $\mathbf{e}_\phi$ component:

$$\frac{1}{r \sin \phi}\left[\frac{\partial F_r}{\partial \theta} - \frac{\partial (rF_\theta)}{\partial r}\sin \phi\right] \tag{5.30}$$

The reader should verify that the results of (5.28), (5.29), and (5.30) can be summarized thus: *the expression for the curl of a vector field in spherical coordinates is given by*

$$\mathbf{curl}\ \mathbf{F} = \frac{1}{r^2 \sin \phi}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & r \sin \phi \, \mathbf{e}_\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial \theta} \\ F_r & rF_\phi & r \sin \phi \, F_\theta \end{vmatrix} \tag{5.31}$$

For reference purposes the formulas derived in this section will be listed, together with their generalizations, at the end of the next section.

Exercises

1. Derive the equations of transformation between cylindrical and spherical coordinates.
2. Use Eqs. (5.1) and (5.2) to derive

$$\mathbf{e}_z = \mathbf{k} \qquad \mathbf{e}_\rho = \frac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{\frac{1}{2}}}$$
$$\mathbf{e}_\theta = \frac{-y\mathbf{i} + x\mathbf{j}}{(x^2 + y^2)^{\frac{1}{2}}} \tag{5.32}$$

3. Verify Eq. (5.19).
4. Use (5.20) and (5.21) to derive

$$\mathbf{e}_r = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$$

$$\mathbf{e}_\phi = \frac{z(x\mathbf{i} + y\mathbf{j}) - (x^2 + y^2)\mathbf{k}}{(x^2 + y^2)^{\frac{1}{2}}(x^2 + y^2 + z^2)^{\frac{1}{2}}}$$

$$\mathbf{e}_\theta = \frac{-y\mathbf{i} + x\mathbf{j}}{(x^2 + y^2)^{\frac{1}{2}}}$$

5. Verify Eq. (5.31).
6. Compute the laplacian $\nabla^2 f$ in cylindrical and spherical coordinates. (*Hint*: Use $\nabla^2 = \text{div}\ \mathbf{grad}$.)
7. Show that if f is a function of r only, then

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

8. Change to cylindrical coordinates and find the divergence and curl of

(a) $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$

(b) $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$

[*Hint*: Observe Eq. (5.32).]

9. What is the arc length of the curve $r = \sin\phi$, $\theta = \pi/2$, for $0 \le \phi \le \pi$?

10. Compute the surface area of the spiral ramp $\rho = u$, $\theta = \pi/2 - v$, $z = v$, for $0 \le u \le 1$, $0 \le v \le 2$. [*Hint*: Use Eqs. (4.26) and (5.5).]

11. With $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{F} = \mathbf{R} \times \mathbf{k}$, compute the flux of $\mathbf{F}$ through the surface of the cylinder $\rho = 1$, $0 \le z \le 1$. Check the divergence theorem in this case.

12. Compute the area of the cone $\phi = \text{constant} = \pi/6$, $0 \le r \le 2$. [*Hint*: Use Eqs. (4.26) and (5.23).]

13. Evaluate $\nabla(r^n)$.

14. Evaluate $\iiint (x^2 + y^2 + z^2)^{\frac{3}{2}}\, dx\, dy\, dz$, integrated over the intersection of the sphere of radius 2 centered at the origin, and the first octant ($x > 0$, $y > 0$, $z > 0$).

15. Letting $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $r = |\mathbf{R}|$, write the vector field $\mathbf{F} = \mathbf{R}/r^3$ in terms of r and $\mathbf{e}_r$.

(a) Show that div $\mathbf{F}$ is identically zero throughout the domain of definition of $\mathbf{F}$.

(b) Show that the surface integral of the normal component of $\mathbf{F}$ over the surface of the unit sphere $r = 1$ is 4π.

(c) Explain why (a) and (b) do not contradict the divergence theorem.

(d) What is the surface integral of the normal component of $\mathbf{F}$ over the surface of a unit sphere with center 4 units away from the origin?

16. Compute the gradient, in spherical coordinates, of $f(r,\phi,\theta) = \cos\phi/r^2$.

17. Compute the divergence and curl, in spherical coordinates, of $\mathbf{F}(r,\phi,\theta) = \mathbf{e}_r + r\mathbf{e}_\phi + r\cos\theta\, \mathbf{e}_\theta$.

18. Compute the flux of $\mathbf{F} = r^n\mathbf{e}_r$ through the surface bounded by the unit upper hemisphere $r = 1$, $0 \le \phi \le \pi/2$, and the equatorial plane. Check the divergence theorem in this case.

19. Verify Stokes' theorem for $\mathbf{F} = x\mathbf{j}$ and the hemispherical surface S: $r = 1$, $0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$. Use spherical coordinates.

20. For what value(s) of n is $\nabla \cdot (r^n\mathbf{e}_r) = 0$?

21. For what value(s) of n is $\nabla \times (r^n\mathbf{e}_r) = \mathbf{0}$?

22. (a) Find a vector field $\mathbf{F} = F_r(r)\mathbf{e}_r$ satisfying $\nabla \cdot \mathbf{F} = r^m$, $m \ge 0$.

(b) Use the divergence theorem to prove

$$\iiint_D r^m\, dV = \frac{1}{m+3} \iint_S r^{m+1}\mathbf{e}_r \cdot d\mathbf{S}$$

(c) Interpret part (b) if $m = 0$. (*Hint*: The volume of a pyramid equals one-third the volume of the parallelepiped having the same base and height.)

5.2 ORTHOGONAL CURVILINEAR COORDINATES

Our experience with the cylindrical and spherical coordinate systems places us in a good position to analyze general coordinate systems, or *curvilinear coordinates.*

The general situation is this: each point in a certain region of space is specified by three numbers (u_1,u_2,u_3), called the curvilinear coordinates of the point. Possibly the numbers can be interpreted as lengths or angles, but no such geometric visualization is required. All that is needed are the transformation equations between the curvilinear coordinates and cartesian coordinates, which we represent by

$$\begin{aligned} x &= x(u_1,u_2,u_3) \qquad & u_1 &= u_1(x,y,z) \\ y &= y(u_1,u_2,u_3) \qquad & u_2 &= u_2(x,y,z) \\ z &= z(u_1,u_2,u_3) \qquad & u_3 &= u_3(x,y,z) \end{aligned} \tag{5.33}$$

Equations (5.33) include (5.1) and (5.20) as special instances.

Observe that it is not feasible to choose the functions u_1, u_2, and u_3 arbitrarily. For example, the system $u_1 = x^2$, $u_2 = y - z$, $u_3 = 2y - 2z$ is unsatisfactory because one cannot invert the equations; in fact, the points $(x,y,z) = (1,2,3)$ and $(x,y,z) = (1,3,4)$ would have identical curvilinear coordinates $(1,-1,-2)$. Therefore, we stipulate that the functions defining u_1, u_2, and u_3 assign different ordered triples to different points in the region of interest. We also assume that they possess continuous partial derivatives of all orders and that at every point P the gradients of these functions are nonzero.

(Sometimes we do not require that the coordinates satisfy these requirements at every point in space. For example, if we pass through the z axis along a line parallel to the x axis, the spherical coordinate θ undergoes a discontinuous jump from 0 to π. We shall generally ignore this difficulty and work only in a domain where the conditions are satisfied.)

These conditions ensure that through any point P in the domain, having curvilinear coordinates equal to (c_1,c_2,c_3), there will pass three isotimic surfaces $u_1(x,y,z) = c_1$, $u_2(x,y,z) = c_2$, $u_3(x,y,z) = c_3$. As illustrated in Fig. 5.10, these surfaces intersect in pairs to give three curves passing through P, along each of which only one coordinate varies: these are the coordinate curves. The normal to the surface $u_i = c_i$ is the gradient:

$$\nabla u_i = \frac{\partial u_i}{\partial x}\mathbf{i} + \frac{\partial u_i}{\partial y}\mathbf{j} + \frac{\partial u_i}{\partial z}\mathbf{k} \tag{5.34}$$

and the tangent to the coordinate curve for u_i is the vector

$$\frac{\partial \mathbf{R}}{\partial u_i} = \frac{\partial x}{\partial u_i}\mathbf{i} + \frac{\partial y}{\partial u_i}\mathbf{j} + \frac{\partial z}{\partial u_i}\mathbf{k} \tag{5.35}$$

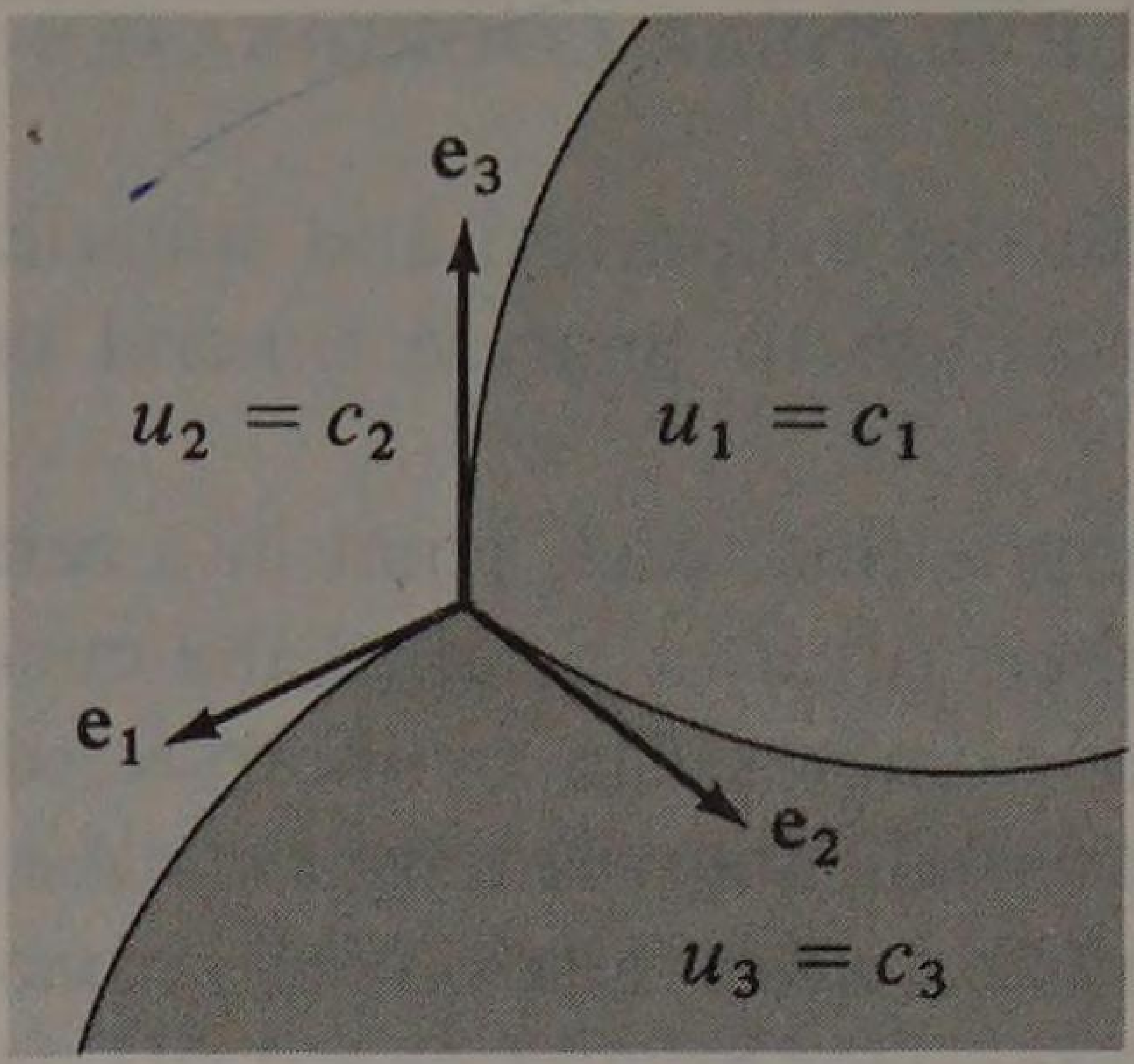

FIGURE 5.10

Recall that in the cases of cylindrical and spherical coordinates, the three normals to the isotimic surfaces were mutually perpendicular. In general, whenever the vectors ∇u_1, ∇u_2, and ∇u_3 are mutually orthogonal at every point, we say that u_1, u_2, u_3 comprise *orthogonal curvilinear coordinates.* In this section we shall restrict our analysis to such systems. Moreover, we assume that the u_i's are numbered so that ∇u_1, ∇u_2, and ∇u_3 (in that order) form a right-handed system.

Another feature of the special coordinate systems we studied in the last section is that each gradient vector ∇u_i was seen to be parallel to the tangent vector $\partial \mathbf{R}/\partial u_i$ for the corresponding coordinate curve. This is always true for orthogonal curvilinear coordinates: *any coordinate curve for u_i intersects the isotimic surface $u_i = c_i$ at right angles when (u_1, u_2, u_3) form orthogonal curvilinear coordinates.* To see this, consider, say, a coordinate curve for u_1.

(*i*) This curve is the intersection of two surfaces $u_2 = c_2$ and $u_3 = c_3$. Hence, its tangent $\partial \mathbf{R}/\partial u_1$ is perpendicular to both surface normals ∇u_2 and ∇u_3.

(*ii*) The vector ∇u_1 is also perpendicular to ∇u_2 and ∇u_3, by definition of orthogonal curvilinear coordinates.

(*iii*) This implies that $\partial \mathbf{R}/\partial u_1$ is parallel or, perhaps, antiparallel to ∇u_1.

Since both point in the direction of increasing u_1, they are parallel.

It follows, of course, that the vectors $\partial \mathbf{R}/\partial u_1$, $\partial \mathbf{R}/\partial u_2$, and $\partial \mathbf{R}/\partial u_3$ also form a right-handed system of mutually orthogonal vectors. In fact, by the chain rule

$$(\nabla u_i) \cdot \left(\frac{\partial \mathbf{R}}{\partial u_j}\right) = \frac{\partial u_i}{\partial x}\frac{\partial x}{\partial u_j} + \frac{\partial u_i}{\partial y}\frac{\partial y}{\partial u_j} + \frac{\partial u_i}{\partial z}\frac{\partial z}{\partial u_j}$$

$$= \frac{\partial u_i}{\partial u_j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{5.36}$$

Thus it is natural to define the right-handed system of mutually orthogonal unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by

$$\mathbf{e}_i = \frac{\nabla u_i}{|\nabla u_i|} = \frac{\partial \mathbf{R}}{\partial u_i} \Bigg/ \left|\frac{\partial \mathbf{R}}{\partial u_i}\right| \qquad (i = 1, 2, 3) \tag{5.37}$$

The vectors $(\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_z)$ and $(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_\theta)$ are special instances of (5.37).

In order to express the vector operations in general orthogonal curvilinear coordinates, we need to evaluate three functions h_i known as the *scale factors. The scale factor h_i is defined to be the rate at which arc length increases on the ith coordinate curve, with respect to u_i.* In other words, if s_i denotes arc length on the ith coordinate curve, measured in the direction of increasing u_i, then

$$h_1 = \frac{ds_1}{du_1} \qquad h_2 = \frac{ds_2}{du_2} \qquad h_3 = \frac{ds_3}{du_3} \tag{5.38}$$

Since arc length in general can be expressed

$$ds = |d\mathbf{R}| = \left|\frac{\partial \mathbf{R}}{\partial u_1}\, du_1 + \frac{\partial \mathbf{R}}{\partial u_2}\, du_2 + \frac{\partial \mathbf{R}}{\partial u_3}\, du_3\right| \tag{5.39}$$

we see that

$$h_i = \left|\frac{\partial \mathbf{R}}{\partial u_i}\right| \qquad (i = 1, 2, 3) \tag{5.40}$$

Combining the last two equations shows that the displacement vector can be expressed in terms of the scale factors by

$$d\mathbf{R} = h_1\, du_1\, \mathbf{e}_1 + h_2\, du_2\, \mathbf{e}_2 + h_3\, du_3\, \mathbf{e}_3 \tag{5.41}$$ ✓

We can get another formula for the scale factor h_i by the following observations:

(*i*) $|\nabla u_i|$ is the rate of change of u_i with respect to distance in the direction of ∇u_i.
(*ii*) The direction of ∇u_i is the direction of the coordinate curve for u_i.
(*iii*) s_i measures distance along the coordinate curve for u_i.

It follows that

$$|\nabla u_i| = \frac{du_i}{ds_i} = \frac{1}{h_i}$$

Therefore,

$$h_1 = \frac{1}{|\nabla u_1|} \qquad h_2 = \frac{1}{|\nabla u_2|} \qquad h_3 = \frac{1}{|\nabla u_3|} \tag{5.42}$$

Example 5.6 Consider the curvilinear coordinate system defined for $z \geq 0$ by

$$\begin{aligned} x &= u_1 - u_2 \\ y &= u_1 + u_2 \\ z &= {u_3}^2 \end{aligned} \tag{5.43}$$

Verify that the system is orthogonal and right-handed, and compute the unit vectors $\mathbf{e}_i$ and the scale factors h_i.

Solution We do not need the inverse equations for this example, but they are easy to derive:

$$u_1 = \frac{x+y}{2}$$

$$u_2 = \frac{y-x}{2}$$

$$u_3 = z^{\frac{1}{2}} \quad \textit{(take the positive square root for definiteness)}$$

We use the right-hand expression in Eq. (5.37) to compute the $\mathbf{e}_i$:

$$\mathbf{e}_1 = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}} \quad = \frac{u''}{\Delta u_1}$$

$$\mathbf{e}_2 = \frac{-\mathbf{i}+\mathbf{j}}{\sqrt{2}}$$

$$\mathbf{e}_3 = \frac{2u_3\mathbf{k}}{|2u_3|} = \mathbf{k} \tag{5.44}$$

Clearly this set is right-handed and orthogonal so that Eqs. (5.43) do, in fact, define orthogonal curvilinear coordinates. The h_i have already been computed in the denominators of (5.44):

$$h_1 = \sqrt{2} \qquad h_2 = \sqrt{2} \qquad h_3 = 2u_3 \tag{5.45}$$

Example 5.7 Compute the scale factors for cylindrical and spherical coordinates.

Solution Comparing Eq. (5.41) and (5.5) and (5.23), respectively, we have

$$h_\rho = 1 \qquad h_\theta = \rho \qquad h_z = 1 \tag{5.46}$$

and

$$h_r = 1 \qquad h_\phi = r \qquad h_\theta = r \sin \phi \tag{5.47}$$

Recall that these were obtained simply by examining the diagrams; the computations (5.40) for the h_i are often unnecessary when the curvilinear coordinates can be visualized.

The scale factors will allow us to write general formulas for arc length, volume, gradient, divergence, and curl in terms of curvilinear coordinates. In this section we will present heuristic derivations for these expressions.

From the discussion above we can say that ds_i is the arc length along the ith coordinate curve, corresponding to a change in the ith coordinate from u_i to $u_i + du_i$. Since an arbitrary displacement $d\mathbf{R}$ is generated by changes du_1, du_2, and du_3, each in mutually perpendicular directions, we can express the element of arc length $|d\mathbf{R}|$ by the Pythagorean theorem as

$$|d\mathbf{R}|^2 = ds_1{}^2 + ds_2{}^2 + ds_3{}^2$$

Using the scale factors from Eq. (5.39), we find that *the arc length along a curve C is given by the line integral*

$$\int_C |d\mathbf{R}| = \int ds = \int \sqrt{(h_1\, du_1)^2 + (h_2\, du_2)^2 + (h_3\, du_3)^2} \tag{5.48}$$

generalizing the formula in Sec. 2.2.

We can immediately generalize the discussion above to conclude that *the line integral of a continuous vector field* $\mathbf{F}$, *expressed in general coordinates by* $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$, *along a curve C, is obtained by*

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_C (F_1 h_1\, du_1 + F_2 h_2\, du_2 + F_3 h_3\, du_3) \tag{5.49}$$

✓

where, in practice, one usually has u_1, u_2, and u_3 given in terms of some parameter t; (5.49) then ultimately becomes an integral involving t and dt.

From Fig. 5.11, we observe that the solid generated by displacements du_1, du_2, and du_3 is approximately a rectangular parallelepiped with edges $h_1\, du_1$, $h_2\, du_2$, and $h_3\, du_3$. Its volume is, therefore,

$$dV = h_1 h_2 h_3\, du_1\, du_2\, du_3 \tag{5.50}$$

Hence, *the volume integral of a function $f(u_1,u_2,u_3)$ is given by*

$$\iiint f(u_1,u_2,u_3)\, dV = \iiint f(u_1,u_2,u_3) h_1 h_2 h_3\, du_1\, du_2\, du_3 \tag{5.51}$$

Again, we can integrate Eq. (5.51) iteratively once the limits are specified. In spherical coordinates, $dV = r^2 \sin\phi\, dr\, d\theta\, d\phi$; in cylindrical coordinates, $dV = \rho\, dp\, d\theta\, dz$.

In Sec. 3.1, it was shown that the component of **grad** f in the $\mathbf{e}_1$ direction is given by df/ds_1, the rate of change of f with respect to distance in the

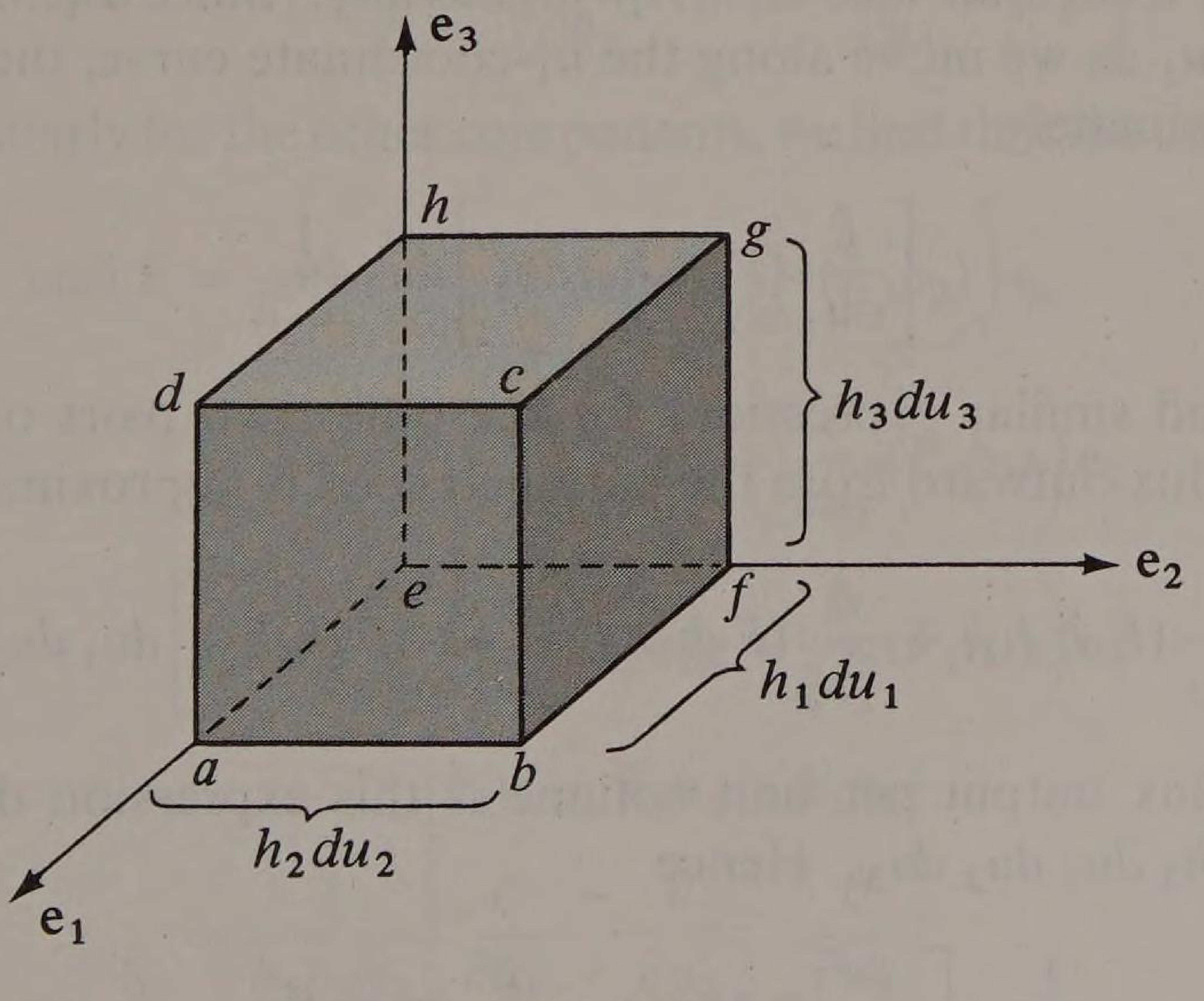

FIGURE 5.11

$\mathbf{e}_1$ direction. Since $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are mutually orthogonal unit vectors, we can immediately express **grad** f in terms of these:

$$\mathbf{grad}\, f = \frac{df}{ds_1}\mathbf{e}_1 + \frac{df}{ds_2}\mathbf{e}_2 + \frac{df}{ds_3}\mathbf{e}_3$$

or, introducing the scale factors,

$$\mathbf{grad}\, f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\mathbf{e}_3$$

Example 5.8 Compute **grad** f in the coordinate system (5.43), for $f(u_1,u_2,u_3) = u_1u_2 + u_3^2$.

Solution From (5.52) and (5.45),

$$\begin{aligned}\nabla f &= \frac{1}{\sqrt{2}}u_2\mathbf{e}_1 + \frac{1}{\sqrt{2}}u_1\mathbf{e}_2 + \frac{1}{2u_3}2u_3\mathbf{e}_3 \\ &= (u_2\mathbf{e}_1 + u_1\mathbf{e}_2)\frac{\sqrt{2}}{2} + \mathbf{e}_3 \qquad (5.53)\end{aligned}$$

The expression for divergence is more complicated. Let

$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$$

be the vector field, given in terms of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. We will calculate div **F** as the flux of **F** out of the sides of the box in Fig. 5.11, divided by the volume of the box, in accordance with the interpretation of divergence given in Secs. 3.3 and 4.9.

The flux density normal to the face *abcd* is $\mathbf{F}\cdot\mathbf{e}_1 = F_1$ and the area of this face is $h_2h_3\,du_2\,du_3$. Therefore the flux outward from that face is $F_1h_2h_3\,du_2\,du_3$. The unit outward normal to face *efgh* is $-\mathbf{e}_1$, so that the flux outward from that face is $-F_1h_2h_3\,du_2\,du_3$. Since F_1, h_2, and h_3 are functions of u_1 as we move along the u_1-coordinate curve, the sum of these two is approximately

$$\left[\frac{\partial}{\partial u_1}(F_1h_2h_3)\,du_1\right]du_2\,du_3$$

From this and similar expressions for the other two pairs of faces we see that the net flux outward from the parallelepiped is approximately

$$\left[\frac{\partial}{\partial u_1}(F_1h_2h_3) + \frac{\partial}{\partial u_2}(F_2h_1h_3) + \frac{\partial}{\partial u_3}(F_3h_1h_2)\right]du_1\,du_2\,du_3$$

and so the flux output per unit volume is this expression divided by the volume $h_1h_2h_3\,du_1\,du_2\,du_3$. Hence

$$\text{div}\,\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(F_1h_2h_3) + \frac{\partial}{\partial u_2}(F_2h_1h_3) + \frac{\partial}{\partial u_3}(F_3h_1h_2)\right] \qquad (5.54)$$

Using Eqs. (5.52) and (5.54), we have the expression for the laplacian: $\nabla^2 f = \text{div}\,\textbf{grad}\, f$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right] \tag{5.55}$$

Now let us find the expression for **curl F**. We shall use the "swirl" characterization described in Exercise 1 of Sec. 4.12. The component of **curl F** in the direction $\mathbf{e}_1$ will be the line integral of the tangential component of **F** around the curve *efghe* in Fig. 5.11, divided by the area enclosed by this curve. [Recall that, by Stokes' theorem, $\int_{efghe} \mathbf{F}\cdot d\mathbf{R}$ equals $\iint (\textbf{curl}\,\mathbf{F})\cdot\mathbf{n}\, dS$, which is approximately $(\textbf{curl}\,\mathbf{F})\cdot\mathbf{e}_1$ times the area.] The integral along the edge *ef* is approximately

$$F_2(u_1,u_2,u_3)h_2(u_1,u_2,u_3)\,du_2$$

Along *gh* we are proceeding in the opposite direction; furthermore, the third coordinate is now $u_3 + du_3$, so the line integral is

$$-F_2(u_1,u_2,u_3 + du_3)h_2(u_1,u_2,u_3 + du_3)\,du_2$$

Thus the net contribution from *ef* and *gh* is given by

$$-\frac{\partial}{\partial u_3}(F_2h_2)\,du_3\,du_2$$

Similarly, the contribution from *fg* and *he* is

$$\frac{\partial}{\partial u_2}(F_3h_3)\,du_2\,du_3$$

Dividing by the area $h_2\,du_2\,h_3\,du_3$, we have

$$(\textbf{curl}\,\mathbf{F})\cdot\mathbf{e}_1 = \frac{1}{h_2h_3}\left(\frac{\partial}{\partial u_2}(F_3h_3) - \frac{\partial}{\partial u_3}(F_2h_2)\right)$$

Reasoning similarly for the other components, we find that the curl is given by

$$\begin{aligned}\textbf{curl}\,\mathbf{F} &= \frac{1}{h_2h_3}\left(\frac{\partial}{\partial u_2}(F_3h_3) - \frac{\partial}{\partial u_3}(F_2h_2)\right)\mathbf{e}_1 \\ &\quad + \frac{1}{h_1h_3}\left(\frac{\partial}{\partial u_3}(F_1h_1) - \frac{\partial}{\partial u_1}(F_3h_3)\right)\mathbf{e}_2 \\ &\quad + \frac{1}{h_1h_2}\left(\frac{\partial}{\partial u_1}(F_2h_2) - \frac{\partial}{\partial u_2}(F_1h_1)\right)\mathbf{e}_3 \\ &= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{u}_1 & h_2\mathbf{u}_2 & h_3\mathbf{u}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ F_1h_1 & F_2h_2 & F_3h_3 \end{vmatrix}\end{aligned} \tag{5.56}$$

Example 5.9 Compute the divergence and curl of the vector field

$$\mathbf{F}(u_1,u_2,u_3) = u_3u_1\mathbf{e}_1 + u_3u_2\mathbf{e}_2 + u_1u_2\mathbf{e}_3$$

in the coordinate system (5.43).

Solution From Eqs. (5.54) and (5.45),

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{4u_3}\left[\frac{\partial}{\partial u_1}(u_3u_1 2\sqrt{2}u_3) + \frac{\partial}{\partial u_2}(u_3u_2 2\sqrt{2}u_3) + \frac{\partial}{\partial u_3}(2u_1u_2)\right] \\ &= \sqrt{2}u_3\end{aligned}$$

From Eq. (5.56),

$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{1}{4u_3}\begin{vmatrix} \sqrt{2}\mathbf{e}_1 & \sqrt{2}\mathbf{e}_2 & 2u_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ \sqrt{2}u_3u_1 & \sqrt{2}u_3u_2 & 2u_1u_2u_3 \end{vmatrix} \\ &= \frac{(2u_1u_3 - \sqrt{2}u_2)\sqrt{2}}{4u_3}\mathbf{e}_1 + \frac{(\sqrt{2}u_1 - 2u_2u_3)\sqrt{2}}{4u_3}\mathbf{e}_2\end{aligned}$$

For reference purposes we list the various vector operations in general orthogonal curvilinear coordinates, cylindrical coordinates, and spherical coordinates together here. The reader is advised to attach a third, and final, permanent bookmark to this page.

GENERAL ORTHOGONAL CURVILINEAR COORDINATES

Scale factors:

$$h_i = \left|\frac{\partial \mathbf{R}}{\partial u_i}\right| = \frac{1}{|\nabla u_i|} \qquad (i = 1, 2, 3)$$

Displacement vector:

$$d\mathbf{R} = h_1\,du_1\,\mathbf{e}_1 + h_2\,du_2\,\mathbf{e}_2 + h_3\,du_3\,\mathbf{e}_3$$

Arc length:

$$ds = (h_1{}^2\,du_1{}^2 + h_2{}^2\,du_2{}^2 + h_3{}^2\,du_3{}^2)^{\frac{1}{2}}$$

Volume element:

$$dV = h_1h_2h_3\,du_1\,du_2\,du_3$$

Gradient:

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\mathbf{e}_3$$

Divergence:

$$\nabla\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(F_1h_2h_3) + \frac{\partial}{\partial u_2}(F_2h_1h_3) + \frac{\partial}{\partial u_3}(F_3h_1h_2)\right]$$

Curl:

$$\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ F_1h_1 & F_2h_2 & F_3h_3 \end{vmatrix}$$

Laplacian:

$$\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

CYLINDRICAL COORDINATES

Displacement vector:

$$d\mathbf{R} = d\rho\,\mathbf{e}_\rho + \rho\,d\theta\,\mathbf{e}_\theta + dz\,\mathbf{e}_z$$

Arc length:

$$ds = (d\rho^2 + \rho^2\,d\theta^2 + dz^2)^{\frac{1}{2}}$$

Volume element:

$$dV = \rho\,d\rho\,d\theta\,dz$$

Gradient:

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$

Divergence:

$$\nabla\cdot\mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

Curl:

$$\nabla\times\mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\theta & F_z \end{vmatrix}$$

* SEE pg 235
FOR UNIT VECTOR
CONVERSIONS
$[\vec{i}, \vec{j}, \vec{k} \Leftrightarrow \vec{e}_\rho, \vec{e}_\theta, \vec{e}_z]$

$\vec{e}_\rho = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$

$\vec{e}_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$

$\vec{e}_z = \vec{k}$

~

$\vec{i} = \cos\theta\,\vec{e}_\rho - \sin\theta\,\vec{e}_\theta$

$\vec{j} = \sin\theta\,\vec{e}_\rho + \cos\theta\,\vec{e}_\theta$

$\vec{k} = \vec{e}_z$

* SEE Pg 231
FOR CONVERSIONS
$(x, y, z) \Leftrightarrow (\rho, \theta, z)$

Laplacian:

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

SPHERICAL COORDINATES

* SEE pg 235 FOR UNIT VECTOR CONVERSIONS

$[\vec{i}, \vec{j}, \vec{k} \Leftrightarrow \vec{e}_r, \vec{e}_\phi, \vec{e}_\theta]$

* SEE pg 231 FOR CONVERSIONS

$(x, y, z) \Leftrightarrow (r, \phi, \theta)$

Displacement vector:

$$d\mathbf{R} = dr\,\mathbf{e}_r + r\,d\phi\,\mathbf{e}_\phi + r\sin\phi\,d\theta\,\mathbf{e}_\theta$$

Arc length:

$$ds = (dr^2 + r^2\,d\phi^2 + r^2\sin^2\phi\,d\theta^2)^{\frac{1}{2}}$$

Volume element:

$$dV = r^2\sin\phi\,dr\,d\phi\,d\theta$$

Gradient:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta$$

Divergence:

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}(F_\phi\sin\phi) + \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial \theta}$$

Curl:

$$\nabla\times\mathbf{F} = \frac{1}{r^2\sin\phi}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & (r\sin\phi)\mathbf{e}_\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial \theta} \\ F_r & rF_\phi & (r\sin\phi)F_\theta \end{vmatrix}$$

Laplacian:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

Exercises

1. Verify that the formulas for the vector operations in cylindrical and spherical coordinates, as computed in Sec. 5.1, are instances of the general formulas derived in this section when the scale factors Eqs. (5.46) and (5.47) are inserted.
2. Verify (5.53) by expressing f in cartesian coordinates, applying ∇, and transforming.

3. Explain why curvilinear coordinates defined by functions of the form

$$u_1 = u_1(z) \qquad u_2 = u_2(x) \qquad u_3 = u_3(y)$$

are automatically orthogonal. What other combinations have this property? How about

$$u_1 = u_1(\rho) \qquad u_2 = u_2(\theta) \qquad u_3 = u_3(z)$$

4. What is the element of volume relative to the coordinate system $u_1 = e^x$, $u_2 = y$, $u_3 = z$?

5. Compute $\nabla^2 g$ if $g = u_1{}^3 + u_2{}^3 + u_3{}^3$ in the coordinate system (5.43).

6. Let $u_1 = x + y$, $u_2 = x - y$, and $u_3 = 2z$.
(a) Is this an orthogonal coordinate system?
(b) Solve for x, y, and z in terms of u_1, u_2, and u_3.
(c) Find ds^2 and hence determine h_1, h_2, and h_3 for this coordinate system.
(d) What is the laplacian relative to this coordinate system?
(e) Let $f(u_1,u_2,u_3) = u_1 + u_2 + 2u_3$. Find **grad** f.

7. Let $u_1 = x + y$, $u_2 = x - 2y$, and $u_3 = 2z$.
(a) Solve for x, y, and z in terms of u_1, u_2, and u_3.
(b) Attempt to determine the scale factors h_1, h_2, and h_3.
(c) What is "wrong"?

8. Consider the transformation

$$x = u_1{}^2 - u_2{}^2$$
$$y = 2u_1u_2$$
$$z = u_3$$

(a) Show that (u_1,u_2,u_3) form right-handed orthogonal curvilinear coordinates.
(b) Compute the scale factors.
(c) Express $\nabla^2 g(u_1,u_2,u_3)$.
(d) Find the divergence and curl of the vector field $\mathbf{F} = u_3\mathbf{e}_1 + u_1\mathbf{e}_2 + u_2\mathbf{e}_3$.

9. Consider the transformation

$$x = u_3$$
$$y = e^{u_2}\cos u_1$$
$$z = e^{u_2}\sin u_1$$

(a) Show that (u_1,u_2,u_3) constitute orthogonal curvilinear coordinates.
(b) Compute the scale factors.
(c) Find $\nabla^2 g$ if $g = u_1{}^2 + u_2{}^2 + u_3{}^2$.
(d) Find the divergence and curl of the vector field

$$\mathbf{F} = -e^{u_2}\mathbf{e}_3 + u_3\mathbf{e}_1$$

10. In cartesian coordinates, $dV = dx\,dy\,dz$. Beginning with Eq. (5.20), differentiate to form dx, dy, and dz in terms of dr, $d\phi$, and $d\theta$, and multiply to obtain $dx\,dy\,dz$.
(a) Does this give dV in spherical coordinates?
(b) Explain this phenomenon.

11. Consider the coordinate system $u_1 = y$, $u_2 = x$, $u_3 = z$. The scale factors are all equal to unity, so that Eq. (5.56) takes an especially simple form.

(a) Let $\mathbf{F} = -u_2\mathbf{e}_1 + u_1\mathbf{e}_2$. Show that (5.56) gives

$$\mathbf{curl\ F} = 2\mathbf{e}_3$$

(b) Obviously $\mathbf{e}_1 = \mathbf{j}$, $\mathbf{e}_2 = \mathbf{i}$, and $\mathbf{e}_3 = \mathbf{k}$, so that $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ and by part (a), **curl F** $= 2\mathbf{k}$. But direct calculation of **curl F** in cartesian coordinates shows that **curl F** $= -2\mathbf{k}$, not $2\mathbf{k}$. What is "wrong"?

12. Suppose that u, v, w are orthogonal curvilinear coordinates for which $ds^2 = v^2\,du^2 + u^2\,dv^2 + dw^2$.

(a) Calculate the divergence of $\mathbf{u}$, where $\mathbf{u}$ is the unit vector tangent to a u curve.

(b) Determine the laplacian of the function $f = uvw$.

13. By using the "lumpiness" definition of the laplacian (Exercise 11, Sec. 4.10) applied to a rectangular parallelepiped, give a direct derivation of (5.55).

5.3 OPTIONAL READING: MATRIX TECHNIQUES IN VECTOR ANALYSIS

In this section we are going to take a quick look at matrix theory, as it is used in vector analysis. The beginner should be aware that he/she is going to learn about a very restricted class of matrices. The subject covers much more ground than we will see here; indeed, most authors devote a whole book to it. However, the matrix calculus has some very nice interpretations when applied to three-dimensional vector analysis, and we shall exploit these features.

Those readers who have already mastered linear algebra should, nonetheless, enjoy reading this section. They will have seen all the formulas before, but the point of view is quite different from the algebraic approach and offers some new insights.

A matrix is a rectangular array of numbers, like a bingo card or the box score of a baseball game. It can have any number of rows and columns. However, in vector analysis only three matrix "sizes" are commonly used: the 1-by-3 row matrix

$$[2,4,1] \tag{5.57}$$

the 3-by-1 column matrix

$$\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \tag{5.58}$$

and the 3-by-3 square matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{5.59}$$

Notice that the dimensions of a matrix are stated "m-by-n", indicating m rows and n columns.

It is useful to think of the entries of a matrix as representing the components of a vector. Thus, Eq. (5.57) represents $2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ as a row and (5.58) represents $2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ as a column. The square matrix (5.59) can either be interpreted as consisting of three rows, representing

$$\begin{aligned} \mathbf{A} &= \mathbf{i} - \mathbf{j} + \mathbf{k} \\ \mathbf{B} &= 2\mathbf{i} + \mathbf{k} \\ \mathbf{C} &= \mathbf{i} + \mathbf{j} + 2\mathbf{k} \end{aligned} \tag{5.60}$$

respectively, or as consisting of three columns, representing

$$\begin{aligned} \mathbf{D} &= \mathbf{i} + 2\mathbf{j} + \mathbf{k} \\ \mathbf{E} &= -\mathbf{i} + \mathbf{k} \\ \mathbf{F} &= \mathbf{i} + \mathbf{j} + 2\mathbf{k} \end{aligned} \tag{5.61}$$

respectively. Both interpretations are useful. To emphasize the point of view, we can write (5.59) as

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \cdots & \mathbf{A} & \cdots \\ \cdots & \mathbf{B} & \cdots \\ \cdots & \mathbf{C} & \cdots \end{bmatrix}$$

if we are thinking of the rows as vectors, or as

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{D} & \mathbf{E} & \mathbf{F} \\ \vdots & \vdots & \vdots \end{bmatrix}$$

if the column-vector interpretation is appropriate.

When we want to address a single number in a matrix $\mathscr{M}$, we use the notation m_{ij} for the entry in row number i and column number j, as illustrated:

$$[m_{11}, m_{12}, m_{13}] \qquad \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} \qquad \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{5.62}$$

Thus in matrix (5.57), $m_{13} = 1$; in (5.58), $m_{21} = 1$; and in (5.59), $m_{12} = -1$, $m_{33} = 2$, and $m_{23} = 1$. (Remember the order: m_{ij} refers to row i, column j. Think of the mnemonic $m_{\text{row column}}$, $m_{\text{Roman Catholic}}$.)

Matrix addition is performed by adding corresponding components.

$$[2 \quad 1 \quad 4] + [7 \quad 11 \quad 3] = [9 \quad 12 \quad 7]$$

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix}$$

This is consistent with the way we add vectors, whether interpreting them as rows or columns. Notice that we only add matrices of the same dimensions; we cannot add, for example, (5.57) to (5.58).

Scalar multiplication also proceeds by entries, in keeping with our vector interpretation.

$$2[2\quad 1\quad 4] = [4\quad 2\quad 8]$$

$$-\tfrac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -1 \end{bmatrix}$$

It follows that matrix addition, like vector addition, has all the usual properties; namely, commutativity, associativity, and distributivity with respect to scalar multiplication. In fact, matrix theory would be dull, indeed, if addition and scalar multiplication were the only operations.

The definition of matrix multiplication is what makes the subject useful and interesting. In general, the product $\mathscr{L}\mathscr{R} = \mathscr{P}$ of two matrices $\mathscr{L}$ and $\mathscr{R}$ is defined only when the number of *columns* of the left factor $\mathscr{L}$ equals the number of *rows* of the right factor $\mathscr{R}$. Denoting this common number by s, the entry in the ith row and jth column of $\mathscr{P}$ is then defined by

$$p_{ij} = \sum_{k=1}^{s} \ell_{ik} r_{kj} \tag{5.63}$$

From the formula (5.63), one can see that $\mathscr{P}$ has the same number of *rows* as $\mathscr{L}$, and the same number of *columns as* $\mathscr{R}$.

If we look at the possibilities for multiplication of the three types of matrices with which we are concerned, we discover that there are four interpretations of (5.63):

(*i*) If $\mathscr{L}$ is a 1-by-3 row matrix and $\mathscr{R}$ is a 3-by-1 column matrix, then the product $\mathscr{L}\mathscr{R}$ is the scalar product of the corresponding vectors:

$$[\cdots\ \mathbf{A}\ \cdots]\begin{bmatrix} \vdots \\ \mathbf{B} \\ \vdots \end{bmatrix} = \mathbf{A}\cdot\mathbf{B} \tag{5.64}$$

For example,

$$[2\quad 4\quad 1]\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = 4 + 4 + 5 = 13$$

(*ii*) If $\mathscr{L}$ is a 3-by-3 square matrix and $\mathscr{R}$ is a 3-by-1 column matrix, then the product $\mathscr{L}\mathscr{R}$ is a 3-by-1 column matrix. The first entry is the scalar product of the first row of $\mathscr{L}$ with $\mathscr{R}$; the second and third entries are the corresponding scalar products of the second and third rows, respectively, with $\mathscr{R}$:

$$\begin{bmatrix} \cdots & \mathbf{A} & \cdots \\ \cdots & \mathbf{B} & \cdots \\ \cdots & \mathbf{C} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \mathbf{D} \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{A} \cdot \mathbf{D} \\ \mathbf{B} \cdot \mathbf{D} \\ \mathbf{C} \cdot \mathbf{D} \end{bmatrix} \tag{5.65}$$

For example,

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 13 \end{bmatrix} \tag{5.66}$$

(*iii*) If $\mathscr{L}$ is a 1-by-3 row matrix and $\mathscr{R}$ is a 3-by-3 square matrix, then the product $\mathscr{L}\mathscr{R}$ is a 1-by-3 row matrix. The first entry is the scalar product of $\mathscr{L}$ with the first column of $\mathscr{R}$; the second and third entries are the corresponding scalar products with the second and third columns, respectively:

$$\begin{bmatrix} \cdots & \mathbf{A} & \cdots \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{B} & \mathbf{C} & \mathbf{D} \\ \vdots & \vdots & \vdots \end{bmatrix} = [\mathbf{A} \cdot \mathbf{B}, \mathbf{A} \cdot \mathbf{C}, \mathbf{A} \cdot \mathbf{D}] \tag{5.67}$$

For example,

$$[2 \quad 4 \quad 1] \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = [11 \quad -1 \quad 8] \tag{5.68}$$

(*iv*) If $\mathscr{L}$ and $\mathscr{R}$ are both 3-by-3 square matrices, then the product $\mathscr{L}\mathscr{R}$ is also a 3-by-3 square matrix. *The* (i, j)*th entry of* $\mathscr{L}\mathscr{R}$ *is the scalar product of row* i *of* $\mathscr{L}$ *with column* j *of* $\mathscr{R}$:

$$\begin{bmatrix} \cdots & \mathbf{A} & \cdots \\ \cdots & \mathbf{B} & \cdots \\ \cdots & \mathbf{C} & \cdots \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{D} & \mathbf{E} & \mathbf{F} \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{A} \cdot \mathbf{D} & \mathbf{A} \cdot \mathbf{E} & \mathbf{A} \cdot \mathbf{F} \\ \mathbf{B} \cdot \mathbf{D} & \mathbf{B} \cdot \mathbf{E} & \mathbf{B} \cdot \mathbf{F} \\ \mathbf{C} \cdot \mathbf{D} & \mathbf{C} \cdot \mathbf{E} & \mathbf{C} \cdot \mathbf{F} \end{bmatrix} \tag{5.69}$$

For example,

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 \\ 4 & 6 & 8 \\ 8 & 8 & 8 \end{bmatrix} \tag{5.70}$$

Notice that the italicized statement in (iv) actually covers cases (i), (ii), and (iii) also. Alert readers may observe that $\mathscr{L}\mathscr{R}$ is also defined when $\mathscr{L}$ is a column and $\mathscr{R}$ is a row, but we have no use for such products here. (They are related to the dyadics mentioned in Sec. 3.6.)

The most common use of matrices is in expressing a system of equations. Two matrices are equal when they have the same dimensions and their corresponding entries are equal.

Example 5.10 Express the system of equations

$$\begin{aligned} x - y + z &= 1 \\ 2x \phantom{{}+y} + z &= 2 \\ x + y + 2z &= 3 \end{aligned} \tag{5.71}$$

as a matrix equation.

Solution The reader can verify that

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{5.72}$$

is equivalent to Eq. (5.71). The square matrix is the same as (5.59), and its rows **A**, **B**, and **C** are identified in (5.60). The equations (5.71) can be written as follows, with $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{R} &= 1 \\ \mathbf{B} \cdot \mathbf{R} &= 2 \\ \mathbf{C} \cdot \mathbf{R} &= 3 \end{aligned}$$

Hence, **R** is the position vector to a point lying simultaneously in three planes, having normals **A**, **B**, and **C**, respectively (recall Sec. 1.10).

Since matrix products can be interpreted as arrays of scalar products, it should come as no surprise that matrix multiplication is linear in each factor; for any scalar s,

$$\begin{aligned} (s\mathscr{M} + \mathscr{N})\mathscr{R} &= s\mathscr{M}\mathscr{R} + \mathscr{N}\mathscr{R} \\ \mathscr{L}(s\mathscr{M} + \mathscr{N}) &= s\mathscr{L}\mathscr{M} + \mathscr{L}\mathscr{N} \end{aligned} \tag{5.73}$$

assuming all products are defined. The proof is left as an exercise.

Since scalar products are commutative, one would expect that matrix multiplication is also. Surprisingly, this is not the case! Consider, for example, the matrices in (5.70) multiplied in reverse order:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 9 \\ 8 & -2 & 7 \\ 8 & 0 & 8 \end{bmatrix}$$

The answer is different! The reason is that when we form $\mathscr{L}\mathscr{R}$ we use vectors from the *rows* of $\mathscr{L}$ and the *columns* of $\mathscr{R}$; but when we form $\mathscr{R}\mathscr{L}$ we use the *columns* of $\mathscr{L}$ and the *rows* of $\mathscr{R}$. We are taking scalar products of different vectors in the two cases. (In fact, if the dimensions do not match, $\mathscr{R}\mathscr{L}$ may be undefined.)

Let us consider the associative law. Is there any difference between $(\mathscr{L}\mathscr{M})\mathscr{R}$ and $\mathscr{L}(\mathscr{M}\mathscr{R})$? Notice that $\mathscr{M}$ must be square 3-by-3 if both products are defined, in the context of "vector analysis matrices."

One begins the computation of $(\mathscr{L}\mathscr{M})\mathscr{R}$ by regarding the *columns* of $\mathscr{M}$ as vectors. However, to compute $\mathscr{L}(\mathscr{M}\mathscr{R})$, one starts with the *row*-vectors of $\mathscr{M}$. Thus, we would expect the products to be different. Astonishingly, they are the same!

Example 5.11 Verify the associative law

$$(\mathscr{L}\mathscr{M})\mathscr{R} = \mathscr{L}(\mathscr{M}\mathscr{R}) \tag{5.74}$$

for $\mathscr{L}$, $\mathscr{M}$, and $\mathscr{R}$ given by Eqs. (5.57), (5.59), and (5.58), respectively.

Solution We computed $\mathscr{L}\mathscr{M}$ in (5.68); hence,

$$(\mathscr{L}\mathscr{M})\mathscr{R} = [11 \quad -1 \quad 8]\begin{bmatrix}2\\1\\5\end{bmatrix} = 22 - 1 + 40 = 61$$

$\mathscr{M}\mathscr{R}$ was evaluated in (5.66); therefore,

$$\mathscr{L}(\mathscr{M}\mathscr{R}) = [2 \quad 4 \quad 1]\begin{bmatrix}6\\9\\13\end{bmatrix} = 12 + 36 + 13 = 61$$

To prove (5.74) in general, we have to abandon momentarily the vector interpretation of matrix multiplication, and work with the explicit formula (5.67) in terms of the entries.

We introduce the notation $\mathscr{N} = \mathscr{L}\mathscr{M}$ and $\mathscr{Q} = \mathscr{M}\mathscr{R}$. Then the (i, j)th entry of the left-hand side of (5.74) is found by using (5.63) twice:

$$\sum_{k=1}^{3} n_{ik} r_{kj} = \sum_{k=1}^{3}\left(\sum_{s=1}^{3} \ell_{is} m_{sk}\right) r_{kj} = \sum_{k=1}^{3}\sum_{s=1}^{3} \ell_{is} m_{sk} r_{kj} \tag{5.75}$$

The (i, j)th element of the right-hand side of (5.74) is

$$\sum_{k=1}^{3} \ell_{ik} q_{kj} = \sum_{k=1}^{3} \ell_{ik}\left(\sum_{s=1}^{3} m_{ks} r_{sj}\right) = \sum_{k=1}^{3}\sum_{s=1}^{3} \ell_{ik} m_{ks} r_{sj} \tag{5.76}$$

Exactly the same terms appear in the final sums in (5.75) and (5.76), so they are equal and (5.74) is proved.

One matrix plays a special role in linear algebra: it is the *identity matrix* $\mathscr{I}$ defined by

$$\mathscr{I} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\cdots & \mathbf{i} & \cdots\\\cdots & \mathbf{j} & \cdots\\\cdots & \mathbf{k} & \cdots\end{bmatrix} = \begin{bmatrix}\vdots & \vdots & \vdots\\\mathbf{i} & \mathbf{j} & \mathbf{k}\\\vdots & \vdots & \vdots\end{bmatrix} \tag{5.77}$$

It is called the "identity" because multiplication by $\mathscr{I}$, on the right or left, produces no change.

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2\\1\\5\end{bmatrix} = \begin{bmatrix}2\\1\\5\end{bmatrix}$$

$$[2 \quad 4 \quad 1]\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = [2 \quad 4 \quad 1]$$

Another way to see this is

$$\begin{bmatrix} \cdots & \mathbf{i} & \cdots \\ \cdots & \mathbf{j} & \cdots \\ \cdots & \mathbf{k} & \cdots \end{bmatrix}\begin{bmatrix} \vdots \\ \mathbf{A} \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{i}\cdot\mathbf{A} \\ \mathbf{j}\cdot\mathbf{A} \\ \mathbf{k}\cdot\mathbf{A} \end{bmatrix} = \begin{bmatrix} \vdots \\ \mathbf{A} \\ \vdots \end{bmatrix}$$

$$[\cdots\ \mathbf{B}\ \cdots]\begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \vdots & \vdots & \vdots \end{bmatrix} = [\mathbf{B}\cdot\mathbf{i}\quad \mathbf{B}\cdot\mathbf{j}\quad \mathbf{B}\cdot\mathbf{k}] = [\cdot\cdot\ \mathbf{B}\ \cdot\cdot]$$

Given a square matrix $\mathscr{M}$, we say $\mathscr{R}$ is a *right inverse* of $\mathscr{M}$ if $\mathscr{M}\mathscr{R} = \mathscr{I}$. For instance, one should check that

$$\overset{\mathscr{M}}{\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}}\overset{\mathscr{R}}{\begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}} = \overset{\mathscr{I}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \tag{5.78}$$

so the second matrix is a right inverse of the first.

Notice the relation between the rows of $\mathscr{M}$ and the columns of $\mathscr{R}$: If we express $\mathscr{M}\mathscr{R} = \mathscr{I}$ as

$$\overset{\mathscr{M}}{\begin{bmatrix} \cdots & \mathbf{A} & \cdots \\ \cdots & \mathbf{B} & \cdots \\ \cdots & \mathbf{C} & \cdots \end{bmatrix}}\overset{\mathscr{R}}{\begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{D} & \mathbf{E} & \mathbf{F} \\ \vdots & \vdots & \vdots \end{bmatrix}} = \overset{\mathscr{I}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

we see that

$$\mathbf{A}\cdot\mathbf{D} = \mathbf{B}\cdot\mathbf{E} = \mathbf{C}\cdot\mathbf{F} = 1$$
$$\mathbf{A}\cdot\mathbf{E} = \mathbf{A}\cdot\mathbf{F} = 0 \qquad \mathbf{B}\cdot\mathbf{D} = \mathbf{B}\cdot\mathbf{F} = 0 \qquad \mathbf{C}\cdot\mathbf{D} = \mathbf{C}\cdot\mathbf{E} = 0 \tag{5.79}$$

Whenever two sets of vectors $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$ and $\{\mathbf{D},\mathbf{E},\mathbf{F}\}$ satisfy relations (5.79), we say that one set of vectors is *reciprocal*, or *dual*, to the other set. The considerations of Chapter 1 will enable us to find formulas for reciprocal vectors (and, as a result, a formula for the right inverse).

Example 5.12 Derive the formulas for the reciprocal vectors for the set $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$.

Solution Examining (5.79), we see that $\mathbf{D}$ must be perpendicular to $\mathbf{B}$ and $\mathbf{C}$. Let us try $\mathbf{D} = s\mathbf{B}\times\mathbf{C}$. To make $\mathbf{A}\cdot\mathbf{D} = 1$, we must choose s so that

$$s\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = s[\mathbf{A},\mathbf{B},\mathbf{C}] = 1$$

Consequently, if $[\mathbf{A},\mathbf{B},\mathbf{C}]$, which is the determinant of $\mathscr{M}$, is not zero, we find

$$\mathbf{D} = \frac{\mathbf{B}\times\mathbf{C}}{[\mathbf{A},\mathbf{B},\mathbf{C}]} \tag{5.80}$$

Similarly,

$$\mathbf{E} = \frac{\mathbf{C}\times\mathbf{A}}{[\mathbf{A},\mathbf{B},\mathbf{C}]} \tag{5.81}$$

and

$$\mathbf{F} = \frac{\mathbf{A}\times\mathbf{B}}{[\mathbf{A},\mathbf{B},\mathbf{C}]} \tag{5.82}$$

On the other hand, if $[\mathbf{A},\mathbf{B},\mathbf{C}] = 0$, then **A**, **B**, and **C** are coplanar, and it is easy to see that no set satisfying (5.79) exists (Exercise 5).

This provides a formula for the right inverse: *if* $[\mathbf{A},\mathbf{B},\mathbf{C}] \neq 0$, *then a right inverse for the matrix*

$$\mathscr{M} = \begin{bmatrix} \cdots & \mathbf{A} & \cdots \\ \cdots & \mathbf{B} & \cdots \\ \cdots & \mathbf{C} & \cdots \end{bmatrix}$$

is given by

$$\mathscr{R} = \frac{1}{[\mathbf{A},\mathbf{B},\mathbf{C}]}\begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{B}\times\mathbf{C} & \mathbf{C}\times\mathbf{A} & \mathbf{A}\times\mathbf{B} \\ \vdots & \vdots & \vdots \end{bmatrix} \tag{5.83}$$

If $[\mathbf{A},\mathbf{B},\mathbf{C}] = 0$, no right inverse exists. The reader should verify (Exercise 7) that the right inverse displayed in (5.78) is consistent with this formula.

Clearly, if we take the reciprocal set of vectors for the *columns* of $\mathscr{M}$ and arrange them in the *rows* of $\mathscr{L}$, we obtain a *left inverse* of $\mathscr{M}$, i.e., $\mathscr{L}\mathscr{M} = \mathscr{I}$. Hence, if we now let **A**,**B**,**C** represent the *columns* of $\mathscr{M}$, we have

$$\frac{1}{[\mathbf{A},\mathbf{B},\mathbf{C}]}\begin{bmatrix} \cdots & \mathbf{B}\times\mathbf{C} & \cdots \\ \cdots & \mathbf{C}\times\mathbf{A} & \cdots \\ \cdots & \mathbf{A}\times\mathbf{B} & \cdots \end{bmatrix}\begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \vdots & \vdots & \vdots \end{bmatrix} = \mathscr{I} \tag{5.84}$$

A quick computation (Exercise 6) reveals that the triple scalar product of the columns of $\mathscr{M}$ equals the triple scalar product of its rows, which in turn equals the determinant. So the existence of a left inverse also hinges on the condition $\det(\mathscr{M}) \neq 0$.

In Exercise 8 the reader is invited to compute the left inverse of the matrix $\mathscr{M}$ in Eq. (5.78). Rather surprisingly, the left inverse $\mathscr{L}$ turns out to be equal to the right inverse $\mathscr{R}$! To see this in general, we use the associative law, together with the equalities $\mathscr{L}\mathscr{M} = \mathscr{I} = \mathscr{M}\mathscr{R}$, to derive

$$\mathscr{L} = \mathscr{L}\mathscr{I} = \mathscr{L}(\mathscr{M}\mathscr{R}) = (\mathscr{L}\mathscr{M})\mathscr{R} = \mathscr{I}\mathscr{R} = \mathscr{R}$$

The fact of the matter, as the reader will show in Exercise 9, is if a matrix $\mathscr{M}$ has any inverse at all, it has only one; and this one is both a left and a right inverse. It can therefore be denoted $\mathscr{M}^{-1}$, unambiguously. The condition for having an inverse is that the determinant of the matrix must be nonzero.

The usefulness of an inverse in "undoing" a system of equations is illustrated in the next example.

Example 5.13 Solve the system of equations given in Example 5.10.

Solution Consider the equations expressed in matrix form (5.72). If we multiply both sides of that equation on the left by the inverse of the matrix, given in (5.78), we find the solution:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-\frac{1}{4}&\frac{3}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{1}{4}&\frac{1}{4}\\\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\1\end{bmatrix}$$

or $x=\frac{1}{2}$, $y=\frac{1}{2}$, $z=1$.

Example 5.14 Derive a general formula for the solution of the system

$$\begin{bmatrix}m_{11}&m_{12}&m_{13}\\m_{21}&m_{22}&m_{23}\\m_{31}&m_{32}&m_{33}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}u\\v\\w\end{bmatrix}\tag{5.85}$$

where $\det \mathscr{M} \neq 0$.

Solution First we form the left inverse of the square matrix. To achieve this, we have to regard it as made up of column vectors:

$$\mathscr{M}=\begin{bmatrix}\vdots&\vdots&\vdots\\\mathbf{D}&\mathbf{E}&\mathbf{F}\\\vdots&\vdots&\vdots\end{bmatrix}\qquad \mathscr{M}^{-1}=\frac{1}{[\mathbf{D},\mathbf{E},\mathbf{F}]}\begin{bmatrix}\cdots&\mathbf{E}\times\mathbf{F}&\cdots\\\cdots&\mathbf{F}\times\mathbf{D}&\cdots\\\cdots&\mathbf{D}\times\mathbf{E}&\cdots\end{bmatrix}$$

Next we multiply on the left by $\mathscr{M}^{-1}$ to obtain

$$\mathscr{I}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{[\mathbf{D},\mathbf{E},\mathbf{F}]}\begin{bmatrix}\cdots&\mathbf{E}\times\mathbf{F}&\cdots\\\cdots&\mathbf{F}\times\mathbf{D}&\cdots\\\cdots&\mathbf{D}\times\mathbf{E}&\cdots\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}$$

Looking at the first component, we find

$$x=\frac{\mathbf{E}\times\mathbf{F}\cdot(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})}{[\mathbf{D},\mathbf{E},\mathbf{F}]}=\frac{\begin{vmatrix}u&v&w\\\cdots&\mathbf{E}&\cdots\\\cdots&\mathbf{F}&\cdots\end{vmatrix}}{\begin{vmatrix}\cdots&\mathbf{D}&\cdots\\\cdots&\mathbf{E}&\cdots\\\cdots&\mathbf{F}&\cdots\end{vmatrix}}=\frac{\begin{vmatrix}u&\vdots&\vdots\\v&\mathbf{E}&\mathbf{F}\\w&\vdots&\vdots\end{vmatrix}}{\begin{vmatrix}\vdots&\vdots&\vdots\\\mathbf{D}&\mathbf{E}&\mathbf{F}\\\vdots&\vdots&\vdots\end{vmatrix}}$$

Identifying the parts of the original matrix $\mathscr{M}$, we have

$$x=\frac{\begin{vmatrix}u&m_{12}&m_{13}\\v&m_{22}&m_{23}\\w&m_{32}&m_{33}\end{vmatrix}}{\begin{vmatrix}m_{11}&m_{12}&m_{13}\\m_{21}&m_{22}&m_{23}\\m_{31}&m_{32}&m_{33}\end{vmatrix}}\tag{5.86}$$

Similarly,

$$y = \frac{\begin{vmatrix} m_{11} & u & m_{13} \\ m_{21} & v & m_{23} \\ m_{31} & w & m_{33} \end{vmatrix}}{\begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} m_{11} & m_{12} & u \\ m_{21} & m_{22} & v \\ m_{31} & m_{32} & w \end{vmatrix}}{\begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix}}$$

These formulas are collectively called *Cramer's rule.*

Exercises

1. Form the indicated products:

(a) $[1 \quad 2 \quad 3]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (d) $[1 \quad 2 \quad 3]\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$

(b) $[1 \quad 2 \quad 3]\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

2. Prove that matrix addition is commutative and associative, and distributes with scalar multiplication. (*Hint*: Exploit the vector interpretation.)

3. Prove the distributive laws (5.73).

4. Construct an example of two square matrices whose product *is* commutative.

5. Show that if **A**, **B**, and **C** are coplanar, then no reciprocal set of vectors exists. (Do not forget to consider the possibility that they are collinear.)

6. Show that the triple scalar product of the columns of a square matrix equals the triple scalar product of its rows.

7. Show that the right inverse in (5.78) agrees with the formula (5.83).

8. Using Eq. (5.84), compute the left inverse of the matrix in (5.59), and compare with (5.78).

9. Prove that if det $\mathscr{M} \neq 0$, $\mathscr{M}$ has only one inverse.

10. Solve by computing the inverse:

$$\begin{aligned} 2x + y + 2z &= 2 \\ 3x + 2z &= 4 \\ x + y + 2z &= 0 \end{aligned}$$

11. Solve Exercise 10 by Cramer's rule.

12. Show that the inverse of a product equals the product of the inverses, in reverse order; i.e., $(\mathscr{L}\mathscr{R})^{-1} = \mathscr{R}^{-1}\mathscr{L}^{-1}$.

The transpose of a matrix $\mathscr{M}$ is a matrix, $\mathscr{M}^T$, whose rows are the columns of the original matrix $\mathscr{M}$.

13. Show that the (j,i)th entry in $\mathscr{M}^T$ equals the (i,j)th entry in $\mathscr{M}$.

14. Show that the transpose of a product is the product of the transposes in reverse order; i.e., $(\mathscr{L}\mathscr{R})^T = \mathscr{R}^T\mathscr{L}^T$.

A symmetric matrix is a matrix that equals its transpose. An antisymmetric matrix is a matrix that equals the negative of its transpose. An orthogonal matrix is a matrix whose inverse equals its transpose.

15. Show that a symmetric matrix must have the form

$$\begin{bmatrix} m_{11} & m_{21} & m_{31} \\ m_{21} & m_{22} & m_{32} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Is the product of two symmetric matrices, symmetric?

16. Show that the columns of an orthogonal matrix are mutually orthogonal unit vectors. Show the same for the rows. What does this say about the reciprocal sets?

17. Show that if $\mathscr{O}$ is orthogonal and $\mathscr{S}$ is symmetric, then $\mathscr{O}^{-1}\mathscr{S}\mathscr{O}$ is symmetric.

18. Show that if A is antisymmetric, it has the form

$$\begin{bmatrix} 0 & -m_{21} & m_{13} \\ m_{21} & 0 & -m_{32} \\ -m_{13} & m_{32} & 0 \end{bmatrix}$$

Then show if $\mathscr{R}$ is a column vector representing $\mathbf{v}$, $\mathscr{A}\mathscr{R}$ represents $(m_{32}\mathbf{i} + m_{13}\mathbf{j} + m_{21}\mathbf{k}) \times \mathbf{v}$.

19. Construct examples of systems of three equations of the form (5.85) whose solutions constitute:
(a) a plane,
(b) a straight line, and
(c) the empty set.
(*Hint*: Remember the interpretation as the intersection of three planes.) What is the determinant of the matrix in these cases?

5.4 OPTIONAL READING: LINEAR ORTHOGONAL TRANSFORMATIONS

We now return to the study of different coordinate systems for describing scalar and vector fields. An important case is the utilization of another *cartesian* coordinate system (right-handed, of course), with axes labeled

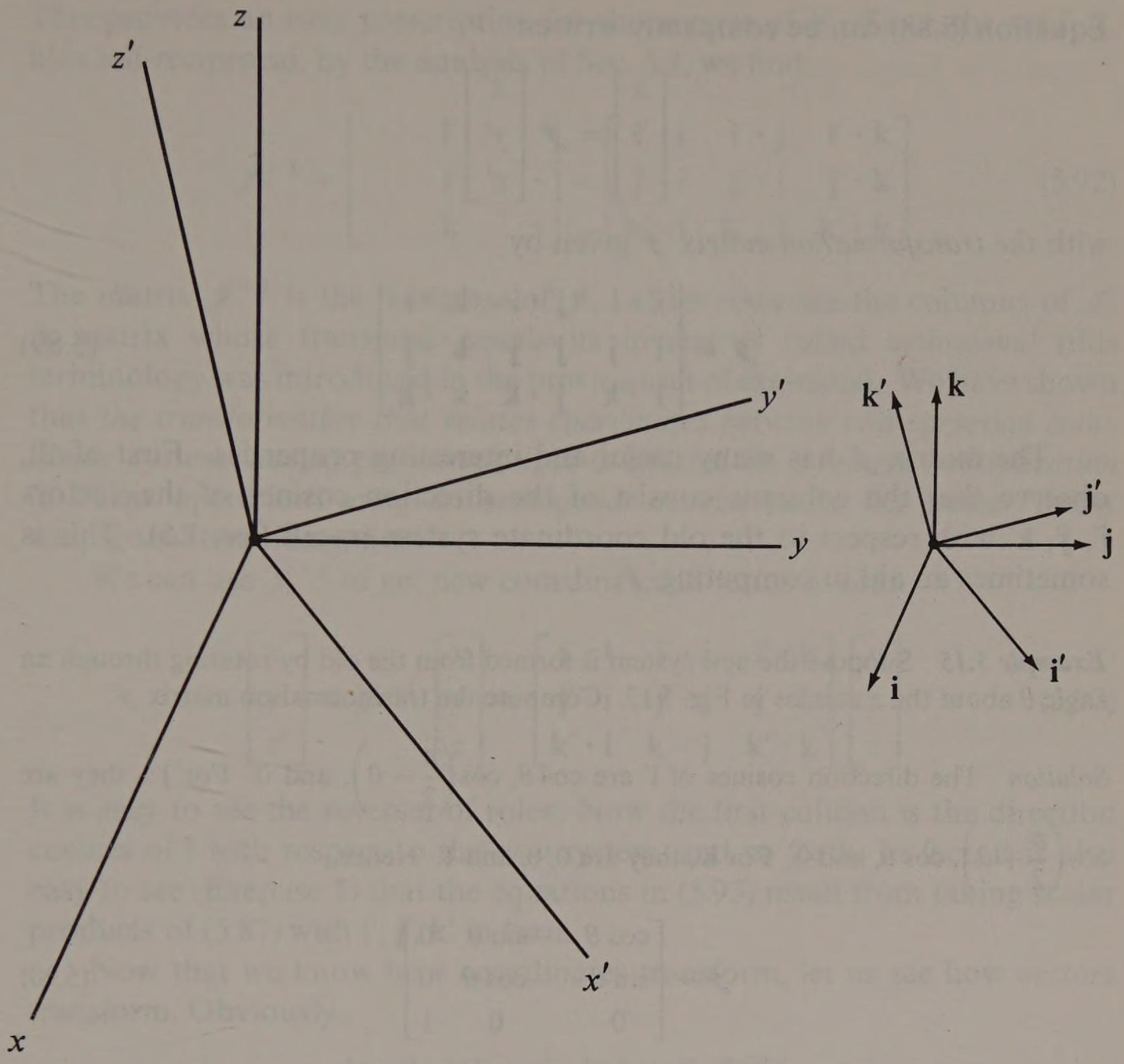

FIGURE 5.12

x', y', z' and their associated unit vectors $\mathbf{i}', \mathbf{j}', \mathbf{k}'$. This "new" coordinate system will have the same origin as the "old" x, y, z system (see Fig. 5.12).

Consider a point in space. The point's coordinates in the old system are (x, y, z) and in the new system they are (x', y', z'). To see how these are related, draw the position vector $\mathbf{R}$ to the point (Fig. 5.12). Then we have two descriptions of $\mathbf{R}$:

$$\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}' \tag{5.87}$$

If we now take scalar products of (5.87) with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in turn, we find

$$\begin{aligned} x &= x'\mathbf{i}' \cdot \mathbf{i} + y'\mathbf{j}' \cdot \mathbf{i} + z'\mathbf{k}' \cdot \mathbf{i} \\ y &= x'\mathbf{i}' \cdot \mathbf{j} + y'\mathbf{j}' \cdot \mathbf{j} + z'\mathbf{k}' \cdot \mathbf{j} \\ z &= x'\mathbf{i}' \cdot \mathbf{k} + y'\mathbf{j}' \cdot \mathbf{k} + z'\mathbf{k}' \cdot \mathbf{k} \end{aligned} \tag{5.88}$$

Equation (5.88) can be compactly written

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathscr{J} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

with *the transformation matrix* $\mathscr{J}$ given by

$$\mathscr{J} = \begin{bmatrix} \mathbf{i}' \cdot \mathbf{i} & \mathbf{j}' \cdot \mathbf{i} & \mathbf{k}' \cdot \mathbf{i} \\ \mathbf{i}' \cdot \mathbf{j} & \mathbf{j}' \cdot \mathbf{j} & \mathbf{k}' \cdot \mathbf{j} \\ \mathbf{i}' \cdot \mathbf{k} & \mathbf{j}' \cdot \mathbf{k} & \mathbf{k}' \cdot \mathbf{k} \end{bmatrix} \tag{5.89}$$

The matrix $\mathscr{J}$ has many useful and interesting properties. First of all, observe that the columns consist of the direction cosines of the vectors $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ with respect to the old coordinate system (recall Sec. 1.5). This is sometimes an aid in computing $\mathscr{J}$.

Example 5.15 Suppose the new system is formed from the old by rotating through an angle θ about the z axis, as in Fig. 5.13. Compute the transformation matrix $\mathscr{J}$.

Solution The direction cosines of $\mathbf{i}'$ are $\cos\theta$, $\cos\left(\frac{\pi}{2} - \theta\right)$, and 0. For $\mathbf{j}'$s, they are $\cos\left(\frac{\pi}{2} + \theta\right)$, $\cos\theta$, and 0. For $\mathbf{k}'$, they are 0, 0, and 1. Hence,

$$\mathscr{J} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.90}$$

From another point of view we can say that the columns of $\mathscr{J}$ represent the vectors $\mathbf{i}'$, $\mathbf{j}'$, and $\mathbf{k}'$ in the old coordinate system

$$\mathscr{J} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{i}' & \mathbf{j}' & \mathbf{k}' \\ \vdots & \vdots & \vdots \end{bmatrix} \tag{5.91}$$

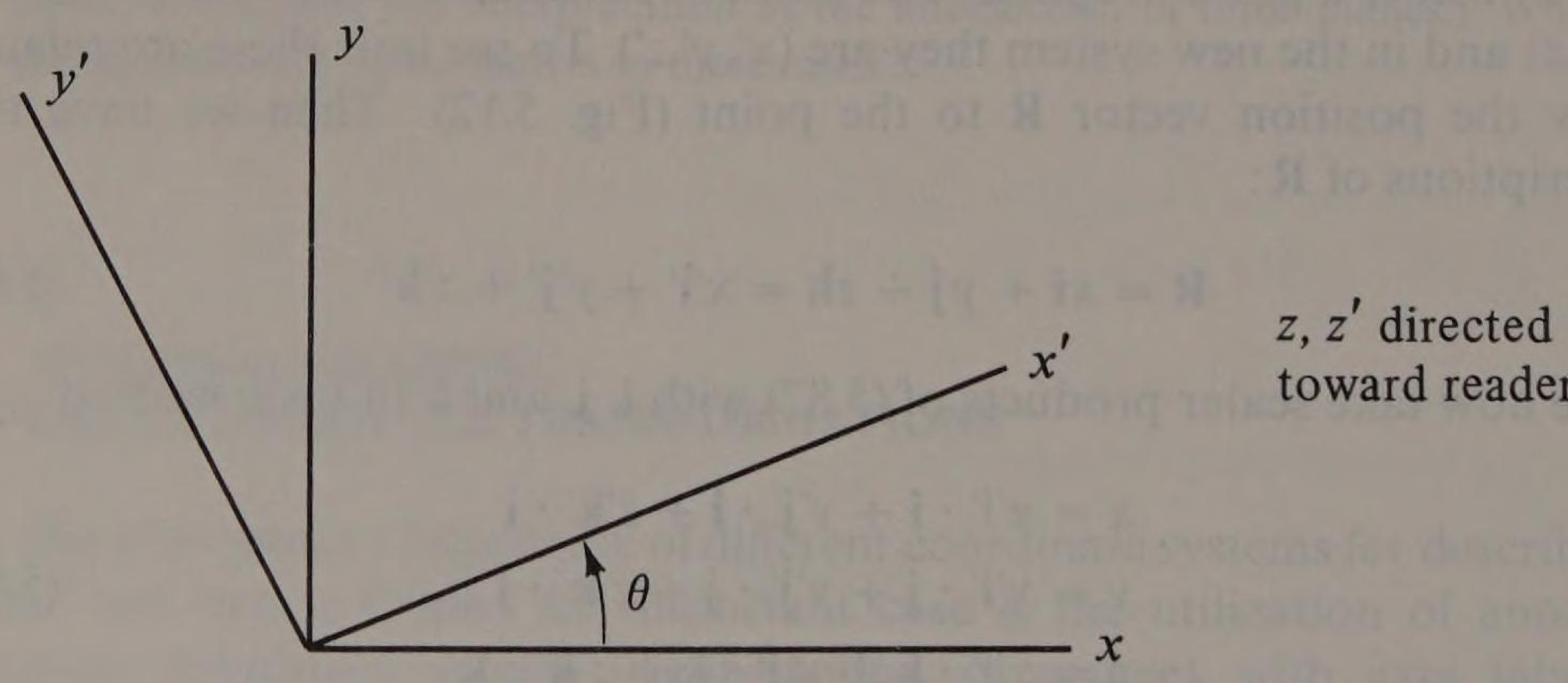

FIGURE 5.13

This provides an easy prescription for the inverse of $\mathscr{I}$. Since the set $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ is self-reciprocal, by the analysis of Sec. 5.3, we find

$$\mathscr{I}^{-1} = \begin{bmatrix} \cdots & \mathbf{i}' & \cdots \\ \cdots & \mathbf{j}' & \cdots \\ \cdots & \mathbf{k}' & \cdots \end{bmatrix} = \begin{bmatrix} \mathbf{i}'\cdot\mathbf{i} & \mathbf{i}'\cdot\mathbf{j} & \mathbf{i}'\cdot\mathbf{k} \\ \mathbf{j}'\cdot\mathbf{i} & \mathbf{j}'\cdot\mathbf{j} & \mathbf{j}'\cdot\mathbf{k} \\ \mathbf{k}'\cdot\mathbf{i} & \mathbf{k}'\cdot\mathbf{j} & \mathbf{k}'\cdot\mathbf{k} \end{bmatrix} \tag{5.92}$$

The matrix $\mathscr{I}^{-1}$ is the *transpose* of $\mathscr{I}$, i.e., its rows are the columns of $\mathscr{I}$. A matrix whose transpose equals its inverse is called *orthogonal* (this terminology was introduced in the previous set of exercises). We have shown that *the transformation that relates coordinates between two cartesian coordinate systems having the same origin is effected by a matrix multiplication* (a linear operation) *using an orthogonal matrix*, hence the name "linear orthogonal transformation."

We can use $\mathscr{I}^{-1}$ to get new coordinates in terms of old:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \mathscr{I}^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{i}'\cdot\mathbf{i} & \mathbf{i}'\cdot\mathbf{j} & \mathbf{i}'\cdot\mathbf{k} \\ \mathbf{j}'\cdot\mathbf{i} & \mathbf{j}'\cdot\mathbf{j} & \mathbf{j}'\cdot\mathbf{k} \\ \mathbf{k}'\cdot\mathbf{i} & \mathbf{k}'\cdot\mathbf{j} & \mathbf{k}'\cdot\mathbf{k} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{5.93}$$

It is easy to see the reversal of roles: Now the first column is the direction cosines of $\mathbf{i}$ with respect to the *new* system, and so forth. In fact, it is also easy to see (Exercise 1) that the equations in (5.93) result from taking scalar products of (5.87) with $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ in turn.

Now that we know how coordinates transform, let us see how *vectors* transform. Obviously,

$$\begin{aligned} \mathbf{i} &= (\mathbf{i}\cdot\mathbf{i}')\mathbf{i}' + (\mathbf{i}\cdot\mathbf{j}')\mathbf{j}' + (\mathbf{i}\cdot\mathbf{k}')\mathbf{k}' \\ \mathbf{j} &= (\mathbf{j}\cdot\mathbf{i}')\mathbf{i}' + (\mathbf{j}\cdot\mathbf{j}')\mathbf{j}' + (\mathbf{j}\cdot\mathbf{k}')\mathbf{k}' \\ \mathbf{k} &= (\mathbf{k}\cdot\mathbf{i}')\mathbf{i}' + (\mathbf{k}\cdot\mathbf{j}')\mathbf{j}' + (\mathbf{k}\cdot\mathbf{k}')\mathbf{k}' \end{aligned} \tag{5.94}$$

An arbitrary vector $\mathbf{V}$ will have the two representations

$$\begin{aligned} \mathbf{V} &= \mathbf{V}\cdot\mathbf{i}\,\mathbf{i} + \mathbf{V}\cdot\mathbf{j}\,\mathbf{j} + \mathbf{V}\cdot\mathbf{k}\,\mathbf{k} = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k} \\ &= \mathbf{V}\cdot\mathbf{i}'\,\mathbf{i}' + \mathbf{V}\cdot\mathbf{j}'\,\mathbf{j}' + \mathbf{V}\cdot\mathbf{k}'\,\mathbf{k}' = V_1'\mathbf{i}' + V_2'\mathbf{j}' + V_3'\mathbf{k}' \end{aligned} \tag{5.95}$$

If we now take scalar products of (5.95) with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, we see that *the components of* $\mathbf{V}$ *transform just like the coordinates of a point*:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \mathscr{I}\begin{bmatrix} V_1' \\ V_2' \\ V_3' \end{bmatrix} \qquad \begin{bmatrix} V_1' \\ V_2' \\ V_3' \end{bmatrix} = \mathscr{I}^{-1}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{5.96}$$

What happens to a scalar field $f(x,y,z)$ under the transformation? If we think of f as representing, say, temperature, it is clear that we do not change the value of f at a given point merely by changing coordinate systems. However, the three numbers describing the point's coordinates *do* change;

so the *formula* for f, or its functional form, will be different. Let us consider an example before attempting the general formulation.

Example 5.16 Suppose $f(x,y,z) = y^2 - x^2$ in the old coordinate system. The linear orthogonal transformation of Example 5.15 is performed, with $\theta = \pi/6$. What is the value of f at the point whose coordinates in the new system are (1,1,0)?

Solution Before we can apply the formula $y^2 - x^2$ for f, we must find x and y, i.e., the original coordinates of (1,1,0). Using the transformation (5.89) with $\mathscr{J}$ given by (5.90) and $\theta = \pi/6$, we find

$$x = x' \cos\theta - y' \sin\theta = 1(\sqrt{3}/2) - 1(1/2) = \frac{\sqrt{3}-1}{2}$$

$$y = x' \sin\theta + y' \cos\theta = 1(1/2) + 1(\sqrt{3}/2) = \frac{\sqrt{3}+1}{2}$$

Hence $y^2 - x^2 = \sqrt{3}$, the value of f.

Notice that the value of f could not be computed by taking $f(x',y',z') = y'^2 - x'^2 = 1 - 1 = 0$. First we have to express the original coordinates in terms of the new, and then apply the formula for f to the original coordinates. In other words, to express a function $f(x,y,z)$ in a new coordinate system, $f'(x',y',z')$, we first express the old coordinates in terms of the new,

$$\begin{aligned} x &= x(x',y',z') \\ y &= y(x',y',z') \\ z &= z(x',y',z') \end{aligned} \tag{5.97}$$

[as in (5.88)], then plug into the original functional form for f:

$$f'(x',y',z') = f(x(x',y',z'),\ y(x',y',z'),\ z(x',y',z')) \tag{5.98}$$

In particular, the transformed function should not be written as $f(x',y',z')$, as this is literally incorrect. (However, this notation is often used for abbreviation when there is no possibility of misinterpretation.)

In Example 5.16, the function $f(x,y,z) = y^2 - x^2$ is computed in the new system by

$$\begin{aligned} f'(x',y',z') &= (x' \sin\theta + y' \cos\theta)^2 - (x' \cos\theta - y' \sin\theta)^2 \\ &= (y'^2 - x'^2)\cos 2\theta + 2x'y' \sin 2\theta \end{aligned} \tag{5.99}$$

The transformation of vector fields is doubly complicated; one must get new *components* in terms of old through the transformation (5.96), but *since the components are functions of position* the rule (5.98) must be employed on each component. Thus, first we use (5.98) to express the old (**i**,**j**,**k**) components of **V** in terms of the new (x',y',z') coordinates of the point in space:

$$\begin{aligned}\mathbf{V} &= V_1(x(x',y',z'),y(x',y',z'),z(x',y,'z'))\mathbf{i}\\ &\quad + V_2(x(x'\cdots)\mathbf{j} + V_3(x(x'\cdots)\mathbf{k}\end{aligned}$$

Then we use $\mathscr{J}^{-1}$ to get the new components of **V**. To summarize, we state the rule for transforming a vector field: *if* **V** *is given in the old system by*

$$\mathbf{V} = V_1(x,y,z)\mathbf{i} + V_2(x,y,z)\mathbf{j} + V_3(x,y,z)\mathbf{k} \tag{5.100}$$

and in the new system by

$$\mathbf{V} = V'_1(x',y',z')\mathbf{i}' + V'_2(x',y',z')\mathbf{j}' + V'_3(x',y',z')\mathbf{k}' \tag{5.101}$$

then the components are related by

$$\begin{bmatrix} V'_1(x',y',z') \\ V'_2(x',y',z') \\ V'_3(x',y',z') \end{bmatrix} = \mathscr{J}^{-1} \begin{bmatrix} V_1(x(x',y',z'),y(x',y',z'),z(x',y',z')) \\ V_2(x(x',y',z'),y(x',y',z'),z(x',y',z')) \\ V_3(x(x',y',z'),y(x',y',z'),z(x',y',z')) \end{bmatrix} \tag{5.102}$$

Example 5.17 Express the vector field

$$\mathbf{V} = \mathbf{i} + (yz)\mathbf{j} + (x^2 + y^2)\mathbf{k} \tag{5.103}$$

in the new coordinate system of Example 5.15.

Solution We use the matrix $\mathscr{J}$ in Eq. (5.90) to transform coordinates via (5.89), and we use its transpose for $\mathscr{J}^{-1}$; hence,

$$\begin{aligned}\begin{bmatrix} V'_1(x',y',z') \\ V'_2(x',y',z') \\ V'_3(x',y',z') \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ (x'\sin\theta + y'\cos\theta)z' \\ (x'\cos\theta - y'\sin\theta)^2 + (x'\sin\theta + y'\cos\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta + x'z'\sin^2\theta + y'z'\sin\theta\cos\theta \\ -\sin\theta + x'z'\sin\theta\cos\theta + y'z'\cos^2\theta \\ x'^2 + y'^2 \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{V} &= (\cos\theta + x'z'\sin^2\theta + y'z'\sin\theta\cos\theta)\mathbf{i}' \\ &\quad + (-\sin\theta + x'z'\sin\theta\cos\theta + y'z'\cos^2\theta)\mathbf{j}' \\ &\quad + (x'^2 + y'^2)\mathbf{k}' \end{aligned} \tag{5.104}$$

Now we turn to the question of what happens to the vector operators **grad**, div, and **curl** when we transform coordinates. As a preliminary experiment, consider the following example.

Example 5.18 Let $f(x,y,z) = y^2 - x^2$. Apply the linear orthogonal transformation of Example 5.15. Compare the expression

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{5.105}$$

with the analogous expression in the new coordinates

$$\frac{\partial f'}{\partial x'}\mathbf{i}' + \frac{\partial f'}{\partial y'}\mathbf{j}' + \frac{\partial f'}{\partial z'}\mathbf{k}' \tag{5.106}$$

Solution The transformed function f' was computed in (5.99), and (5.106) becomes

$$(-2x' \cos 2\theta + 2y' \sin 2\theta)\mathbf{i}' + (2x' \sin 2\theta + 2y' \cos 2\theta)\mathbf{j}'$$

Now let us compute (5.105) and then transform the resulting vector field to the new system.

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{2y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = -2x\mathbf{i} + 2y\mathbf{j}$$

Applying (5.102),

$$\mathscr{J}^{-1}\begin{bmatrix} -2x(x',y',z') \\ 2y(x',y',z') \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2x' \cos\theta + 2y' \sin\theta \\ 2x' \sin\theta + 2y' \cos\theta \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2x' \cos 2\theta + 2y' \sin 2\theta \\ 2x' \sin 2\theta + 2y' \cos 2\theta \\ 0 \end{bmatrix}$$

Hence, (5.105) transforms into (5.106) and they describe the same vector field.

Of course, both expressions in the last example represent ∇f, computed in different coordinate systems. The reason they resulted in the same vector field is that **grad** f can be characterized without reference to any coordinate system; it is a vector pointing in the direction of maximum rate of change of f with respect to distance, and having length equal to this maximum rate of change. The same argument that showed **grad** f could be computed by (5.105) in the old coordinate system also shows it can be computed by (5.106) in the new coordinate system.

Similarly, the divergence of a vector field can be defined in a coordinate free manner, thanks to the divergence theorem (recall the last paragraph of Sec. 4.10). If we compute div $\mathbf{V}$ as the flux per unit volume out of a box with sides perpendicular to the (x,y,z) axes, we get

$$\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \tag{5.107}$$

If we use a box with sides perpendicular to the (x',y',z) axes, we get

$$\frac{\partial V'_1}{\partial x'} + \frac{\partial V'_2}{\partial y'} + \frac{\partial V'_3}{\partial z'} \tag{5.108}$$

Since they both represent div $\mathbf{V}$, (5.107) and (5.108) are the same scalar field, expressed in different coordinates.

Example 5.19 Verify this statement for **V** as in Example 5.17.

Solution Applying (5.107) to (5.103), we obtain $\mathbf{V}\cdot\mathbf{V} = z$. Applying (5.108) to (5.104), we obtain $z'\sin^2\theta + z'\cos^2\theta = z'$. Since $z' = z$ for this transformation, these are the same.

Of course, we can say the same about **curl V**. Because of Stokes' theorem, the component of **curl V** in any direction is the "swirl" of **V** in that direction, a concept defined without reference to a coordinate system (recall Exercise 1 following Sec. 4.12). Hence, the vector field

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} \tag{5.109}$$

transforms to the vector field

$$\begin{vmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' \\ \dfrac{\partial}{\partial x'} & \dfrac{\partial}{\partial y'} & \dfrac{\partial}{\partial z'} \\ V'_1 & V'_2 & V'_3 \end{vmatrix} \tag{5.110}$$

and they both equal **curl V**. Exercise 7 asks the reader to verify this for the vector field (5.103).

The following example reveals another interpretation of the matrix $\mathcal{J}$ in (5.89).

Example 5.20 Verify that (5.105) and (5.106) describe the same vector field for a *general* linear orthogonal transformation.

Solution The functional forms f and f' are related by Eq. (5.98); hence,

$$\begin{aligned} \frac{\partial f'}{\partial x'} &= \frac{\partial f(x(x',y',z'), y(x',y',z'), z(x',y',z'))}{\partial x'} \\ &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x'} \end{aligned} \tag{5.111}$$

by the chain rule. Similar equations hold for $\partial f'/\partial y'$ and $\partial f'/\partial z'$.

From the equations (5.88), we find

$$\frac{\partial x}{\partial x'} = \mathbf{i}'\cdot\mathbf{i} \qquad \frac{\partial x}{\partial y'} = \mathbf{j}'\cdot\mathbf{i} \qquad \frac{\partial x}{\partial z'} = \mathbf{k}'\cdot\mathbf{i} \qquad \frac{\partial y}{\partial x'} = \mathbf{i}'\cdot\mathbf{j} \quad (etc.)$$

In other words, the partial derivatives of the old coordinates with respect to the new coordinates are the entries of the matrix $\mathscr{J}$:

$$\mathscr{J} = \begin{bmatrix} \dfrac{\partial x}{\partial x'} & \dfrac{\partial x}{\partial y'} & \dfrac{\partial x}{\partial z'} \\[2ex] \dfrac{\partial y}{\partial x'} & \dfrac{\partial y}{\partial y'} & \dfrac{\partial y}{\partial z'} \\[2ex] \dfrac{\partial z}{\partial x'} & \dfrac{\partial z}{\partial y'} & \dfrac{\partial z}{\partial z'} \end{bmatrix} \tag{5.112}$$

Hence, we can see that (5.111), and the corresponding y' and z' equations, can be used to express the vector (5.106) by way of the following matrix product:

$$\begin{bmatrix} \dfrac{\partial f'}{\partial x'} & \dfrac{\partial f'}{\partial y'} & \dfrac{\partial f'}{\partial z'} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{bmatrix} \mathscr{J}$$

Taking transposes and remembering that $\mathscr{J}$ is orthogonal, we find

$$\begin{bmatrix} \dfrac{\partial f'}{\partial x'} \\[2ex] \dfrac{\partial f'}{\partial y'} \\[2ex] \dfrac{\partial f'}{\partial z'} \end{bmatrix} = \mathscr{J}^{-1} \begin{bmatrix} \dfrac{\partial f}{\partial x} \\[2ex] \dfrac{\partial f}{\partial y} \\[2ex] \dfrac{\partial f}{\partial z} \end{bmatrix} \tag{5.113}$$

But the right side of (5.113) is precisely what we obtain by applying the transformation rule (5.102) to the vector (5.105)! Hence, (5.105) and (5.106) describe the same vector field.

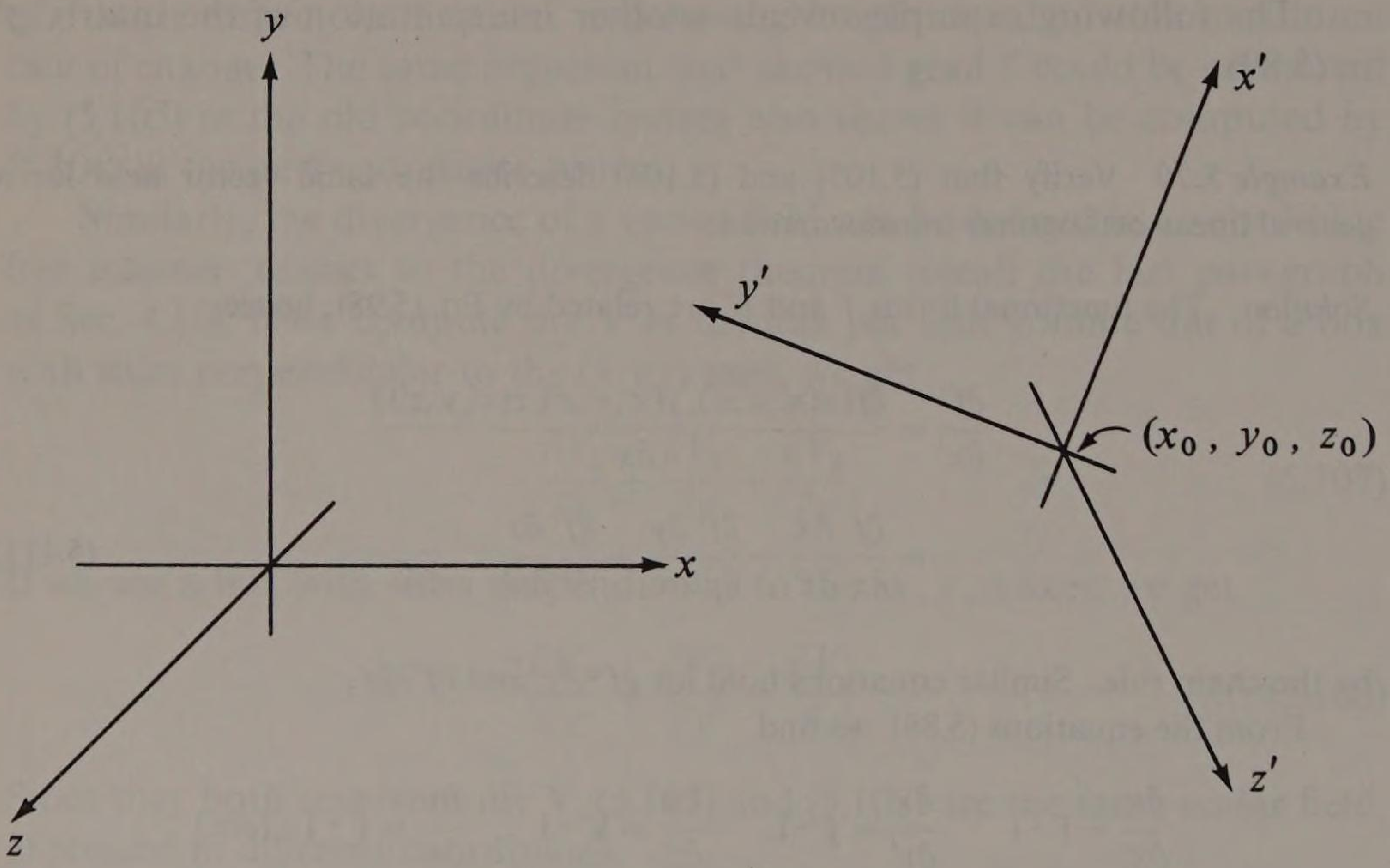

FIGURE 5.14

The analogous verifications of the "invariance" of the formulas for divergence and curl are left to the Exercises.

The fact that $\mathscr{J}$ is the matrix of partial derivatives is reflected by the following, alternative notation:

$$\mathscr{J} = \frac{\partial(x,y,z)}{\partial(x',y',z')}.$$

The terminology, "Jacobian matrix," is often used when one refers to the matrix of partial derivatives—hence the symbol "$\mathscr{J}$".

Finally we would like to extend the analysis of this section to a slightly more general type of transformation. Suppose that the origin of the new coordinate system does not coincide with the origin of the first; it is located at, say, the point with coordinates (x_0, y_0, z_0) in the old system (see Fig. 5.14). Then if a point P has old coordinates (x, y, z) and new coordinates (x', y', z'), the vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ in Fig. 5.15 are related by

$$\mathbf{R}_1 = \mathbf{R}_0 + \mathbf{R}_2$$

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}' \tag{5.114}$$

Taking scalar products of (5.114) with **i**, **j**, and **k** in turn lead to

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \mathscr{J} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{5-115}$$

with $\mathscr{J}$ defined as before, from Eq. (5.89). Thus, coordinates of a point do not transform by a linear orthogonal transformation in this case; Eq. (5.115) describes an *affine transformation.*

However, vectors still transform according to the old rules (5.95) and (5.96) because they are free; shifting the origin makes no difference to them.

Moreover, the vector operators **grad**, div, and **curl** retain their "invariant" formulas, again because they can be defined without reference to a coordinate

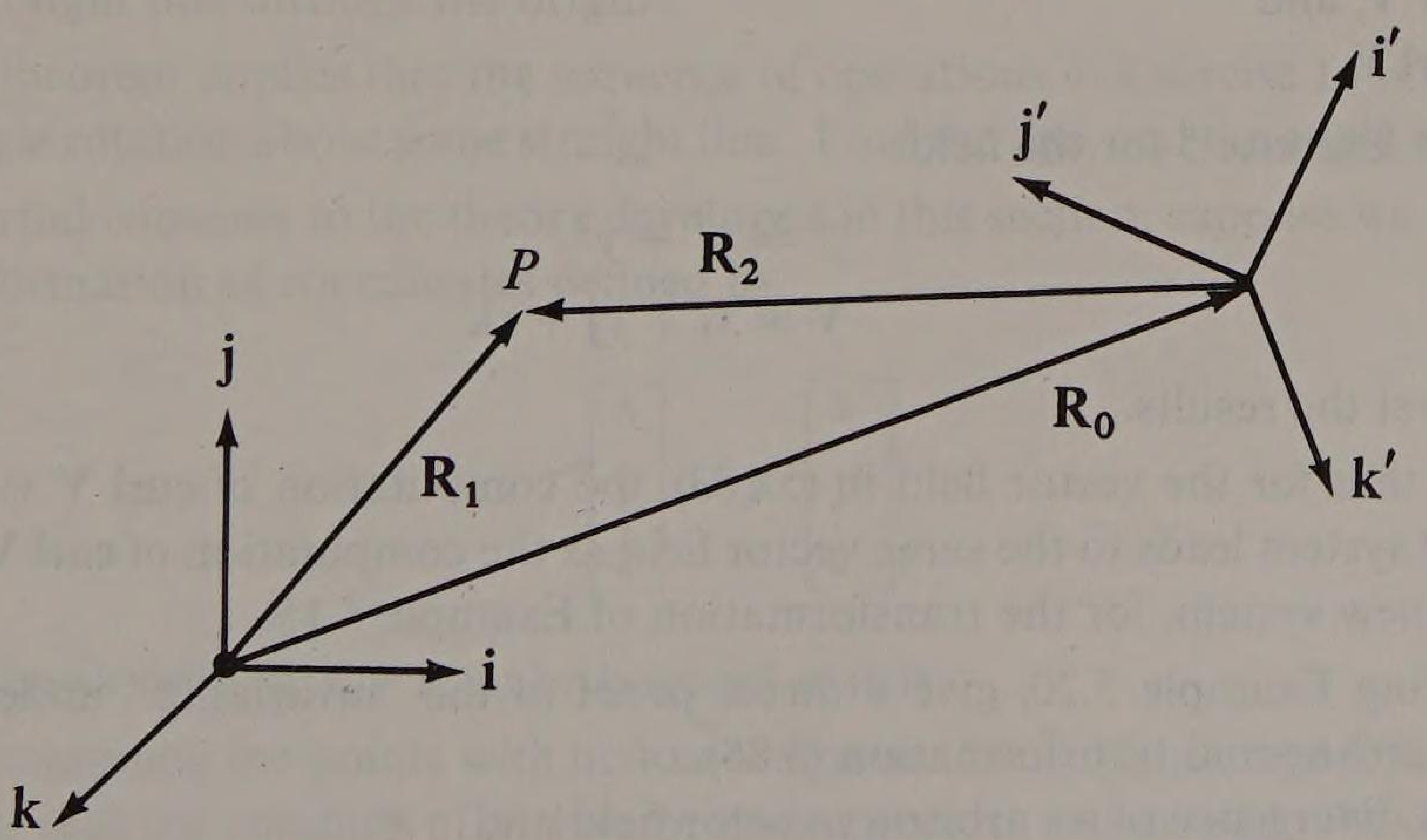

FIGURE 5.15

system. Even the direct proofs of their invariance, as exemplified by Example 5.20, can be carried out unchanged because the matrix of (5.89) is still the matrix of partial derivatives. The reader should review Example 5.20 to be sure this point is understood.

Consequently, most of the analysis we have developed for transformations between cartesian coordinate systems having the same origin holds for transformations between cartesian coordinate systems located at different origins. This includes the orthogonality of the matrix $\mathscr{J}$, the rules (5.96) for transforming vectors, the rules (5.98) and (5.102) for transforming scalar and vector fields, and the expressions (5.106), (5.108), and (5.110) for **grad**, div, and **curl**. The only difference is the appearance of the nonhomogeneous "shifts" in the equation (5.115) relating point coordinates.

Exercises

1. Show that Eq. (5.93) can be derived by taking scalar products of (5.87) with $\mathbf{i}'$, $\mathbf{j}'$, and $\mathbf{k}'$ in turn.
2. (a) Derive the matrix for the transformation generated by rotating the (x, y, z) system about the x axis through an angle ϕ.
   (b) Repeat part (a) if the (x, y, z) system is rotated about the y axis through an angle ψ.
3. Verify that the transpose of $\mathscr{J}$ in (5.90) equals its inverse.
4. If $\mathbf{V} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $\mathbf{W} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, compute $\mathbf{V}$ and $\mathbf{W}$ in the new coordinate system of Example 5.15. Verify that the scalar product $\mathbf{V} \cdot \mathbf{W}$ remains the same.
5. Consider the scalar and vector fields

$$f(x, y, z) = xyz$$
$$\mathbf{V}(x, y, z) = xz\mathbf{i} + \mathbf{j} + xyz\mathbf{k}$$

   If the coordinate transformation of Example 5.15 is performed with $\theta = \pi/6$, express the following fields in the new coordinate system:
   (a) the scalar field f
   (b) the vector field $\mathbf{V}$,
   (c) **grad** f,
   (d) div $\mathbf{V}$, and
   (e) **curl** $\mathbf{V}$.
6. Repeat Exercise 5 for the fields

$$f = x^2 + y^2$$
$$\mathbf{V} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

   Interpret the results.
7. Verify that for the vector field in (5.103), the computation of **curl V** via (5.109) in the old system leads to the same vector field as the computation of **curl V** via (5.110) in the new system, for the transformation of Example 5.15.
8. Modeling Example 5.20, give a direct proof of the "invariance" under a general linear orthogonal transformation (5.88), of
   (a) the divergence of an arbitrary vector field and
   (b) the curl of an arbitrary vector field.

9. Repeat Exercise 8 for a linear orthogonal transformation plus shift (5.115).

10. Is the laplacian of a scalar "invariant" under the transformations considered in this section? Consider your answer on the basis of
(a) laplacian = div **grad** and
(b) Exercise 11 following Sec. 4.10.

11. What is the element of arc length, $ds = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}}$, in the new coordinate system?

12. What is the volume element, $dV = dx\,dy\,dz$, in the new coordinate system?

13. Suppose a new x', y', z' coordinate system is related to the original system by a linear orthogonal transformation described by a matrix $\mathscr{J}$, as in (5.89), and a "still newer" x'', y'', z'' coordinate system is related to the "new" system by a linear orthogonal transformation generated by a matrix $\mathscr{K}$. Show that the x'', y'', z'' system is related to the original x, y, z system by a linear orthogonal transformation, with the associated matrix given by $(\mathscr{J}\mathscr{K})$. Prove directly that the product of two orthogonal matrices is orthogonal. What interpretation can you give to the columns of $(\mathscr{J}\mathscr{K})$?

14. Compute the matrix $\mathscr{J}$ associated with the following sequence of operations, taking the x, y, z axes into the corresponding x', y', z' axes:
(a) First rotate through an angle $\pi/4$ about the z axis.
(b) Then rotate through an angle $\pi/2$ about the "current" y axis.
(c) Finally rotate through an angle $(-\pi/4)$ about the "current" x axis.

15. Show directly that scalar products are preserved under the general linear orthogonal transformation (5.96); i.e., show that $V_1W_1 + V_2W_2 + V_3W_3 = V'_1W'_1 + V'_2W'_2 + V'_3W'_3$ when the components of **V** and **W** are related by (5.96). This provides verification of the (obvious) fact that lengths and angles are preserved under these transformations.

16. Show that under the linear orthogonal transformation (5.88) there is a straight line through the origin; all of whose points have the same coordinates before and after the transformation. That is, for all points on this line, $x = x'$, $y = y'$ and $z = z'$. [*Hint*: If **R** is the position vector of a point on the line, then $\mathbf{R} \cdot \mathbf{i} = \mathbf{R} \cdot \mathbf{i}'$; hence $\mathbf{R} \cdot (\mathbf{i} - \mathbf{i}') = 0$. Similarly, $\mathbf{R} \cdot (\mathbf{j} - \mathbf{j}') = \mathbf{R} \cdot (\mathbf{k} - \mathbf{k}') = 0$. If $\mathbf{i} = \mathbf{i}'$ or $\mathbf{j} = \mathbf{j}'$, there is nothing to prove, so try $\mathbf{R} = (\mathbf{i} - \mathbf{i}') \times (\mathbf{j} - \mathbf{j}')$.]

17. Based on the last two exercises, can you derive *Euler's theorem*: every transformation of the form (5.88) can be described as a rotation of the coordinate system about some straight line through the origin?

18. Euler's theorem implies that the sequence of operations in Exercise 14 is equivalent to a single rotation about some straight line. Find the line, and the angle of rotation.

19. As a partial converse to the theory developed in this section, suppose we *begin* with a transformation of coordinates defined by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathscr{J} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{5.116}$$

and we know only that $\mathscr{J}$ is an orthogonal matrix.
(a) By examining the points with new coordinates (1,0,0), (0,1,0), and (0,0,1) in turn, show that the columns of $\mathscr{J}$, interpreted as vectors expressed in the old system, point along the new x', y', and z' axes.

(b) Exploit the orthogonality of $\mathscr{J}$ to prove that the new axes are mutually orthogonal and that $(x'^2 + y'^2 + z'^2)^{\frac{1}{2}}$ equals the distance of (x', y', z') from the origin.

(c) From (a) and (b) we may conclude that the new system is a bonafide cartesian coordinate system. However, it may be left-handed, as the simple example

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

shows. What modifications of rules (5.96), (5.98), (5.102), (5.106), (5.108), and (5.110) have to be made when transforming to a left-handed system?

(d) How can you determine, from the matrix $\mathscr{J}$, whether or not the new system is right-handed?

20. Generalize Exercise 18: Given an arbitrary orthogonal matrix $\mathscr{J}$ that generates a transformation via (5.116) and given that the new system is right-handed, show how to compute the angle and axis of rotation in terms of $\mathscr{J}$.

21. Derive the matrix for the transformation generated by rotating the x, y, z system through an angle $\pi/2$ about the straight line through the origin parallel to $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

22. Generalize the previous exercise: Derive the matrix for a rotation of the x, y, z system through an angle θ about a straight line through the origin parallel to $\mathbf{n}$.

23. (*Significance of the Jacobian*). Obviously the transformation equations (5.33) for orthogonal coordinates are, in general, nonlinear. However, the relation

$$d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial u_1} du_1 + \frac{\partial \mathbf{R}}{\partial u_2} du_2 + \frac{\partial \mathbf{R}}{\partial u_3} du_3 \tag{5.117}$$

between the differentials can be viewed as a "local linearization" of Eq. (5.33).

(a) Show that (5.117) can be expressed

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial u_1} & \dfrac{\partial x}{\partial u_2} & \dfrac{\partial x}{\partial u_3} \\ \dfrac{\partial y}{\partial u_1} & \dfrac{\partial y}{\partial u_2} & \dfrac{\partial y}{\partial u_3} \\ \dfrac{\partial z}{\partial u_1} & \dfrac{\partial z}{\partial u_2} & \dfrac{\partial z}{\partial u_3} \end{bmatrix} \begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} \tag{5.118}$$

Recall that the matrix of partial derivatives in (5.118) was identified in this section as the Jacobian of the transformation (5.33). It is abbreviated

$$\mathscr{J} = \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)}$$

(b) Show that the chain rule implies that the inverse of the Jacobian is

$$\mathscr{J}^{-1} = \frac{\partial(u_1, u_2, u_3)}{\partial(x, y, z)} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x} & \dfrac{\partial u_1}{\partial y} & \dfrac{\partial u_1}{\partial z} \\ \dfrac{\partial u_2}{\partial x} & \dfrac{\partial u_2}{\partial y} & \dfrac{\partial u_2}{\partial z} \\ \dfrac{\partial u_3}{\partial x} & \dfrac{\partial u_3}{\partial y} & \dfrac{\partial u_3}{\partial z} \end{bmatrix} \tag{5.119}$$

(c) Show that the requirement of (u_1, u_2, u_3) forming orthogonal curvilinear coordinates forces the rows of $\mathscr{J}$ to be orthogonal. Nonetheless, $\mathscr{J}$ is not an orthogonal matrix. Why?

(d) Show that the determinant of $\mathscr{J}$ is $h_1 h_2 h_3$, the factor appearing in the volume element (5.50). This prompts the mnemonic

$$dx\,dy\,dz = \left|\frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)}\right| du_1\,du_2\,du_3$$

Review Problems

1. Vectors from the origin O to four points A, B, C, D are given as follows:

$$\mathbf{A} = 2\mathbf{i} \qquad \mathbf{B} = 3\mathbf{j} \qquad \mathbf{C} = 4\mathbf{k} \qquad \mathbf{D} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

   (a) Find the length of the perpendicular drawn from A to the plane BCD.
   (b) Find the length of the common perpendicular to the lines AB and CD.
   (c) Find a vector parallel to this perpendicular.

2. The vertices of a regular tetrahedron are $OABC$. Prove that the vector $OA + OB + OC$ is perpendicular to the plane ABC.

3. Find the angle which the plane OAB makes with the z axis, if A is the point (1,3,2) and B is (2,1,1).

4. Given the points $O(0,0,0)$, $A(1,2,3)$, $B(0,-1,1)$, $C(2,0,2)$.
   (a) Find a vector perpendicular to the plane OAB.
   (b) Find the distance from C to the plane OAB.

5. Determine the shortest distance from the point (3,4,5) to the line through the origin parallel to the vector $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

6. Write the scalar equations of the line parallel to the intersection of the planes $3x + y + z = 5$, $x - 2y + 3z = 1$ and passing through the point (4,2,1).

7. Given the points $P_1(2,-1,4)$, $P_2(-1,0,3)$, $P_3(4,3,1)$, and $P_4(3,-5,0)$, determine
   (a) the volume of the tetrahedron $P_1P_2P_3P_4$;
   (b) the equation of the plane containing the points P_1, P_2, and P_3;
   (c) the cosine of the angle between the line segments P_1P_2 and P_1P_3.

8. Write an expression for a vector 5 units long, parallel to the plane $3x + 4y + 5z = 10$ and perpendicular to the vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

9. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ be position vectors of the points $A(1,3,-2)$, $B(3,5,-3)$, $C(-5,9,-5)$, and $D(4,-1,10)$, respectively. Find
   (a) $|\mathbf{A} - \mathbf{D}|$
   (b) $\mathbf{A} \times \mathbf{B}$
   (c) $(\mathbf{A} - \mathbf{C}) \cdot (\mathbf{A} - \mathbf{B})$
   (d) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$.

10. Given the four points specified in Exercise 9, determine
   (a) the area of the triangle OAB;
   (b) the volume of the tetrahedron $OABC$;
   (c) the angle CAB.

11. By vector methods, prove that the angle subtended at the circumference by a diameter of a circle is a right angle.

12. Let PQR be a triangle. By vector methods, show there exists a triangle whose sides are parallel and equal in length to the medians of PQR.

13. (a) How many unit vectors make equal angles with the vectors $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 4\mathbf{k}$?
   (b) Find the unit vector $\mathbf{u}$ that bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

14. Write $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v})$ as a determinant involving only scalar products.

15. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, is it necessarily true that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$?

16. Given that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors having the same magnitude and $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$, what can you say about $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$?

17. Find the distance from the point $A(3,7,2)$ to the plane passing through $B(5,10,8)$ that is perpendicular to the line AB.

18. Find the distance from the origin to the plane through (3,2,6) that is perpendicular to the z axis.

19. Find the distance from the origin to the plane passing through (3,4,2) that is perpendicular to the line joining (1,2,3) and (3,5,9).

20. A plane has intercepts (4,0,0), (0,6,0) and (0,0,12). Find the equations of another plane through $(6,-2,4)$ that is parallel to this plane.

21. Show that the curve $x = t$, $y = 2t^2$, $z = t^3$ intersects the plane $x + 8y + 12z = 162$ at right angles.

22. Find the point on the sphere $x^2 + y^2 + z^2 = 84$ that is nearest the plane $x + 2y + 4z = 77$.

23. Find the point on the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$ that is nearest to the plane $x + 2y + 3z = 8$.

24. By vector methods, find the point on the curve $x = t$, $y = t^2$, $z = 2$ at which the temperature $\phi(x,y,z) = x^2 - 6x + y^2$ takes its minimum value.

25. What point on the curve $x = t$, $y = t^2$, $z = 2$ is closest to the surface $x^2 - 6x + y^2 + 7 = 0$?

26. At what angle does the curve $x = t$, $y = 2t - t^2$, $z = 2t^4$ intersect the surface $x^2 + y^3 + 3z^2 = 14$ at the point (1,1,2)?

27. The velocity field of a fluid is described by Fig. 3.6. A quantity of fluid occupies a spherical region centered at P at time $t = 0$. Describe the region occupied by the same particles a short time thereafter. Will the region be spherical?

28. Let $\mathbf{R} = R_1\mathbf{i} + R_2\mathbf{j} + R_3\mathbf{k}$ be a vector function of the time t, and let

$$\left.\frac{d\mathbf{R}}{dt}\right|_m = \frac{dR_1}{dt}\mathbf{i} + \frac{dR_2}{dt}\mathbf{j} + \frac{dR_3}{dt}\mathbf{k}$$

be the time rate of change of $\mathbf{R}$ computed on the assumption that $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ do not vary with time. Now suppose that $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ do vary with the time t, but only as a rigid system (they remain mutually perpendicular unit vectors). Show that, at any instant of time t, there exists a vector $\boldsymbol{\omega}$ such that the actual rate of change of R is

$$\frac{d\mathbf{R}}{dt} = \left.\frac{d\mathbf{R}}{dt}\right|_m + \boldsymbol{\omega} \times \mathbf{R}$$

(The letter m denotes *moving*; $(d\mathbf{R}/dt)|_m$ is the rate of change relative to the "moving frame" **i**,**j**,**k**.)

29. If $\mathbf{R}_1$ denotes the position vector of a point P relative to an origin O_1 in the xy plane and $\mathbf{R}_2$ denotes the position vector of the same point relative to another origin O_2, then $|\mathbf{R}_1| + |\mathbf{R}_2| = $ constant is the equation of an ellipse with foci O_1 and O_2. Use this observation to prove that lines O_1P and O_2P make equal angles with the tangent to the ellipse at P. [*Hint*: **grad** $(|\mathbf{R}_1| + |\mathbf{R}_2|)$ is normal to the ellipse.]

30. Find the angle between the surfaces $z = x^2 + y^2$ and $x^2 + y^2 + (z - 3)^2 = 9$ at the point $(2,-1,5)$.

31. Given $f(x,y,z) = 2x^2 + y$ and $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, find (a) ∇f, (b) $\nabla \cdot \mathbf{R}$ (c) $\nabla^2 f$, (d) $\nabla \times (f\mathbf{R})$.

32. If $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$, evaluate each of the following at the point $(-1,2,3)$: (a) $\nabla^2\mathbf{F}$, (b) $\nabla \times \mathbf{F}$, (c) $\nabla \cdot \mathbf{F}$.

33. Evaluate $\nabla^2[(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times \nabla(\mathbf{R} \cdot \mathbf{R})^2]$, where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

34. Evaluate $\nabla \ln(xyz - 5)$ at the point $(1,2,3)$.

35. Evaluate $\mathbf{A} \cdot \nabla\mathbf{R} + \nabla(\mathbf{A} \cdot \mathbf{R}) + \mathbf{A} \cdot \nabla \times \mathbf{R}$, where $\mathbf{A}$ is a constant vector field and $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

36. If $r^2 = x^2 + y^2 + z^2$, $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $\mathbf{A}$ is a constant vector field, find

(a) $\nabla \cdot (r^2\mathbf{A})$
(b) $\nabla \times (r^2\mathbf{A})$
(c) $\mathbf{R} \cdot \nabla(r^2\mathbf{A})$
(d) $\nabla(\mathbf{A} \cdot \mathbf{R})^4$
(e) $\nabla \cdot (r\mathbf{A})$
(f) $\mathbf{R} \cdot \nabla(\mathbf{A} \cdot \mathbf{R}\mathbf{A})$
(g) $\nabla \cdot (\mathbf{A} \times \mathbf{R})$
(h) $\nabla \times (\mathbf{A} \times \mathbf{R})$
(i) $\nabla^2(\mathbf{R} \cdot \mathbf{R})$

37. Consider the potential $\phi(x,y,z) = xyz$.
(a) Find a vector normal to the equipotential surface through the point $(1,2,3)$.
(b) Find $d\phi/ds$ at the same point, if s is measured in the direction of the vector $6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

38. Given $\phi(x,y,z) = z^2 - x - y$, determine
(a) an equation of the plane tangent to the surface $\phi = 2$ at the point $(-2,4,2)$;
(b) equations of the line normal to the surface $\phi = 2$ at the point $(-2,4,2)$;
(c) the derivative of ϕ at $(-2,4,2)$ in the direction of the vector $3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$.

39. What angle does the vector $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ make with the surface $xy - z^2 = 3$ at the point $(3,4,3)$?

40. If r is the distance from the origin to the point (x,y,z) and $\mathbf{A}$ is a constant vector, evaluate

$$\nabla\left(\mathbf{A} \cdot \nabla\frac{1}{r}\right) + \nabla \times \left(\mathbf{A} \times \nabla\frac{1}{r}\right)$$

41. For what value of the constant C is the vector field $\mathbf{V} = (x + 4y)\mathbf{i} + (y - 3z)\mathbf{j} + Cz\mathbf{k}$ the curl of some vector field $\mathbf{F}$?

42. Find **curl** $[f(r)\mathbf{R}]$ where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = |\mathbf{R}|$, and f is a differentiable function,
(a) by direct calculation,
(b) by geometrical interpretation.

43. Given Maxwell's equations in free space,

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}$$

show that **E** and **H** both satisfy the wave equation

$$\nabla^2 \mathbf{u} = \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

44. Given $\phi = \tan^{-1} x + \tan^{-1} y$ and $\psi = \dfrac{x+y}{1-xy}$, show that $\nabla\phi \times \nabla\psi = \mathbf{0}$.

45. Do the preceding exercise without explicitly calculating $\nabla\phi$ or $\nabla\psi$.

46. If $\nabla\phi \times \nabla\psi = \mathbf{0}$ throughout space, but neither $\nabla\phi$ nor $\nabla\psi$ is zero anywhere, what can you conclude about the isotimic surfaces $\phi =$ constant and $\psi =$ constant?

47. Given $w = uv$, where u and v are scalar fields, show that $\nabla w \cdot \nabla u \times \nabla v = 0$,
(a) by direct calculation,
(b) without calculation.

48. Generalize the result of the preceding exercise.

49. If **F** and **G** are conservative fields, is **F** × **G** necessarily conservative? If not, what *can* you say about **F** × **G**?

50. Find the surface integral $\iint \mathbf{F} \cdot d\mathbf{S}$ over the surface of the cylinder $x^2 + y^2 = 9$ included in the first octant between $z = 0$ and $z = 4$, given that $\mathbf{F} = y\mathbf{i} + (x+2)\mathbf{j} + (x^3 \sin yz)\mathbf{k}$. (By using the divergence theorem this can be reduced to a simple problem in arithmetic.)

51. By a symmetry argument, or otherwise, show that $\int (x^2 - y^2)\, ds = 0$, when the line integral is taken around a circle $x^2 + y^2 = a^2$ in the xy plane.

52. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ where $\mathbf{F} = y\mathbf{i} + (x - 2x^3 z)\mathbf{j} + xy^3\mathbf{k}$ and S is the surface of a sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane.

53. Use Green's theorem to derive the formula $A = \pi ab$ for the area of an ellipse. [*Hint*: If $\mathbf{F} = \frac{1}{2}(-y\mathbf{i} + x\mathbf{j})$, $\mathbf{curl}\ \mathbf{F} = \mathbf{k}$.]

54. Evaluate $\int_{(0,0)}^{(1,2)} (15x^4 - 3x^2y^2)\, dx - 2x^3 y\, dy$ along the path $2x^4 - 6xy^3 + 23y = 0$.

55. Surface and volume integrals of vector-valued functions are defined as for numerically valued functions. Alternatively, they can be defined by simply integrating the x, y, and z components (which are numerical) separately. Show formally that $\iiint_D \nabla\phi\, dV = \iint_S \phi\mathbf{n}\, dS$ by applying the divergence theorem to $\mathbf{F} = \phi\mathbf{C}$ where **C** is a constant vector field.

56. Similarly, derive the identity

$$\iiint_D \nabla \times \mathbf{A}\, dV = \iint_S \mathbf{n} \times \mathbf{A}\, dS$$

where **n** is the outer normal to S, the boundary of D.

57. Give a vector interpretation of each of the following. The notation is that used in Sec. 4.10.

(a) $\displaystyle \lim_{V \to 0} \frac{\iint_S \mathbf{n} f\, dS}{V}$

(b) $\displaystyle \lim_{V \to 0} \frac{\iint_S \mathbf{n} \times \mathbf{F}\, dS}{V}$

58. Parabolic cylindrical coordinates (u,v,z) are defined by $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$, $z = z$ where $-\infty < u < \infty$, $v \geq 0$, $-\infty < z < \infty$. In order to make use of the formulas in Sec. 5.2, it is necessary to know the scale factors h_u, h_v, and h_z. Determine these scale factors.

59. What is the element of volume in parabolic cylindrical coordinates?

60. (a) Write div **A** in parabolic cylindrical coordinates.
(b) Write Laplace's equation $\nabla^2\phi = 0$ in parabolic cylindrical coordinates.

APPENDIX A

Historical Notes

It is not really possible to appreciate the history of vector analysis without knowing something of the history of mathematics in general, and this is too broad a topic for us to discuss here. We shall confine our remarks to certain specific topics, and let the interested reader pursue the subject further elsewhere.

The word "vector" comes from a Latin word meaning "to carry" and is still sometimes used to mean "that which carries". For example, one says "the mosquito is the vector of malaria." The word entered mathematics via astronomy, where it was originally used with a somewhat different meaning. The notion of vector addition was arrived at independently by Möbius and others in the early part of the nineteenth century, thus giving rise to vector *algebra.* Vector *analysis* is somewhat more recent. For example, the notion of *curl* apparently was introduced by J. C. Maxwell in his *Treatise on Electricity and Magnetism* (1873).

The notation used in this book is essentially due to J. Willard Gibbs, whose book on vector analysis was printed privately in the early 1880's, and Oliver Heaviside, whose book on *Electromagnetic Theory* (1893) makes hilarious reading because of his jibes at mathematicians.

One of the most interesting events in the history of vector analysis is the controversy that once existed between exponents of vector analysis and a few other mathematicians who felt that *quaternions* were more suitable for solving problems in physics. Before proceeding, let us briefly discuss the algebra of quaternions. Quaternions are formally similar to complex numbers, so let us first consider the background of the idea of a complex number.

As long ago as 1545 a mathematician (Cardan) "solved" a problem in algebra that has no real solutions. The problem is to find two numbers whose sum is 10 and whose product is 40. Cardan gave a formal solution, involving the square root of a negative number, and verified by substitution that these "fictitious numbers" have the required properties. As early as 1629, Girard suggested that such "impossible solutions" should be considered for three reasons: one can give a general rule for finding roots of certain equations, these solutions supply the lack of other solutions, and they may in any event have their own usefulness. In 1673, Wallis pointed out that numbers such as $\sqrt{-1}$ should be just as legitimate in mathematics as negative numbers. One cannot have $\sqrt{-1}$ eggs in a basket, but then neither can one have -7 eggs in a basket. Wallis came very close to giving the usual geometrical interpretation of complex numbers. It remained for a Norwegian surveyor Wessel to do this in 1797. (Argand did it independently in 1806, which is why the term Argand diagram is used. Wessel published his work in an obscure journal and did not receive credit during his lifetime.)

It was not until 1831 that Gauss put complex numbers on a respectable basis. Since some readers of this book may have learned complex numbers from a viewpoint that predates 1831, let us briefly review complex numbers. A complex number is an ordered pair (x, y) of real numbers. They are added and multiplied by (real) scalars as if they were row matrices. However, the product $(x_1, y_1)(x_2, y_2)$ of two row matrices is not defined in matrix algebra. We define the product of two complex numbers according to

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1)$$

If we identify (1,0) with the real number 1, and let i denote (0,1), then

$$(x, y) = (x,0) + (0, y) = x(1,0) + y(0,1) = x + yi$$

which is the usual notation for a complex number (except for electrical engineers, who use j instead of i). Moreover, we have $i^2 = (0,1)(0,1) = (-1,0) = -1$, so it is now possible to square a number and obtain a negative number.

Now we recall that multiplication by $\cos\theta + i\sin\theta$ has the effect of rotating a complex number through an angle θ. Hence, rotations in a plane can be obtained by identifying the plane with the Argand diagram and the rotation with the operation of multiplying by $\cos\theta + i\sin\theta$. This suggested to W. R. Hamilton that rotations in space might be similarly obtained, if there were some way to multiply *triples* of numbers to obtain a system of "hypercomplex" numbers that would provide a three-dimensional analog of the complex number system. Apparently this problem troubled him for a period of fifteen years. This is not too surprising when one considers that, up to the time of Hamilton, it was generally assumed that the commutative law $xy = yx$ was a necessary condition for the consistency of the rules of algebra. Hamilton is credited with the realization that this is not the case; actually, Gauss had the same idea earlier but did not publish his work.

The realization that a noncommutative algebra is needed is still not enough. Hamilton was still trying to do the impossible. It was proved later, by Frobenius in 1878, that it is impossible to multiply ordered triples in such a manner that the resulting algebraic system will have all of the properties Hamilton desired. Evidently Hamilton suspected this himself. It was on a famous day, October 16, 1843, when he was out walking with his wife, that Hamilton, in a great flash of insight, conceived of the quaternions. It is said that he carved the fundamental formulas of this new algebra in the stone of Brougham Bridge, on which he happened to be at the moment. He immediately recognized the importance of his discovery (some might say invention) and devoted the remainder of his life to quaternions.

At first glance, a quaternion looks like a cross between a complex number and a vector. The usual form for writing a quaternion is

$$x = x_0 + x_1 i + x_2 j + x_3 k$$

Quaternions are added and multiplied by real numbers in the obvious manner. The product of two quaternions is defined by formally multiplying them out according to the usual rules of algebra (except that we must be careful to preserve the order) and then simplifying the resulting expression by using the following rules:

$$\begin{array}{lll} i^2 = -1 & j^2 = -1 & k^2 = -1 \\ ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

An example will illustrate the procedure. If, for instance, $x = 3 - i + 2j + k$ and $y = 3j - 2k$, we have

$$\begin{aligned} xy &= (3 - i + 2j + k)(3j - 2k) \\ &= 9j - 6k - 3ij + 2ik + 6j^2 - 4jk + 3kj - 2k^2 \\ &= 9j - 6k - 3k - 2j - 6 - 4i - 3i + 2 \\ &= -4 - 7i + 7j - 9k \end{aligned}$$

It can be shown that the quaternions constitute a division algebra. That is, to each quaternion $x \neq 0$, there is a quaternion x^{-1} such that $xx^{-1} = x^{-1}x = 1$. We shall not digress to show how an inverse is computed. It is important to note, however, that we cannot write y/x, since this would be ambiguous. We must write either $x^{-1}y$ or yx^{-1} and, since multiplication of quaternions is not commutative, these two expressions may not be equal.

The *real part* of a quaternion $x_0 + x_1 i + x_2 j + x_3 k$ is the number x_0. If the real part of a quaternion is zero, the quaternion is called a *pure quaternion.* In applying quaternions to problems in physics or geometry, pure quaternions are identified with ordinary vectors in three-dimensional space, as the notation suggests.

If x and y are pure quaternions, the real part of xy turns out to be the negative of the scalar product $x \cdot y$ (computed by the usual formula), and the pure quaternionic part represents the vector product $x \times y$. Thus it is possible to do with quaternions many of the things one ordinarily does in vector analysis by using scalar and vector products.

Although the quaternions comprise a four-dimensional division algebra, rather than a three-dimensional one, it turned out that quaternions fulfilled the needs envisaged by Hamilton. It is possible to represent rotations by the use of quaternions, although not so simply as one might have wished, and in general there is a certain awkwardness in the use of quaternions. After working for ten years, Hamilton published his *Lectures on Quaternions* (1853); his *Elements of Quaternions* appeared in 1866, the year after his death. Incidentally, the earliest use of the word *vector* (in the mathematical sense), according to the Oxford dictionary, is in this work.

Hamilton had one devoted disciple, P. G. Tait, who mastered all the tricks of quaternions, and devoted himself to the cause of convincing one and all that quaternions were the ultimate tool for geometers and physicists. There were others who disagreed.

At about the same time that Hamilton made his remarkable discovery, H. G. Grassmann published a work called the *Ausdehnungslehre*, or the *Theory of Extension*. In this remarkable book, both matrix theory and tensor algebra are developed implicitly, but because he filled the book with philosophical abstractions, and because of its difficulty, the book was essentially ignored by mathematicians. A second edition was published in 1862, but the work was not much appreciated until the twentieth century.

The vector analysis of Gibbs and Heaviside, and the various generalities in this chapter, are more closely related to the *Ausdehnungslehre* than to anything Hamilton did. Grassmann introduced various types of "products" of vectors, and set things up for Gibbs to invent dyadics, and discussed linear transformations in general. The notion of a linear associative algebra was developed by Benjamin Peirce in the 1860's. The only other name we shall mention is that of Cayley, who was eminent for (among other things) conceiving of n-dimensional space (as did Grassmann) and who published a *Memoir on the Theory of Matrices* in 1858.

A delightful controversy took place between Gibbs and Tait concerning the merits of the use of quaternions in solving problems in geometry and physics. There is a certain beauty and mathematical elegance in the quaternions, but they are not very well adapted to practical use. Tait viewed vector analysis as a "hermaphroditic monster" and did not hesitate to express this view in print. The replies of Gibbs can be found in his collected works, available in any library, and they are both entertaining and instructive to read. By the beginning of the twentieth century, vector analysis was well established, and it was amply demonstrated that Hamilton and Tait were overly optimistic in their thought that quaternions would be as revolutionary to mathematics as was the invention of calculus. The revolutionary idea contributed by Hamilton was simply that it is possible to have a self-consistent algebra in which multiplication is not commutative.

Exercises

1. Show that, if u and v are pure quaternions,

$$uv = -\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\times\mathbf{v}$$

2. Show that if the vectors **u** and **v** are identified with pure quaternions,

$$\mathbf{u}\cdot\mathbf{v} = -\frac{uv + vu}{2}$$

and

$$\mathbf{u}\times\mathbf{v} = \frac{uv - vu}{2}$$

3. (a) Let **n** denote a unit vector that is perpendicular to a plane P. Thinking of P as a mirror, show that the reflected image of a vector **v** in the mirror is given by

$$\mathbf{v}' = \mathbf{v} - 2(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$$

(b) Show that this can be written in quaternionic form as

$$v' = nvn$$

4. Let P and P' be two planes intersecting in a line L and let $\theta/2$ be the angle between the two planes. Choose unit normals **n** and **n**′ respectively so that the angle between **n** and **n**′ is $\theta/2$ and let **u** be a unit vector along L so chosen that $\mathbf{n}\times\mathbf{n}' = \sin(\theta/2)\mathbf{u}$.

(a) Letting **v**′ denote the reflected image of **v** in P and **v**″ denote the reflected image of **v**′ in P', show that **v**″ is the vector obtained by rotating **v** through an angle θ about the axis L. (The positive sense of rotation is related to the direction of **u** by the right-hand rule.)

(b) Derive the quaternionic relation

$$v'' = n'nvnn' \tag{A.1}$$

(c) Writing (A.1) in the form $v'' = (-n'n)v(-nn')$ derive the relation

$$v'' = (\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta\, u)v(\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta\, u) \tag{A.2}$$

5. If z is a complex number, the exponential e^z is defined by the infinite series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Using the same expression to define e^z when z is a quaternion, let $z = \phi u$ where u is a pure quaternion representing a unit vector and ϕ is an angle, and derive

$$e^{\phi u} = -\cos\phi + \sin\phi\, u$$

6. Rewrite (A.2) in exponential notation. (This is the formula for rotations that Hamilton was seeking when he developed the algebra of quaternions.)

APPENDIX **B**

Two Theorems of Advanced Calculus

In this appendix we prove two theorems of advanced calculus which are important to vector analysis.

THEOREM B.1 *Let $f(x,y,z)$ be a scalar function possessing continuous first partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ in some domain D. Also let* **h** *be a unit vector with components (h_1,h_2,h_3). Then the directional derivative of f in the direction* **h** *exists in D and is given by*

$$\frac{df}{ds} = \frac{\partial f}{\partial x} h_1 + \frac{\partial f}{\partial y} h_2 + \frac{\partial f}{\partial z} h_3$$

Proof The directional derivative, if its exists, is given by the limit of

$$\frac{f(x + sh_1, y + sh_2, z + sh_3) - f(x,y,z)}{s} \tag{B.1}$$

as s approaches zero.

By adding and subtracting equal terms, we rewrite (B.1) as

$$\frac{f(x + sh_1, y + sh_2, z + sh_3) - f(x + sh_1, y + sh_2, z)}{s}$$

$$+ \frac{f(x + sh_1, y + sh_2, z) - f(x + sh_1, y, z)}{s}$$

$$+ \frac{f(x + sh_1, y, z) - f(x,y,z)}{s} \tag{B.2}$$

Each of the terms of (B.2) involves differences of values of f when *only one* coordinate is changed. Hence we can use the powerful tools of the ordinary calculus of one variable; in particular, the mean value theorem applies to each term (since the partial derivatives are continuous). For the first term in (B.2), we conclude that there is a number α between 0 and 1 such that

$$\begin{aligned} f(x + sh_1, y + sh_2, z + sh_3) - f(x + sh_1, y + sh_2, z) \\ = sh_3 \frac{\partial f}{\partial z}(x + sh_1, y + sh_2, z + \alpha sh_3) \end{aligned}$$

Analyzing the other terms similarly, we find numbers β and γ also between 0 and 1 such that the expression (B.1) is equal to

$$\begin{aligned} &\frac{sh_3 \dfrac{\partial f}{\partial z}(x + sh_1, y + sh_2, z + \alpha sh_3)}{s} \\ &\qquad + \frac{sh_2 \dfrac{\partial f}{\partial y}(x + sh_1, y + \beta sh_2, z)}{s} \\ &\qquad + \frac{sh_1 \dfrac{\partial f}{\partial x}(x + \gamma sh_1, y, z)}{s} \end{aligned} \tag{B.3}$$

Now we let s approach zero. The numbers α, β, and γ are always between 0 and 1, and since the partials are continuous, we conclude that the limit exists and is given by

$$\frac{\partial f}{\partial x} h_1 + \frac{\partial f}{\partial y} h_2 + \frac{\partial f}{\partial z} h_3$$

(Q.E.D)

In Theorem B.2 the notation is rather confusing. We advise the reader that the following symbols mean the same thing:

$$\partial^2 f/\partial y\, \partial x = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

THEOREM B.2 *Let $f(x,y)$ be a scalar function possessing continuous first partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ in some domain D. Furthermore let the second derivative $\partial^2 f/\partial y\, \partial x$ exist and be continuous in D. Then the second derivative $\partial^2 f/\partial x\, \partial y$ also exists in D and*

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}$$

Proof The second derivative, $\partial^2 f/\partial x\,\partial y$, if it exists, is the limit as s goes to zero of

$$\frac{\dfrac{\partial f}{\partial y}(x+s,y)-\dfrac{\partial f}{\partial y}(x,y)}{s} \tag{B.4}$$

On the other hand, the y derivatives in (B.4) can also be expressed as limits; (B.4) is equal to

$$\frac{1}{s}\left(\lim_{t\to 0}\frac{f(x+s,y+t)-f(x+s,y)}{t}-\lim_{t\to 0}\frac{f(x,y+t)-f(x,y)}{t}\right) \tag{B.5}$$

Since limits and sums are interchangeable, we can write (B.5) as

$$\lim_{t\to 0}\frac{[f(x+s,y+t)-f(x+s,y)]-[f(x,y+t)-f(x,y)]}{st} \tag{B.6}$$

The numerator in (B.6) can be regarded as the difference between the values of a function $F(u)$ evaluated at $u = x + s$ and $u = s$; this complicated function $F(u)$ is defined by

$$F(u) = f(u, y+t) - f(u, y)$$

As a function of u, F has a continuous derivative given by

$$F'(u) = \frac{\partial f}{\partial u}(u, y+t) - \frac{\partial f}{\partial u}(u, y)$$

and is thus vulnerable to the mean value theorem of one-dimensional calculus. We conclude that there is a number α between 0 and 1 such that

$$F(x+s) - F(x) = sF'(x+\alpha s)$$

Thus the expression (B.6) is equal to

$$\lim_{t\to 0}\frac{s\left(\dfrac{\partial f}{\partial x}(x+\alpha s, y+t)-\dfrac{\partial f}{\partial x}(x+\alpha s, y)\right)}{st} \tag{B.7}$$

We can now apply the mean value theorem to the function $\partial f/\partial x$ in (B.7), since only the second argument is changing; by hypothesis, $\partial f/\partial x$ has a continuous derivative with respect to its second argument, namely, $\partial^2 f/\partial y\,\partial x$! We conclude that there is a number β between 0 and 1 such that expression (B.7) is equal to

$$\lim_{t\to 0}\frac{st\dfrac{\partial^2 f}{\partial y\,\partial x}(x+\alpha s, y+\beta t)}{st} \tag{B.8}$$

Since $\partial^2 f/\partial y\,\partial x$ is continuous, the expression (B.8) equals

$$\frac{\partial^2 f}{\partial y\,\partial x}(x + \alpha s, y) \tag{B.9}$$

We have shown that (B.4) equals (B.9). Now taking limits as s goes to zero, again invoking the continuity of $\partial^2 f/\partial y\,\partial x$, we see that (B.4) does indeed have a limit, and that it is given by

$$\frac{\partial^2 f}{\partial y\,\partial x}(x, y)$$

(Q.E.D.)

APPENDIX **C**

The Vector Equations of Classical Mechanics

C.1 MECHANICS OF PARTICLES AND SYSTEMS OF PARTICLES

The basic equation of classical mechanics is expressed by Newton's second law, *force equals mass times acceleration.* More explicitly, if a particle, or a "point mass," is located at the position $\mathbf{r}$ in an inertial coordinate system, and if it has mass m, then its motion will be governed by the equation

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \tag{C.1}$$

Here $\mathbf{F}$ is the *vector* sum of all the forces acting on the particle, and $\mathbf{v}$ and $\mathbf{a}$ are velocity and acceleration, respectively (t, of course, is time). All of the equations in this section will be derived from (C.1).

The *momentum* $\mathbf{p}$ of the particle is defined by

$$\mathbf{p} = m\mathbf{v} = m\frac{d\mathbf{r}}{dt} \tag{C.2}$$

Its *angular momentum* ℓ is defined by

$$\ell = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v} \tag{C.3}$$

The *torque* $\mathbf{T}$ of a force $\mathbf{F}$ exerted at the point $\mathbf{r}$ is defined by

$$\mathbf{T} = \mathbf{r} \times \mathbf{F} \tag{C.4}$$

Combining these equations, we find that *the torque on a particle equals its rate of change of angular momentum*:

$$\frac{d\boldsymbol{\ell}}{dt} = \frac{d}{dt}\left(\mathbf{r} \times m\frac{d\mathbf{r}}{dt}\right) = \frac{d\mathbf{r}}{dt} \times m\frac{d\mathbf{r}}{dt} + \mathbf{r} \times m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{r} \times \mathbf{F} = \mathbf{T} \tag{C.5}$$

The *kinetic energy* $\mathscr{K}$ of the particle is defined by

$$\mathscr{K} = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}m\left|\frac{d\mathbf{r}}{dt}\right|^2 = \frac{|\mathbf{p}|^2}{2m} \tag{C.6}$$

It obeys the equation

$$\frac{d\mathscr{K}}{dt} = m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = m\mathbf{a} \cdot \mathbf{v} = \mathbf{F} \cdot \mathbf{v} \tag{C.7}$$

The quantity $\mathbf{F} \cdot \mathbf{v}$ is called the *power* delivered by the force $\mathbf{F}$, and it equals the rate of change of kinetic energy.

If a particular force acting on the particle can be expressed as a function of the position $\mathbf{r}$, so that $\mathbf{F}$ is a vector field, it is natural to ask if this vector field is conservative (Sec. 4.3). If so, we introduce the *potential energy*, $\mathscr{V}(\mathbf{r})$, satisfying

$$\mathbf{F}^{(\text{cons})}(\mathbf{r}) = -\nabla\mathscr{V}(\mathbf{r}) \tag{C.8}$$

(Notice that a minus sign is incorporated in the definition so that $\mathscr{V}(\mathbf{r}) = -\phi(\mathbf{r})$, as in Sec. 4.3). The total force on the particle is then expressed as the sum of the conservative and the non-conservative forces

$$\begin{aligned}\mathbf{F} &= \sum_{\alpha} \mathbf{F}_{\alpha}^{(\text{cons})} + \mathbf{F}^{(\text{non-cons})} \\ &= -\sum_{\alpha} \nabla\mathscr{V}_{\alpha}(\mathbf{r}) + \mathbf{F}^{(\text{non-cons})} \\ &= -\nabla\mathscr{V}(\mathbf{r}) + \mathbf{F}^{(\text{non-cons})}\end{aligned}$$

where $\mathscr{V} = \sum_{\alpha} \mathscr{V}_{\alpha}$ is the total potential energy. Observe that the power generated by the conservative forces can be written as the time derivative of $-\mathscr{V}(\mathbf{r})$ taken along the trajectory $\mathbf{r}(t)$:

$$\begin{aligned}\sum_{\alpha} \mathbf{F}_{\alpha}^{(\text{cons})} \cdot \mathbf{v} &= -\nabla\mathscr{V}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{dt} \\ &= -\frac{\partial\mathscr{V}}{\partial x}\frac{dx}{dt} - \frac{\partial\mathscr{V}}{\partial y}\frac{dy}{dt} - \frac{\partial\mathscr{V}}{\partial z}\frac{dz}{dt} \\ &= -\frac{d}{dt}\mathscr{V}(\mathbf{r})\end{aligned}$$

Consequently, (C.7) can be expressed

$$\frac{d\mathscr{K}}{dt} = -\frac{d\mathscr{V}(\mathbf{r}(t))}{dt} + \mathbf{F}^{(\text{non-cons})} \cdot \mathbf{v}$$

or introducing the *total energy* $\mathscr{K} + \mathscr{V} = \mathscr{E}$,

$$\frac{d\mathscr{E}}{dt} = \mathbf{F}^{(\text{non-cons})} \cdot \mathbf{v} \tag{C.9}$$

In the important case when all of the forces are conservative, we see that the total energy of the particle is constant along the trajectory.

Now let us derive some of the basic laws governing the motion of a *system* of N particles located at the positions $\mathbf{r}_\alpha$ and having masses m_α ($\alpha = 1, 2, \ldots, N$); the total force on the αth particle will be denoted $\mathbf{F}_\alpha$. We introduce the total mass $M = \sum_\alpha m_\alpha$, the total momentum $\mathbf{P} = \sum_\alpha \mathbf{p}_\alpha$, the total angular momentum $\mathbf{L} = \sum_\alpha \boldsymbol{\ell}_\alpha$, and the *position vector* $\mathbf{R}_{\text{cm}}$ *of the center of mass*, defined by

$$\mathbf{R}_{\text{cm}} = \frac{\sum_\alpha m_\alpha \mathbf{r}_\alpha}{M} \tag{C.10}$$

Several useful theorems emerge from these definitions.

THEOREM C.1 *The total momentum* $\mathbf{P}$ *of the system equals the momentum of a single particle of mass* M *moving with the center of mass.*

Proof

$$M\frac{d\mathbf{R}_{\text{cm}}}{dt} = \sum_\alpha m_\alpha \frac{d\mathbf{r}_\alpha}{dt} = \mathbf{P} \tag{C.11}$$

THEOREM C.2 *The motion of the center of mass* $\mathbf{R}_{\text{cm}}$ *is the same as that of a single particle of mass* M *subjected to all the forces on all the particles simultaneously.*

Proof

$$M\frac{d^2\mathbf{R}_{\text{cm}}}{dt^2} = \sum_\alpha m_\alpha \frac{d^2\mathbf{r}_\alpha}{dt^2} = \sum_\alpha \mathbf{F}_\alpha$$

It is sometimes convenient to refer to certain vector quantities (position, velocity) *measured relative to the center of mass*, or measured "in the center of mass system." The center of mass system (hereafter known as the "C-M system") is a moving coordinate system whose origin is located at $\mathbf{R}_{\text{cm}}$ and whose axes remain parallel to the corresponding axes of the inertial system. Clearly, the position vector of the αth particle in the C-M system is $\mathbf{r}_\alpha - \mathbf{R}_{\text{cm}}$; and its velocity in the C-M system, *defined as the derivative of this vector*, is the difference of the inertial velocities:

$$\frac{d}{dt}(\mathbf{r}_\alpha - \mathbf{R}_{\text{cm}}) = \frac{d\mathbf{r}_\alpha}{dt} - \frac{d\mathbf{R}_{\text{cm}}}{dt} = \mathbf{v}_\alpha - \frac{d\mathbf{R}_{\text{cm}}}{dt}$$

Its angular momentum is

$$\ell_\alpha^{\text{c-m}} = (\mathbf{r}_\alpha - \mathbf{R}_{\text{cm}}) \times m_\alpha \left(\mathbf{v}_\alpha - \frac{d\mathbf{R}_{\text{cm}}}{dt} \right)$$

THEOREM C.3 *The total angular momentum of the system equals the sum of the angular momenta of the individual particles in the C-M system, plus the angular momentum of a single particle of mass M moving with the center of mass.*

Proof We have to show that

$$\mathbf{L} \equiv \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \frac{d\mathbf{r}_\alpha}{dt} = \sum_\alpha (\mathbf{r}_\alpha - \mathbf{R}_{\text{cm}}) \times m_\alpha \left(\frac{d\mathbf{r}_\alpha}{dt} - \frac{d\mathbf{R}_{\text{cm}}}{dt} \right) + \mathbf{R}_{\text{cm}} \times M \frac{d\mathbf{R}_{\text{cm}}}{dt} \tag{C.12}$$

The *sum* on the right equals

$$\sum_\alpha \mathbf{r}_\alpha \times m_\alpha \frac{d\mathbf{r}_\alpha}{dt} - \mathbf{R}_{\text{cm}} \times \left(\sum_\alpha m_\alpha \frac{d\mathbf{r}_\alpha}{dt} \right) - \left(\sum_\alpha m_\alpha \mathbf{r}_\alpha \right) \times \frac{d\mathbf{R}_{\text{cm}}}{dt} + \mathbf{R}_{\text{cm}} \times \sum_\alpha m_\alpha \frac{d\mathbf{R}_{\text{cm}}}{dt}$$

Using (C.10), we express this as

$$\mathbf{L} - \mathbf{R}_{\text{cm}} \times M \frac{d\mathbf{R}_{\text{cm}}}{dt} - M\mathbf{R}_{\text{cm}} \times \frac{d\mathbf{R}_{\text{cm}}}{dt} + \mathbf{R}_{\text{cm}} \times M \frac{d\mathbf{R}_{\text{cm}}}{dt} = \mathbf{L} - \mathbf{R}_{\text{cm}} \times M \frac{d\mathbf{R}_{\text{cm}}}{dt}$$

and inserting this back into (C.12) verifies the identity.

We invite the reader to supply the proof of the following similar result.

THEOREM C.4 *The total kinetic energy of the system equals the sum of the kinetic energies of the individual particles in the C-M system, plus the kinetic energy of a single particle of mass M moving with the center of mass:*

$$\sum_\alpha \frac{1}{2} m_\alpha |\mathbf{v}_\alpha|^2 = \sum_\alpha \frac{1}{2} m_\alpha \left| \mathbf{v}_\alpha - \frac{d\mathbf{R}_{\text{cm}}}{dt} \right|^2 + \frac{1}{2} M \left| \frac{d\mathbf{R}_{\text{cm}}}{dt} \right|^2$$

Usually it is convenient to separate the forces on the αth particle into two categories, namely, *internal* and *external* forces. The internal forces are produced by interaction with the other particles, while the external forces are produced outside the system. The advantages of this separation accrue from the following facts:

(*i*) Most internal forces between particles are *two-particle interactions*, so that if the force on the αth particle due to the βth particle is $\mathbf{F}_\alpha^{(\beta)}$, the total internal force on the αth particle is

$$\mathbf{F}_\alpha^{\text{int}} = \sum_{\beta \neq \alpha} \mathbf{F}_\alpha^{(\beta)}$$

(*ii*) Most two-particle interactions obey Newton's third law, *every action is accompanied by an equal and opposite reaction*, which is interpreted to mean that

$$\mathbf{F}_\alpha^{(\beta)} = -\mathbf{F}_\beta^{(\alpha)} \tag{C.13}$$

and that both $\mathbf{F}_\alpha^{(\beta)}$ and $\mathbf{F}_\beta^{(\alpha)}$ are directed along the line connecting particles α and β:

$$(\mathbf{r}_\beta - \mathbf{r}_\alpha) \times \mathbf{F}_\alpha^{(\beta)} = (\mathbf{r}_\beta - \mathbf{r}_\alpha) \times \mathbf{F}_\beta^{(\alpha)} = 0 \tag{C.14}$$

Systems having these properties will be said to "satisfy NTL." (Observe that if a system has some internal forces that violate NTL, one can use the artifice of categorizing these as *external* forces and interpreting the theorems accordingly).

Clearly, if a system satisfies NTL, then one can omit the internal forces from the sum in Theorem C.2, because they cancel pairwise. A more profound result is the following:

THEOREM C.5 *If a system satisfies NTL, then the rate of change of its total angular momentum measured in the C-M system equals the sum of the torques of the external forces, measured relative to the center of mass.*

Proof The theorem states that

$$\frac{d\mathbf{L}^{\text{c-m}}}{dt} \equiv \frac{d}{dt} \sum_\alpha (\mathbf{r}_\alpha - \mathbf{R}_{\text{cm}}) \times m_\alpha \left(\mathbf{v}_\alpha - \frac{d\mathbf{R}_{\text{cm}}}{dt} \right) = \sum_\alpha (\mathbf{r}_\alpha - \mathbf{R}_{\text{cm}}) \times \mathbf{F}_\alpha^{\text{ext}} \tag{C.15}$$

where the force on the αth particle satisfies

$$\begin{aligned} \mathbf{F}_\alpha &= \mathbf{F}_\alpha^{\text{ext}} + \mathbf{F}_\alpha^{\text{int}} \\ &= \mathbf{F}_\alpha^{\text{ext}} + \sum_{\beta \neq \alpha} \mathbf{F}_\alpha^{(\beta)} \end{aligned}$$

To derive this, we sum (C.5) over all particles:

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt} \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \frac{d\mathbf{r}_\alpha}{dt} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha \tag{C.16}$$

By (C.12), the left-hand side can be written

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{L}^{\text{c-m}}}{dt} + \frac{d}{dt}\left(\mathbf{R}_{\text{cm}} \times M\frac{d\mathbf{R}_{\text{cm}}}{dt}\right) = \frac{d\mathbf{L}^{\text{c-m}}}{dt} + \mathbf{R}_{\text{cm}} \times M\frac{d^2\mathbf{R}_{\text{cm}}}{dt^2}$$

and Theorem C.2 for NTL systems reduces this to

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{L}^{\text{c-m}}}{dt} + \mathbf{R}_{\text{cm}} \times \sum_{\alpha} \mathbf{F}_{\alpha}^{\text{ext}} \tag{C.17}$$

The right-handed side of (C.16) can also be reduced:

$$\begin{aligned}\sum_{\alpha} \mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha} &= \sum_{\alpha} \mathbf{r}_{\alpha} \times (\mathbf{F}_{\alpha}^{\text{ext}} + \mathbf{F}_{\alpha}^{\text{int}}) \\ &= \sum_{\alpha} \mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}^{\text{ext}} + \sum_{\alpha}\sum_{\beta\neq\alpha} \mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}^{(\beta)}\end{aligned}$$

The terms in the second sum cancel pairwise by (C.13) and (C.14):

$$\mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}^{(\beta)} + \mathbf{r}_{\beta} \times \mathbf{F}_{\beta}^{(\alpha)} = (\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}) \times \mathbf{F}_{\alpha}^{(\beta)} = \mathbf{0}$$

while the other sum splits into

$$\sum_{\alpha} (\mathbf{r}_{\alpha} - \mathbf{R}_{\text{cm}}) \times \mathbf{F}_{\alpha}^{\text{ext}} + \mathbf{R}_{\text{cm}} \times \sum_{\alpha} \mathbf{F}_{\alpha}^{\text{ext}} \tag{C.18}$$

Equating (C.17) and (C.18) and dropping the common term, we arrive at (C.15).

Theorems C.2 and C.5 are very useful for NTL systems with many particles and complicated internal forces. They frequently permit a general overall description of the motion (position of C-M, and angular momentum about C-M) in terms of the external forces only. As we shall see, these data provide a complete characterization of the motion of rigid bodies.

The concept of potential energy can be applied to *systems* as well as to single particle motions. If any of the forces on the αth particle are functions of $\mathbf{r}_{\alpha}$ only, and if the field $\mathbf{F}_{\alpha}(\mathbf{r}_{\alpha})$ is conservative, we introduce the corresponding potential energy $\mathscr{V}_{\alpha}(\mathbf{r}_{\alpha})$ as before:

$$\mathbf{F}_{\alpha}^{(\text{cons})}(\mathbf{r}_{\alpha}) = -\nabla^{(\alpha)}\mathscr{V}_{\alpha}(\mathbf{r}_{\alpha})$$

with, obviously,

$$\nabla^{(\alpha)} = \mathbf{i}\frac{\partial}{\partial x_{\alpha}} + \mathbf{j}\frac{\partial}{\partial y_{\alpha}} + \mathbf{k}\frac{\partial}{\partial z_{\alpha}}$$

The two-particle interaction forces, however, generally depend on the coordinates of *both* particles,

$$\mathbf{F}_1^{(2)} = \mathbf{F}_1^{(2)}(\mathbf{r}_1,\mathbf{r}_2)$$

and even worse situations are imaginable.

To get a convenient theory for systems, therefore, *we shall classify as conservative those forces* $\mathbf{F}_\alpha$ *that are derivable from a single potential energy function* $\mathscr{V}(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)$ *via*

$$\mathbf{F}_\alpha^{(\text{cons})} = -\nabla^{(\alpha)}\mathscr{V}(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N) \qquad (\alpha = 1,2,\ldots,N) \tag{C.19}$$

Here one regards $\mathbf{r}_\beta$ as constant when we apply $\nabla^{(\alpha)}$, if $\beta \neq \alpha$. Fortunately, many physical forces are conservative in the sense of (C.19).

The classification of conservative and non-conservative system forces leads to a generalization of (C.9). If $\mathscr{E}$ denotes the total system energy, defined by

$$\mathscr{E} = \sum_\alpha \frac{1}{2} m_\alpha |\mathbf{v}_\alpha|^2 + \mathscr{V}(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)$$

then *the rate of change of energy along the system trajectory equals the power generated by the non-conservative forces*:

$$\begin{aligned}\frac{d\mathscr{E}}{dt} &= \sum_\alpha (\mathbf{F}_\alpha^{(\text{cons})} + \mathbf{F}_\alpha^{(\text{non-cons})}) \cdot \frac{d\mathbf{r}_\alpha}{dt} + \sum_\alpha \nabla^{(\alpha)}\mathscr{V} \cdot \frac{d\mathbf{r}_\alpha}{dt} \\ &= \sum_\alpha \mathbf{F}_\alpha^{(\text{non-cons})} \cdot \frac{d\mathbf{r}_\alpha}{dt}\end{aligned} \tag{C.20}$$

C.2 MECHANICS OF RIGID BODIES

A *rigid body* is a system of particles whose internal forces are so strong that they hold the interparticle distances fixed:

$$|\mathbf{r}_\alpha - \mathbf{r}_\beta| = \textit{constant} \tag{C.21}$$

Rigid bodies are theoretical idealizations, but their motions provide very accurate descriptions of the mechanics of real physical solids. The conditions of rigidity (C.21) can be enforced, in theory, by forces obeying (C.13) and (C.14), so we will treat rigid bodies as NTL systems.

Observe that Eq. (C.21) implies that the *angles* between the interparticle vectors $(\mathbf{r}_\alpha - \mathbf{r}_\beta)$ are also fixed, because the angles can be expressed in terms of the lengths via the law of cosines. Hence, also, the dot products stay constant.

Furthermore, *the center of mass of a rigid body stays fixed in the body.* To see that $\mathbf{R}_{\text{cm}}$ stays a constant distance from, say, $\mathbf{r}_5$, we express $|\mathbf{R}_{\text{cm}} - \mathbf{r}_5|$ in terms of these fixed dot products:

$$\begin{aligned}|\mathbf{R}_{\text{cm}} - \mathbf{r}_5|^2 &= \left|\frac{\sum_\alpha m_\alpha \mathbf{r}_\alpha}{M} - \mathbf{r}_5\right|^2 = \left|\frac{\sum_\alpha m_\alpha \mathbf{r}_\alpha}{M} - \frac{\sum_\alpha m_\alpha \mathbf{r}_5}{M}\right|^2 \\ &= \left|\frac{\sum_\alpha m_\alpha(\mathbf{r}_\alpha - \mathbf{r}_5)}{M}\right|^2 = \frac{\sum_\alpha \sum_\beta m_\alpha(\mathbf{r}_\alpha - \mathbf{r}_5) \cdot m_\beta(\mathbf{r}_\beta - \mathbf{r}_5)}{M^2}\end{aligned}$$

Hence, the distance from $\mathbf{R}_{\text{cm}}$ to each of the particles stays constant, and $\mathbf{R}_{\text{cm}}$ is fixed in the body.

Let us first consider the motion of a rigid body that moves *with one point held in place.* In this case it is convenient to use an inertial coordinate system with its origin at the stationary point. Also, we define an auxiliary coordinate system with the same origin, *but with its axes fixed in the body.* The position of the body is then completely specified by determining the position of the body-fixed axes.

In the subsequent development we shall draw freely upon the techniques discussed in Secs 5.3 and 5.4, and the corresponding Exercises.

If we let **i**, **j**, and **k** be unit vectors along the inertial axes and $\mathbf{i}^b$, $\mathbf{j}^b$, and $\mathbf{k}^b$ be unit vectors along the body-fixed axes, then any vector **h** has the two representations

$$\begin{aligned}\mathbf{h} &= h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k}\\ &= h_1{}^b\mathbf{i}^b + h_2{}^b\mathbf{j}^b + h_3{}^b\mathbf{k}^b\end{aligned}$$

where the h_i, $h_j{}^b$ are related by an *orthogonal transformation* expressed through the orthogonal 3-by-3 matrix $\mathcal{O}$:

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \mathcal{O} \begin{bmatrix} h_1{}^b \\ h_2{}^b \\ h_3{}^b \end{bmatrix} \tag{C.22}$$

If **h** changes with time, then we have the two expressions for its derivative

$$\begin{aligned}\frac{d\mathbf{h}}{dt} &= \frac{dh_1}{dt}\mathbf{i} + \frac{dh_2}{dt}\mathbf{j} + \frac{dh_3}{dt}\mathbf{k}\\ &= \frac{dh_1{}^b}{dt}\mathbf{i}^b + h_1{}^b\frac{d\mathbf{i}^b}{dt} + \frac{dh_2{}^b}{dt}\mathbf{j}^b + h_2{}^b\frac{d\mathbf{j}^b}{dt} + \frac{dh_3{}^b}{dt}\mathbf{k}^b + h_3{}^b\frac{d\mathbf{k}^b}{dt}\end{aligned}$$

(Of course, the inertial axes stay fixed.)

Now suppose an observer *turns with the body* and computes the rate of change of **h**. This person is unaware that the body axes are turning and the only way to detect a change in **h** is *if its body-fixed components change.* Hence, he/she computes

$$\left(\frac{d\mathbf{h}}{dt}\right)^b = \frac{dh_1{}^b}{dt}\mathbf{i}^b + \frac{dh_2{}^b}{dt}\mathbf{j}^b + \frac{dh_3{}^b}{dt}\mathbf{k}^b \tag{C.23}$$

What is the relation between $d\mathbf{h}/dt$ and $(d\mathbf{h}/dt)^b$? To answer this question, we turn to (C.22). The components of $d\mathbf{h}/dt$ in the inertial system satisfy

$$\frac{d}{dt}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \mathcal{O}\frac{d}{dt}\begin{bmatrix} h_1{}^b \\ h_2{}^b \\ h_3{}^b \end{bmatrix} + \frac{d\mathcal{O}}{dt}\begin{bmatrix} h_1{}^b \\ h_2{}^b \\ h_3{}^b \end{bmatrix} \tag{C.24}$$

(Here $d\mathcal{O}/dt$ has the obvious interpretation as the 3-by-3 matrix whose elements are the derivatives of the corresponding elements of $\mathcal{O}$). The first term on the right gives the components, in the inertial system, of the vector $(d\mathbf{h}/dt)^b$ in Eq. (C.23). To unravel the meaning of the second term, we first rewrite it as

$$\frac{d\mathcal{O}}{dt}\mathcal{O}^T\mathcal{O}\begin{bmatrix} h_1{}^b \\ h_2{}^b \\ h_3{}^b \end{bmatrix} \tag{C.25}$$

keeping in mind that $\mathcal{O}^T\mathcal{O} = \mathcal{I} = \mathcal{O}\mathcal{O}^T$. Differentiating this last relationship reveals that

$$\frac{d\mathcal{O}}{dt}\mathcal{O}^T + \mathcal{O}\frac{d\mathcal{O}^T}{dt} = 0 \tag{C.26}$$

but since

$$\left(\frac{d\mathcal{O}}{dt}\mathcal{O}^T\right)^T = \mathcal{O}\frac{d\mathcal{O}^T}{dt}$$

Eq. (C.26) says that $(d\mathcal{O}/dt)\mathcal{O}^T$ is antisymmetric. Thus, we can write it as

$$\frac{d\mathcal{O}}{dt}\mathcal{O}^T = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{C.27}$$

Now insert (C.27) and (C.22) into (C.25). According to problem 18 of the Exercises in Sec. 5.3, the resulting expression gives the inertial components of $\boldsymbol{\omega} \times \mathbf{h}$, where

$$\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$$

Hence, we understand (C.24) to say

$$\frac{d\mathbf{h}}{dt} = \left(\frac{d\mathbf{h}}{dt}\right)^b + \boldsymbol{\omega} \times \mathbf{h} \tag{C.28}$$

To determine the meaning of the vector $\boldsymbol{\omega}$, we let $\mathbf{h}$ be the vector from the origin to the αth particle: $\mathbf{h} = \mathbf{r}_\alpha$. Then $\mathbf{h}$ is fixed in the body, so we have

$$\frac{d\mathbf{r}_\alpha}{dt} = \mathbf{0} + \boldsymbol{\omega} \times \mathbf{r}_\alpha \tag{C.29}$$

But the left-hand side is the velocity, in the inertial system, of the particle at $\mathbf{r}_\alpha$. Hence, (C.29) is the same as Eq. (1.21), and we identify $\boldsymbol{\omega}$ as *the angular velocity of the body-fixed system with respect to the inertial system.* The result (C.28) is known as *Coriolis' law*, and the formula (C.27) tells how to compute the angular velocity vector $\boldsymbol{\omega}$ from the transformation matrix.

Let us use the angular velocity vector to express the angular momentum. We have

$$\begin{aligned}\mathbf{L} &= \sum_\alpha \mathbf{r}_\alpha \times m_\alpha\mathbf{v}_\alpha = \sum_\alpha \mathbf{r}_\alpha \times m_\alpha(\boldsymbol{\omega} \times \mathbf{r}_\alpha) \\ &= \sum_\alpha m_\alpha|\mathbf{r}_\alpha|^2\boldsymbol{\omega} - \sum_\alpha m_\alpha\mathbf{r}_\alpha\mathbf{r}_\alpha \cdot \boldsymbol{\omega}\end{aligned}$$

by (1.29). To study this, we must use tensor notation. If we let $r_i^{(\alpha)}$ denote the ith component of $\mathbf{r}_\alpha$, then the **i**th component of **L** can be written

$$\begin{aligned} L_i &= \sum_\alpha m_\alpha r_\ell^{(\alpha)2}\omega_i - \sum_\alpha m_\alpha r_i^{(\alpha)} r_j^{(\alpha)}\omega_j \\ &= I_{ij}\omega_j \end{aligned} \tag{C.30}$$

(recall the summation convention.) Here the I_{ij} are the components of a tensor of rank two, called the *inertia tensor*:

$$I_{ij} = \sum_\alpha m_\alpha [r_\ell^{(\alpha)^2}\,\delta_{ij} - r_i^{(\alpha)} r_j^{(\alpha)}]$$

Observe that I is symmetric: $I_{ij} = I_{ji}$.

Now that we have an expression for the angular momentum, we can write the equations of motion, derived by summing (C.5) over all the particles: *the total torque equals the rate of change of the total angular momentum.* Hence, in tensor form,

$$T_i = \frac{d}{dt}(I_{ij}\omega_j)$$

Notice that since the inertial components of $\mathbf{r}^{(\alpha)}$ change as the body moves, I_{ij} is a function of time. However, if we work in the body-fixed coordinate system, all the $r_i^{(\alpha)}$ are fixed, and I_{ij} is constant! Hence, acknowledging Coriolis' law, we find that in the body-fixed system the equations of motion are

$$T_i = I_{ij}\frac{d\omega_j}{dt} + \varepsilon_{ijk}\omega_j I_{k\ell}\omega_\ell \tag{C.31}$$

Furthermore, it is well-known that because I is a symmetric tensor, there exist body-fixed axis systems wherein the off-diagonal components of I are zero; in such *principal axis systems*

$$I = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$$

Writing (C.31) accordingly, we derive *Euler's equations of motion* for a body-fixed principal axis system:

$$\begin{aligned} T_1 &= I_{11}\frac{d\omega_1}{dt} - (I_{22} - I_{33})\omega_2\omega_3 \\ T_2 &= I_{22}\frac{d\omega_2}{dt} - (I_{33} - I_{11})\omega_3\omega_1 \\ T_3 &= I_{33}\frac{d\omega_3}{dt} - (I_{11} - I_{22})\omega_1\omega_2 \end{aligned} \tag{C.32}$$

In principal, one determines the motion by solving the Euler equations for $\boldsymbol{\omega}$ and then finding the transformation $\mathcal{O}$ from Eq. (C.27). The latter task is made somewhat easier by choosing a convenient parametrization for $\mathcal{O}$ in terms of "Eulerian angles."

The Eulerian angles are discussed in the references below. Suffice it to say that the rotation described by $\mathcal{O}$ can be decomposed into a sequence of three successive rotations about the z, x, and z (again) body axes through the angles ϕ, θ, and ψ, respectively; hence,

$$\mathcal{O} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}$$

and Eq. (C.27) becomes, in the body-fixed system,

$$\begin{aligned} \omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 &= \dot{\phi}\cos\theta + \dot{\psi} \end{aligned} \tag{C.33}$$

Once $\boldsymbol{\omega}$ is determined from the Euler equations, the Eulerian angles ϕ, θ, and ψ are determined by solving (C.33). Then $\mathcal{O}$ can be computed.

It should be mentioned that other conventions are sometimes used to define the Eulerian angles.

It is sometimes useful to express the kinetic energy of this motion in terms of the inertia tensor. We have

$$\begin{aligned} \mathscr{K} &= \frac{1}{2}\sum_\alpha m_\alpha|\mathbf{v}|^2 = \frac{1}{2}\sum_\alpha m_\alpha|\boldsymbol{\omega}\times\mathbf{r}_\alpha|^2 \\ &= \frac{1}{2}\sum_\alpha m_\alpha(\boldsymbol{\omega}\times\mathbf{r}_\alpha)\cdot(\boldsymbol{\omega}\times\mathbf{r}_\alpha) \\ &= \frac{1}{2}\sum_\alpha m_\alpha[|\mathbf{r}_\alpha|^2\,|\boldsymbol{\omega}|^2 - (\mathbf{r}_\alpha\cdot\boldsymbol{\omega})^2] \end{aligned}$$

by (1.32). In tensor notation we find

$$\begin{aligned} \mathscr{K} &= \frac{1}{2}\sum_\alpha m_\alpha[r_\ell{}^{(\alpha)2}\omega_j{}^2 - r_j{}^{(\alpha)}\omega_j r_i{}^{(\alpha)}\omega_i] \\ &= \frac{1}{2}\omega_i\sum_\alpha m_\alpha[r_\ell{}^{(\alpha)2}\,\delta_{ij} - r_i{}^{(\alpha)}r_j{}^{(\alpha)}]\omega_j \\ &= \frac{1}{2}\omega_i I_{ij}\omega_j \end{aligned}$$

If we let $\boldsymbol{\omega} = \omega\mathbf{n}$, where $\mathbf{n}$ is a unit vector, then $\mathscr{K} = \frac{1}{2}(n_i I_{ij} n_j)\omega^2$. The quantity $n_i I_{ij} n_j$ is called the *moment of inertia* about the $\mathbf{n}$ direction. If

$\dot{\mathbf{n}} = \mathbf{k}^b$, we have $n_i = \delta_{i3}$ and the moment of inertia about the z-axis becomes

$$\delta_{i3} I_{ij} \delta_{j3} = I_{33} = \sum_{\alpha} m_\alpha (r_\ell{}^{(\alpha)2} - r_3{}^{(\alpha)2}) \quad = \sum_{\alpha} m_\alpha (x^{(\alpha)2} + y^{(\alpha)2})$$

a formula familiar to most calculus students.

We now turn our attention to the general motion of the rigid body, dropping the assumption about the stationary point. Theorem C.2 allows us to write the equations of motion for the center of mass:

$$\sum_{\alpha} \mathbf{F}_\alpha{}^{\text{ext}} = M \frac{d^2 \mathbf{R}_{\text{cm}}}{dt^2} \tag{C.34}$$

(invoking NTL for the internal forces.) Also, Theorem C.5 says that the external torque, measured about the center of mass, equals the rate of change of angular momentum measured in the C-M system. However, since the center of mass is a body-fixed point, and relativizing all quantities to the center of mass is equivalent to treating the center of mass as stationary, the theory we just discussed can be used to analyze this motion. In particular, if I denotes the inertia tensor *with components computed in a body fixed principal axis system with its origin at the center of mass*, the three Euler equations (C.32) describe the angular velocity of the body system with respect to the C-M system, which is parallel to the inertial system. Hence, the six scalar equations (C.34) and (C.32) determine the three components of $\mathbf{R}_{\text{cm}}$ and the three Eulerian angles, and the position of the rigid body can thus be completely specified.

Exercises

1. As a rule, the angular momentum $\mathbf{L}$ is *not* parallel to the angular velocity $\boldsymbol{\omega}$; but if $\boldsymbol{\omega}$ is directed along the ith principal axis of I, $\mathbf{L} = I_{ii}\boldsymbol{\omega}$(not summed). Demonstrate this.
2. Prove: the moment of inertia I_o in the direction $\mathbf{n}$ measured about the point $\mathbf{R}_o$ is related to the corresponding moment of inertia I_{cm} measured about $\mathbf{R}_{\text{cm}}$ by the formula

$$I_o = I_{\text{cm}} + M|(\mathbf{R}_{\text{cm}} - \mathbf{R}_o)| \times \mathbf{n}^2$$

3. Prove Theorem C.4.

References

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APPENDIX D

The Vector Equations of Electromagnetism

D.1 ELECTROSTATICS

It is well known that two electrically charged particles at rest will exert forces on each other. The forces vary inversely with the square of the distance separating them (the "inverse square law"), and directly with the charge on each particle. They are attractive or repulsive accordingly as they are of opposite or of equal polarity. This body of facts is known as "Coulomb's law," and it can be formulated vectorially as follows.

Let q_1 and q_2 denote the charge on each particle (signed according to polarity), and let $\mathbf{r}_1$ and $\mathbf{r}_2$ designate the respective position vectors. Then the force on particle 1 due to particle 2 equals

$$\mathbf{F}_1^{(2)} = \frac{kq_1q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} = kq_1q_2\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$$

where the (positive) constant k depends on the system of units.

If there are N stationary particles with charges q_i, then the forces they exert on a "test charge" q located at $\mathbf{r}$ add vectorially:

$$\mathbf{F} = \sum_{i=1}^{N} kqq_i\frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3} \tag{D.1}$$

Historically it has proved convenient to interpret Eq. (D.1) as

$$\mathbf{F} = q\mathbf{E}(\mathbf{r}) \tag{D.2}$$

where the *electric field* $\mathbf{E}(\mathbf{r})$ is the force, per unit charge, that would be exerted by the charged particles 1 through N on a charged particle located at $\mathbf{r}$. Hence, $\mathbf{E}(\mathbf{r})$ is a vector field given by

$$\mathbf{E}(\mathbf{r}) = k \sum_{i=1}^{N} q_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3} \tag{D.3}$$

The electric field is irrotational for $\mathbf{r} \neq \mathbf{r}_i$, as the following computation shows:

$$\begin{aligned}\nabla \times \mathbf{E} &= \sum k q_i \nabla \times \frac{(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} \\ &= \sum k q_i \left[\frac{\nabla \times (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} + \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}_i|^3} \right) \times (\mathbf{r} - \mathbf{r}_i) \right] \\ &= \sum k q_i \left[\mathbf{0} - \frac{3(\mathbf{r} - \mathbf{r}_i) \times (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^5} \right] \\ &= \mathbf{0}\end{aligned} \tag{D.4}$$

using identities (3.21), (3.24), and (3.25). In fact, $\mathbf{E}$ is the negative gradient of the *electrostatic potential* $\mathscr{V}(\mathbf{r})$, given by

$$\mathscr{V}(\mathbf{r}) = \sum k q_i / |\mathbf{r} - \mathbf{r}_i| \tag{D.5}$$

The equation

$$\mathbf{E} = -\nabla \mathscr{V} \tag{D.6}$$

follows from identity (3.25).

The divergence of the electric field, for $\mathbf{r} \neq \mathbf{r}_i$, is computed by using (3.20), (3.23), and (3.25):

$$\nabla \cdot \mathbf{E} = k \sum q_i \left[\frac{3}{|\mathbf{r} - \mathbf{r}_i|^3} - \frac{3(\mathbf{r} - \mathbf{r}_i) \cdot (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^5} \right] = 0$$

However, $\nabla \cdot \mathbf{E}$ is undefined (infinite?) if the tip of $\mathbf{r}$ coincides with one of the point charges. To investigate this more fully, we consider the electric field due to a single point charge q_1 located at the origin $\mathbf{r}_1 = \mathbf{0}$, and compute its flux through a closed surface S:

$$\text{flux} = \iint_S \mathbf{E} \cdot d\mathbf{S} = k q_1 \iint \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S} \tag{D.7}$$

(where, as usual, $r = |\mathbf{r}|$). If the domain D enclosed by S does not contain the origin, then $\nabla \cdot \mathbf{E} = 0$ throughout D and the net flux out of S is zero, by the divergence theorem. If the origin lies inside D, then we consider a sphere S_ε around the origin, whose radius ε is so small that the interior of S_ε lies inside D (see Fig. D.1). Since $\nabla \cdot \mathbf{E} = 0$ in the intervening region bounded by S and S_ε, the flux out of S equals the flux of S_ε (compare Exercise 15, Sec.

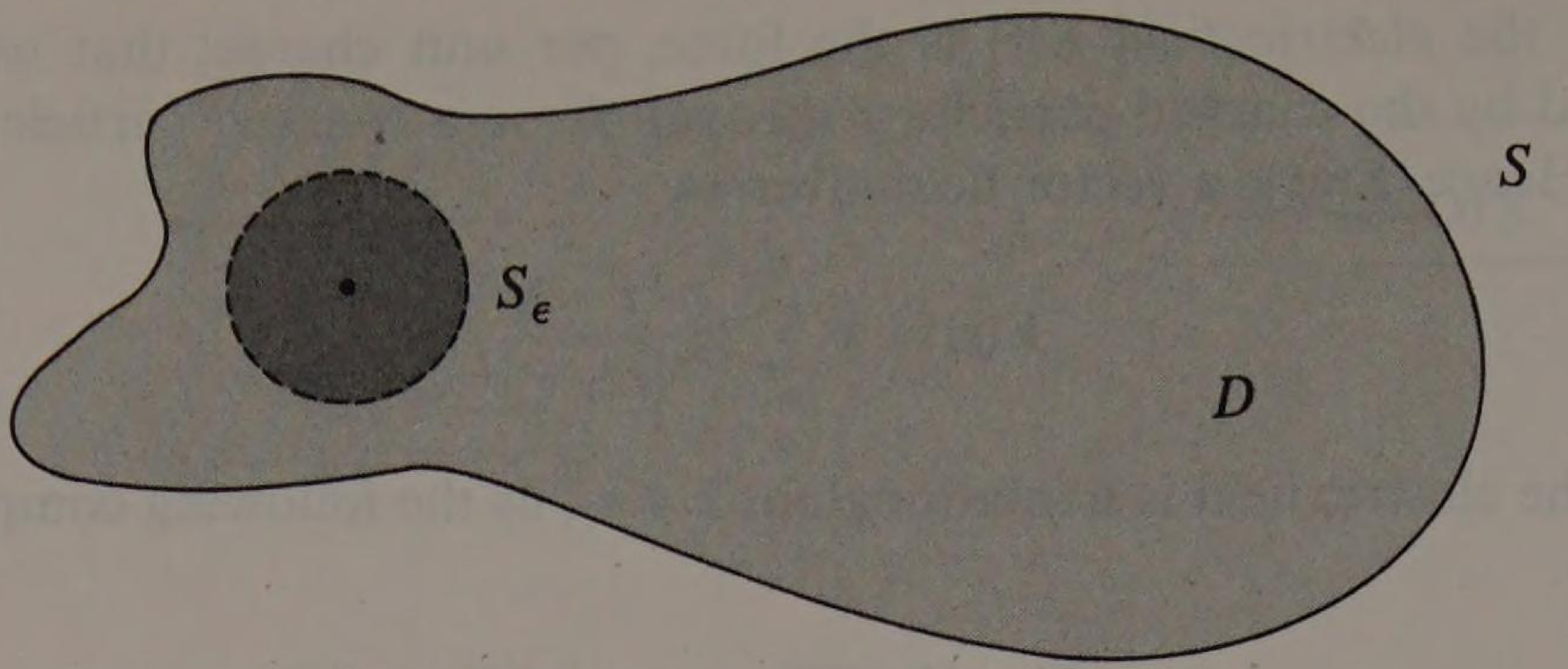


FIGURE D.1

4.10). Parametrizing S_ε by the spherical coordinates θ and ϕ (with $r = \varepsilon =$ constant), we have

$$d\mathbf{S} = \left(\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}\right) d\phi\, d\theta$$

$$= r^2 \sin\phi\, d\phi\, d\theta \frac{\mathbf{r}}{r}$$

Thus the flux out of S_ε, and hence S, equals

$$kq_1 \int_0^{2\pi} \int_0^{\pi} \frac{\mathbf{r}}{r^3} \cdot \frac{\mathbf{r}}{r} r^2 \sin\phi\, d\phi\, d\theta = 4\pi k q_1$$

Shifting the charge q_1 to position $\mathbf{r}_1$, we conclude that

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \begin{cases} 4\pi k q_1 & \text{if } S \text{ encloses } \mathbf{r}_1 \\ 0 & \text{otherwise.} \end{cases}$$

For N charges, we add all the individual contributions to the flux and derive *Gauss' law of electrostatics*

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi k \sum (q_i \text{ or } 0) = 4\pi k \text{ (total charge enclosed by } S) \qquad \text{(D.8)}$$

[recall Eq. (4.39)]. Here we are assuming that none of the charges q_i actually lies on the surface S.

Point charges do occur in nature as electrons, protons, etc., but in most microscopic physical situations involving matter so many particles are present that it becomes necessary to use a continuum approximation, i.e., to replace sums over point charges by integrals over charge densities. Thus, we introduce the *charge density function* $\rho(\mathbf{r})$ with units of charge per unit volume, so that the total charge q in a region D is given by

$$q = \iiint_D \rho(x, y, z)\, dx\, dy\, dz = \iiint \rho(\mathbf{r})\, dV \qquad \text{(D.9)}$$

This leads us to consider the analogs of Eqs. (D.3), (D.5), (D.6), and (D.7) for the continuous case:

$$\mathbf{E}(\mathbf{r}) = k \iiint \rho(\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\, dV' \tag{D.10}$$

$$\mathscr{V}(\mathbf{r}) = k \iiint \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dV' \tag{D.11}$$

$$\mathbf{E}(\mathbf{r}) = -\nabla \mathscr{V}(\mathbf{r}) \tag{D.12}$$

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi k \iint_D \int_{(\text{enclosed by } S)} \rho(\mathbf{r})\, dV \tag{D.13}$$

However, we now encounter some mathematical difficulties. The troubles arise from the appearance of zeroes in the denominators. For the discrete-particle case, we simply excluded the points $\mathbf{r} = \mathbf{r}_i$ from consideration in equations like (D.3); these points were finite in number and we were willing to work around them. However, our continuum model will have whole regions of space where ρ is nonzero and the integrands in (D.10) and (D.11) will diverge if $\mathbf{E}$ or $\mathscr{V}$ is to be evaluated in such regions. Thus, the questions arise:

(*i*) Are the improper integrals (D.10) and (D.11) well-defined at points where $\rho(\mathbf{r}) \neq 0$?
(*ii*) If so, are (D.12) and (D.13) still valid?

These questions form the basis of *potential theory*, a subject that is treated in the textbook by Jeffreys and Jeffreys. Without going into details, we can get some insight by considering the following example: Suppose $f(\mathbf{r})$ is a continuous function and we wish to integrate $f(\mathbf{r})/r^p$ over some region containing the origin, where $\mathbf{r} = \mathbf{0}$. If we use spherical coordinates we have

$$\iiint \frac{f(\mathbf{r})}{r^p}\, dV = \iiint \frac{f(r,\phi,\theta)}{r^p}\, r^2 \sin\phi\, dr\, d\phi\, d\theta$$

$$= \iiint [f(r,\phi,\theta) r^{2-p}\, dr] \sin\phi\, d\phi\, d\theta$$

Since the improper integral $\int_0^a r^q dr$ converges for $q > -1$, we conclude that the inner integral over r will be finite for $p < 3$. The other integrals present no difficulty, so we propose the following rule of thumb for manipulating the improper integrals that arise due to the continuum approximation: *All the usual operations and theorems may be applied in a straightforward manner unless one encounters integrals of the form*

$$\iiint f(\mathbf{r}')|\mathbf{r} - \mathbf{r}'|^{-p}\, dV' \qquad \textit{with } p \geq 3$$

Of course such a glib statement is mathematically treacherous, but it roughly summarizes the results of the more rigorous potential theory.

Accordingly, we see that the potential $\mathscr{V}(\mathbf{r})$ in Eq. (D.11) is well defined at all points $\mathbf{r}$ where the charge density $\rho(\mathbf{r})$ is continuous. (In fact, jump discontinuities in ρ are permissible). So also is the electric field $\mathbf{E}(\mathbf{r})$: keep in mind that the net component of $|\mathbf{r}-\mathbf{r}'|$ in the denominator is 2 in Eq. (D.10). Furthermore, since (D.10) is obtained from (D.11) by formal differentiation, and all the integrals are convergent, the relation

$$\mathbf{E}(\mathbf{r}) = -\nabla\mathscr{V}(\mathbf{r}) \tag{D.12}$$

is true. Thus, $\mathbf{E}(\mathbf{r})$ is still irrotational and (D.4) holds. Moreover, both integrals in Gauss' law, Eq. (D.13), are quite regular, and the law remains valid.

Now let us compute the divergence of $\mathbf{E}$, which by Eq. (D.12) equals the negative laplacian of $\mathscr{V}$. Modeling the computations we made earlier, we are tempted to proceed formally from (D.10) to obtain

$$\begin{aligned}\nabla\cdot\mathbf{E} &= k\iiint \rho(\mathbf{r}')\nabla\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,dV'\\ &= k\iiint \rho(\mathbf{r}')\left[\frac{3}{|\mathbf{r}-\mathbf{r}'|^3}-\frac{3(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^5}\right]dV'\\ &= 0 \qquad\qquad (\mathit{WRONG!})\end{aligned}$$

but, as the warning flag indicates, this calculation is suspect because of the exponents in the denominators; the rule of thumb is violated. The correct expression is obtained by applying the divergence theorem to the left-hand side of Eq. (D.13), resulting in

$$\iiint_D \nabla\cdot\mathbf{E}\,dV = 4\pi k\iiint \rho(\mathbf{r})\,dV$$

Since this equation holds for any domain D, we conclude

$$\nabla\cdot\mathbf{E} = 4\pi k\rho \tag{D.14}$$

the "differential form of Gauss' law." Expressing $\mathbf{E}$ in terms of $\mathscr{V}$, we have *Poisson's equation for the electrostatic potential*

$$\nabla^2\mathscr{V} = -4\pi k\rho \tag{D.15}$$

A typical situation in electrostatics is the following: One is explicitly given (*i*) the charge distribution ρ in a certain domain D and (*ii*) the values of the potential $\mathscr{V}(\mathbf{r})$ on the boundary surface S of the domain; the problem is to find $\mathscr{V}(\mathbf{r})$ everywhere inside the domain. Observe that one cannot simply use (D.11) because the charge density function ρ is only known inside D. All one knows about the charges outside D is that they, together with the interior charges, give rise to the specified values of $\mathscr{V}$ on the surface S. (For example, the charges within an electric conductor always distribute themselves along its surface so that the conductor is an equipotential, in the electrostatic case). Thus, one has to solve Poisson's equation in D, with the specified boundary conditions, to find $\mathscr{V}(\mathbf{r})$. Taking the gradient then gives the electric field $\mathbf{E}(\mathbf{r})$.

D.2 MAGNETOSTATICS

Just as stationary charges produce an electric field that can be detected as a force on a test charge, moving charges or *currents* produce *magnetic fields* that exert forces on "test currents." However, the geometric properties of these fields are somewhat more complicated.

A point charge q_1 located at $\mathbf{r}_1$, and moving with a velocity $\mathbf{v}_1$, produces a *magnetic induction field* **B** whose magnitude and direction at the point **r** are given (for nonrelativistic speeds) by

$$\mathbf{B}(\mathbf{r}) = \gamma q_1 \frac{\mathbf{v}_1 \times (\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3} \tag{D.16}$$

where γ is a positive constant depending on the system of units. The magnetic induction vector exerts a force on a particle with charge q and velocity **v** given by

$$\mathbf{F} = \eta q \mathbf{v} \times \mathbf{B} \tag{D.17}$$

Here again η is a dimensional constant.

Note the dependence on velocity in these equations; a stationary particle produces no magnetic field, nor is it influenced by a magnetic field. A moving particle produces a field perpendicular to its velocity, and is forced by an external field in a direction perpendicular both to its velocity and the field.

Furthermore, the mutual interaction between two moving particles does not satisfy Newton's third law; the force on particle 2 $(q_2,\mathbf{r}_2,\mathbf{v}_2)$ due to particle 1 $(q_1,\mathbf{r}_1,\mathbf{v}_1)$ is

$$\mathbf{F}_2^{(1)} = \eta q_2 \mathbf{v}_2 \times \frac{q_1 \mathbf{v}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$$

whereas

$$\mathbf{F}_1^{(2)} = \eta q_1 \mathbf{v}_1 \times \frac{q_2 \mathbf{v}_2 \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$$

and a little experimentation will reveal that $\mathbf{F}_1^{(2)}$ is not equal to $\mathbf{F}_2^{(1)}$, nor are these forces directed along $\mathbf{r}_2 - \mathbf{r}_1$.

Once again it is necessary to make a continuum approximation to solve most physical problems. Thus, we shall consider a continuum of moving charges, with a charge density function $\rho_m(\mathbf{r})$ and a velocity field $\mathbf{v}(\mathbf{r})$. (In many situations one has moving charges flowing through a background of stationary charges; e.g., in a current carrying conductor the conducting electrons flow past the stationary ions. Thus, we use a subscript to distinguish between the density of moving charges, ρ_m, and the total charge density ρ).

These moving charges give rise to a current density $\mathbf{j}(\mathbf{r})$:

$$\mathbf{j}(\mathbf{r}) = \rho_m(\mathbf{r})\mathbf{v}(\mathbf{r}) \tag{D.18}$$

According to the discussion in Sec. 3.3, the flux of $\mathbf{j}$ through a surface S equals the amount of charge crossing the surface per unit time, or, in other words, the current I through the surface.

$$I = \iint_S \mathbf{j} \cdot d\mathbf{S}$$

Moreover, the conservation of charge can be expressed, according to Sec. 3.3, as

$$\nabla \cdot \mathbf{j} = -\frac{\partial \rho_m}{dt} = -\frac{\partial \rho}{dt} \tag{D.19}$$

(since the stationary charges do not change). For magnetostatics, $\partial\rho/\partial t = 0$, so $\mathbf{j}$ is solenoidal.

The total magnetic induction field due to a steady (that is, time-independent) current distribution is obtained by superposition:

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \gamma \iiint \rho_m(\mathbf{r}')\mathbf{v}(\mathbf{r}') \times \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, dV' \\ &= \gamma \iiint \mathbf{j}(\mathbf{r}') \times \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, dV' \end{aligned} \tag{D.20}$$

In some physical experiments the current producing the **B** field is carried by a wire filament. It is convenient to model this situation by letting the cross-sectional area of the wire go to zero, and the current density to infinity, in such a manner that the flux of $\mathbf{j}$ along the wire, i.e., the current I, is constant. Then the wire becomes a space curve $\mathbf{r}' = \mathbf{r}'(s)$ carrying a current I. The volume integration in (D.20) is regarded as an iterated integral, integrating first over the cross-sectional area, then along the length of the wire. In our model the first integral yields the current I, and (D.20) becomes

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \gamma I \int \frac{d\mathbf{r}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \\ &\left(= \gamma I \int \frac{d\mathbf{r}'(s)}{ds} \times \frac{\mathbf{r}-\mathbf{r}'(s)}{|\mathbf{r}-\mathbf{r}'(s)|^3}\, ds \right) \end{aligned} \tag{D.21}$$

In magnetostatic situations, of course, the current is carried around a closed curve; otherwise, $\mathbf{j}$ could not be solenoidal. Thus, the integral in (D.21) could be more accurately written $\oint$.

It is very instructive to compute the total force exerted on one current loop by another. The effect of the field $\mathbf{B}(\mathbf{r})$ on the loop $\mathbf{r}_1(s)$ carrying current I_1 is given by

$$\begin{aligned}\mathbf{F}_1 &= \eta \iiint \mathbf{j} \times \mathbf{B}\, dV \\ &= \eta I_1 \oint d\mathbf{r}_1 \times \mathbf{B}(\mathbf{r}_1)\end{aligned}$$

If $\mathbf{B}$ is produced by loop $\mathbf{r}_2(s)$ carrying current I_2, then the interaction force is

$$\mathbf{F}_1{}^{(2)} = \eta\gamma I_1 I_2 \oint_2 \oint_1 d\mathbf{r}_1 \times \frac{[d\mathbf{r}_2 \times (\mathbf{r}_1 - \mathbf{r}_2)]}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$$

By identities (1.29) and (3.25)

$$\begin{aligned}\mathbf{F}_1{}^{(2)} &= \eta\gamma I_1 I_2 \oint_2 \oint_1 \left\{ \frac{d\mathbf{r}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}\, d\mathbf{r}_2 - d\mathbf{r}_1 \cdot d\mathbf{r}_2 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \right\} \\ &= \eta\gamma I_1 I_2 \oint_2 d\mathbf{r}_2 \oint_1 \nabla_1 \left(\frac{-1}{|\mathbf{r}_1 - \mathbf{r}_2|} \right) \cdot d\mathbf{r}_1 \\ &\quad - \eta\gamma I_1 I_2 \oint_2 \oint_1 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}\, d\mathbf{r}_1 \cdot d\mathbf{r}_2\end{aligned}$$

The first term vanishes by Theorem 4.2, and the resulting expression

$$\mathbf{F}_1{}^{(2)} = -\eta\gamma I_1 I_2 \oint_2 \oint_1 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}\, d\mathbf{r}_1 \cdot d\mathbf{r}_2$$

is antisymmetric; i.e., on interchanging indices we find that $\mathbf{F}_1{}^{(2)} = -\mathbf{F}_2{}^{(1)}$. Therefore, the total magnetic force between two current loops satisfies a form of Newton's third law, even though the interparticle forces do not!

It can be shown that a current loop does not exert a net force on itself, but that any given portion of the loop is, in general, subjected to a force. We delete this analysis because the improper integrals are quite complicated.

Returning to (D.20), we use (3.25) to rewrite the equation for the magnetic induction:

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= -\gamma \iiint \mathbf{j}(\mathbf{r}') \times \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dV' \\ &= \gamma \nabla \times \iiint \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dV' \qquad \text{(D.22)}\end{aligned}$$

(Since ∇ operates only on $\mathbf{r}$.) It follows immediately that $\mathbf{B}$ is solenoidal

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(D.23)}$$

Hence, $\mathbf{B}$ is derivable from a vector potential $\mathbf{A}$,

$$\mathbf{B} = \nabla \times \mathbf{A}$$

and, in fact, we can read off from (D.22) what **A** should be:

$$\mathbf{A}(\mathbf{r}) = \gamma \iiint \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV' + \nabla\psi(\mathbf{r})$$

where ψ is an arbitrary scalar function. (This degree of freedom is known as *gauge invariance*.)

To find the curl of **B**, we apply identity (3.30) to (D.22):

$$\begin{aligned}\nabla \times \mathbf{B} &= \gamma\nabla \times \left(\nabla \times \iiint \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV'\right) \\ &= \gamma\nabla\left(\nabla \cdot \iiint \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV'\right) - \gamma\nabla^2 \iiint \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV'\end{aligned}$$

Reconsidering Eqs. (D.11) and (D.15), we can see that the components of the last term equal those of $4\pi\gamma\mathbf{j}(\mathbf{r})$. For the "divergence" term, we bring ∇ inside the integral and appeal to

$$\nabla f(\mathbf{r}-\mathbf{r}') = -\nabla' f(\mathbf{r}-\mathbf{r}')$$

(here ∇ operates on **r**, and ∇' operates on **r**′) to write

$$\begin{aligned}\nabla \cdot \iiint \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV' &= -\iiint \mathbf{j}(\mathbf{r}') \cdot \nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dV' \\ &= -\iiint \nabla' \cdot \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV' \\ &\quad + \iiint \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla' \cdot \mathbf{j}(\mathbf{r}')\, dV'\end{aligned} \tag{D.24}$$

For magnetostatics, $\nabla \cdot \mathbf{j} = 0$, and the divergence theorem produces

$$\nabla \cdot \iiint \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV' = -\iint_S \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \cdot d\mathbf{S}' \tag{D.25}$$

Since we are integrating over a region containing all the current sources $\mathbf{j}(\mathbf{r})$, the enclosing surface has no currents; thus, $\mathbf{j} = \mathbf{0}$ on the right in (D.25) and we ultimately have *Ampere's law*:

$$\nabla \times \mathbf{B}(\mathbf{r}) = 4\pi\gamma\mathbf{j}(\mathbf{r}) \tag{D.26}$$

Eqs. (D.23) and (D.26) are the basic laws of magnetostatics. It is sometimes convenient to apply Stokes' theorem to (D.26), yielding the equation

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = 4\pi\gamma \iint_S \mathbf{j} \cdot d\mathbf{S} = 4\pi\gamma I$$

where the surface S is bounded by the curve C and I is the total current crossing S.

The typical problem in magnetostatics involves solving (D.23) and (D.26) subject to certain boundary conditions between different media. We defer discussion of the latter for the moment.

D.3 ELECTRODYNAMICS

In time-varying (nonstatic) situations, the previous equations must be modified. First of all, observe that, in accordance with the equation of continuity (D.19), the term $\nabla \cdot \mathbf{j}$ in Eq. (D.24) should not be dropped, but in general that it should be replaced by $-\partial\rho/\partial t$. Hence, instead of (D.26), we find

$$\begin{aligned}
\nabla \times \mathbf{B}(\mathbf{r}) &= \gamma\nabla\left(\nabla \cdot \iiint \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV'\right) - \gamma\nabla^2 \iiint \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' \\
&= -\gamma\nabla \iiint \left(\frac{\partial\rho(\mathbf{r}')/\partial t}{|\mathbf{r}-\mathbf{r}'|}\right) dV' + 4\pi\gamma\mathbf{j}(\mathbf{r}) \\
&= -\gamma\frac{\partial}{\partial t}\nabla \iiint \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' + 4\pi\gamma\mathbf{j}(\mathbf{r}) \\
&= \frac{\gamma}{k}\frac{\partial\mathbf{E}(\mathbf{r})}{\partial t} + 4\pi\gamma\mathbf{j}(\mathbf{r})
\end{aligned}$$

by Eq. (D.11) and (D.12). Thus, we write

$$\nabla \times \mathbf{B} = 4\pi\gamma\mathbf{j} + \frac{\gamma}{k}\frac{\partial\mathbf{E}}{\partial t} \tag{D.27}$$

for dynamic situations. Notice that (D.27) implies that magnetic induction fields are produced not only by currents **j** but also by changing electric fields. Maxwell, the discoverer of this effect, called $(1/4\pi k)(\partial\mathbf{E}/\partial t)$ the "displacement current."

Another necessary modification in the equations was discovered experimentally by Faraday. It involves the magnetic flux Φ across an oriented surface S:

$$\Phi = \iint_S \mathbf{B} \cdot d\mathbf{S}$$

Faraday observed that when the flux through S changes, an electric field is produced around the curve C forming the boundary of S, in accordance with

$$\frac{d\Phi}{dt} = \iint_S \frac{\partial\mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{1}{\alpha}\oint_C \mathbf{E} \cdot d\mathbf{r}$$

where α is another positive constant.

Applying Stokes' theorem, we find

$$\alpha \iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\iint_S \nabla \times \mathbf{E} \cdot d\mathbf{S}$$

and since this holds for arbitrary surfaces S we conclude that a changing magnetic induction field produces an electric field, so that Eq. (D.4) must be modified for dynamic situations to read

$$\nabla \times \mathbf{E} = -\alpha \frac{\partial \mathbf{B}}{\partial t}$$

Actually Faraday's law is more general than we have described. We have assumed that the flux Φ through S changed because the $\mathbf{B}$ field, itself, changed. In fact, Φ can also change if the surface S is moving or turning, in which case the Eq. (4.72) must be used to compute $d\Phi/dt$. Faraday observed that the same electric field is induced regardless of the mechanism that produces the change in Φ. This situation is relativistic and we refer the reader to the references for elaboration, but we remark that one consequence of the analysis is the identification of the constants α and η,

$$\alpha = \eta \tag{D.28}$$

and we incorporate this fact in our subsequent equations.

The four main equations that we have examined above,

$$\nabla \cdot \mathbf{E} = 4\pi k \rho \tag{D.14}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{D.23}$$

$$\nabla \times \mathbf{E} = -\eta \frac{\partial \mathbf{B}}{\partial t} \tag{D.29}$$

$$\nabla \times \mathbf{B} = 4\pi\gamma \mathbf{j} + \frac{\gamma}{k} \frac{\partial \mathbf{E}}{\partial t} \tag{D.27}$$

are known as *Maxwell's equations*, and they can be used to find $\mathbf{E}$ and $\mathbf{B}$ when the charges ρ and currents $\mathbf{j}$ are known throughout all space. The charges, in turn, are subjected to the *Lorentz force*, obtained by adding (D.2) and (D.17):

$$\mathbf{F} = \iiint \rho(\mathbf{E} + \eta \mathbf{v} \times \mathbf{B})\, dV \tag{D.30}$$

So, in general, the coupled system of Maxwell's equations and Lorentz's equation describes how the charges produce, and are influenced by, the fields.

If the charge sources are known only in a region D, Maxwell's equations have to be supplemented with boundary conditions. These can be derived from the equations themselves, as follows: Suppose the region D is bounded by the smooth surface S. Consider an infinitesimal "Gaussian pillbox", i.e.,

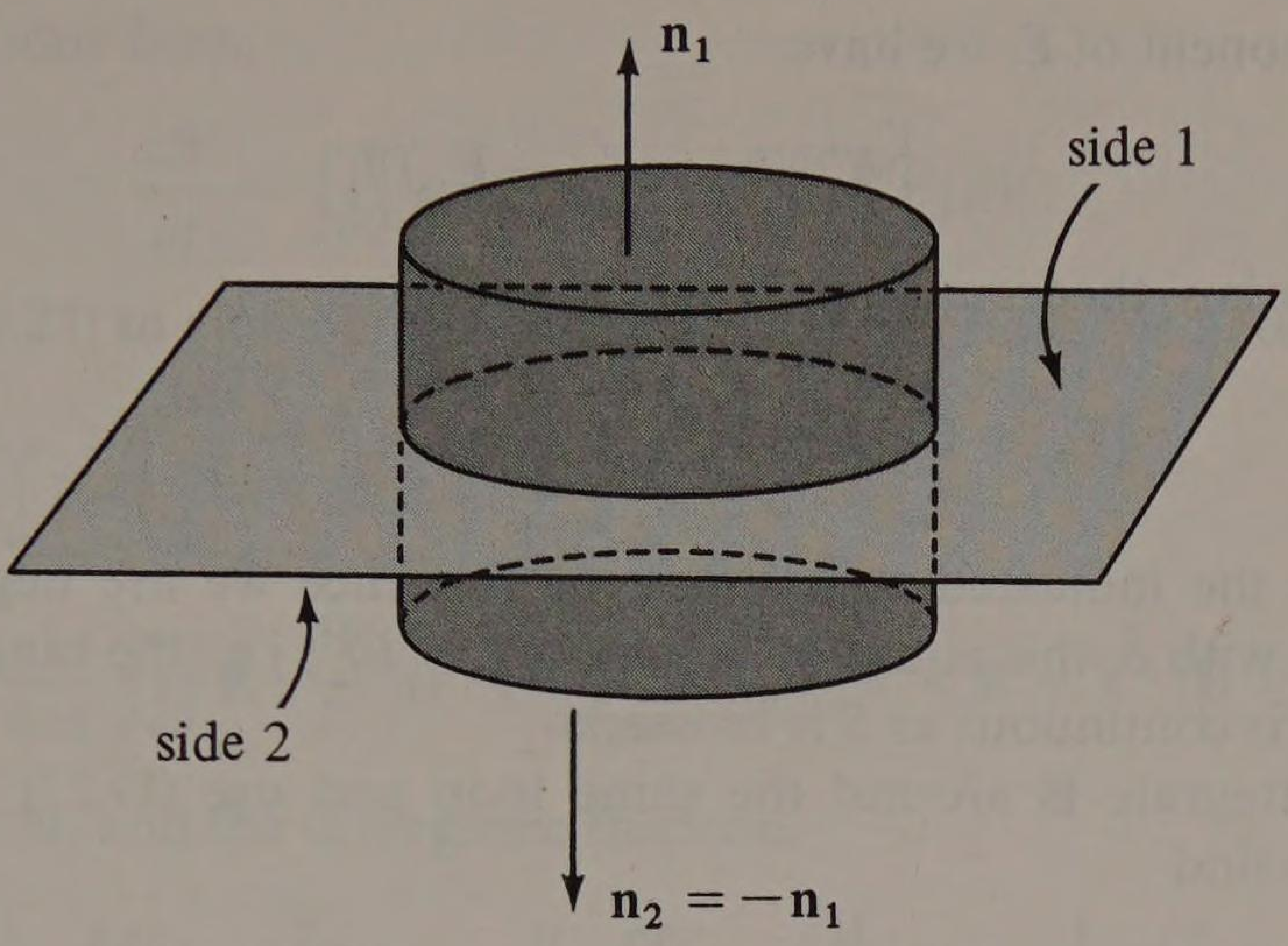


FIGURE D.2

a very short circular cylinder with axis normal to S and with a face on either side of S, as in Fig. D.2.

Regarding the height of the cylinder as much shorter than the diameter of the faces, we apply Gauss' law (D.8) to find

$$(\mathbf{E}_1 \cdot \mathbf{n}_1 + \mathbf{E}_2 \cdot \mathbf{n}_2)(\text{area of base}) = 4\pi k\ (\text{charge enclosed})$$

$$\mathbf{n}_1 \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 4\pi k \left(\frac{\text{charge}}{\text{area}}\right) = 4\pi k\ (\text{surface charge density})$$

Thus, the normal component of **E** jumps by an amount $4\pi k$ times the surface charge density as the surface is crossed.

Since $\nabla \cdot \mathbf{B} = 0$, the analogous argument shows that the normal component of **B** is continuous as the surface is crossed.

Now we consider an infinitesimal loop crossing the surface, as in Fig. (D.3). We compute the line integral of **E** around this path, again treating the height ε as negligible compared to the length δ. If $\mathbf{E}_t$ denotes the relevant

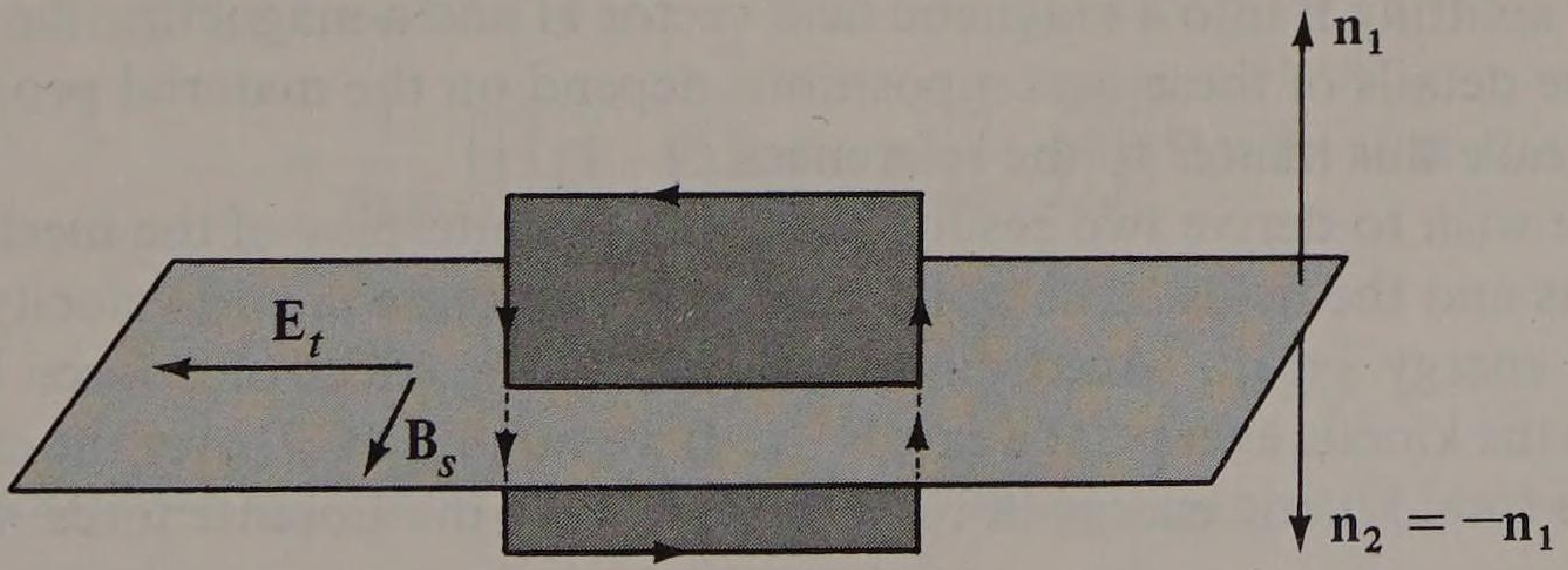


FIGURE D.3

vector component of **E**, we have

$$\oint \mathbf{E} \cdot d\mathbf{r} = (\mathbf{E}_{t_1} - \mathbf{E}_{t_2})\,\delta$$

Applying Stokes' theorem and Eq. (D.29), we have

$$(\mathbf{E}_{t_1} - \mathbf{E}_{t_2})\,\delta = -\eta \iint \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{\partial \mathbf{B}_s}{\partial t}\,\delta\varepsilon$$

where B_s is the indicated component of **B**. Since we are neglecting ε in comparison with δ, this equation implies $E_{t_1} = E_{t_2}$; i.e., the tangential component of **E** is continuous as S is crossed.

If we integrate **B** around the same loop and use (D.27) and Stokes' theorem, we find

$$(B_{t_1} - B_{t_2})\,\delta = 4\pi\gamma(j_s\varepsilon)\,\delta + \frac{\gamma}{k}\frac{\partial E_s}{\partial t}\,\delta\varepsilon$$

Again, we neglect the last term; however, the term $j_s\varepsilon$ might be appreciable if there is a *surface current density*. Checking the orientations, we conclude that the tangential component of **B** jumps by an amount $4\pi\gamma$ times the surface current density, in a direction perpendicular to the latter, as S is crossed.

Summarizing, we introduce σ as the surface charge density and **K** as the surface current density, and we say that the normal and tangential components of **E** and **B** jump by amounts

$$\begin{aligned} \Delta\mathbf{E}_{\text{normal}} &= 4\pi k\sigma \\ \Delta\mathbf{B}_{\text{normal}} &= 0 \\ \Delta\mathbf{E}_{\text{tangential}} &= 0 \\ \Delta\mathbf{B}_{\text{tangential}} &= 4\pi\gamma\mathbf{K} \times \mathbf{n} \end{aligned}$$

as we go from side 2 to side 1.

When there are material media inside the region of interest, the physics of the situation often makes it convenient to distinguish between free and bound charges, and free and bound currents. This is aided by splitting **E** into an electric displacement vector **D** and an electric polarization vector **P**, and by splitting **B** into a magnetic field vector **H** and a magnetization vector **M**. The details of these decompositions depend on the material properties, so we leave this matter to the references.

We wish to derive two results involving the interplay of the mechanical motions and the fields. The point charge q with mass m and velocity **v** has kinetic energy $\frac{1}{2}m|\mathbf{v}|^2$. According to Eq. (C.7), the effect of a force **F** is to change the kinetic energy, at a rate $\mathbf{F} \cdot \mathbf{v}$. If we sum Eq. (C.7) over the charges, call the total kinetic energy $\mathscr{K}$, and replace **F** by the Lorentz force, we find

$$\frac{d\mathscr{K}}{dt} = \sum q(\mathbf{E} + \eta\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = \sum q\mathbf{E} \cdot \mathbf{v}$$

or, in continuous form,

$$\frac{d\mathscr{K}}{dt} = \iiint \rho_m \mathbf{E} \cdot \mathbf{v}\, dV = \iiint \mathbf{E} \cdot \mathbf{j}\, dV$$

Using (D.27) to eliminate j and then invoking (3.28), we find

$$\begin{aligned}\frac{d\mathscr{K}}{dt} &= \iiint \mathbf{E} \cdot \left(\frac{1}{4\pi\gamma} \nabla \times \mathbf{B} - \frac{1}{4\pi k} \frac{\partial \mathbf{E}}{\partial t}\right) dV \\ &= -\frac{1}{4\pi k} \iiint \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}\, dV + \frac{1}{4\pi\gamma} \iiint (\mathbf{B} \cdot \nabla \times \mathbf{E} + \nabla \cdot (\mathbf{B} \times \mathbf{E}))\, dV\end{aligned}$$

Applying (D.29) and the divergence theorem,

$$\frac{d\mathscr{K}}{dt} = -\frac{1}{4\pi k} \iiint \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}\, dV - \frac{\eta}{4\pi\gamma} \iiint \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}\, dV + \frac{1}{4\pi\gamma} \iint_S \mathbf{B} \times \mathbf{E} \cdot d\mathbf{S}$$

where the surface integral is taken over the boundary of the region. Consequently,

$$\frac{d}{dt}\left[\mathscr{K} + \iiint \left(\frac{|\mathbf{E}|^2}{8\pi k} + \frac{\eta |\mathbf{B}|^2}{8\pi\gamma}\right) dV\right] = -\iint_S \frac{\mathbf{E} \times \mathbf{B}}{4\pi\gamma} \cdot d\mathbf{S}$$

This equation leads one to postulate that the electromagnetic field itself has an energy distributed throughout space with a density

$$\frac{|\mathbf{E}|^2}{8\pi k} + \eta \frac{|\mathbf{B}|^2}{8\pi\gamma}$$

and that the energy of the electromechanical system is carried off by the field, with a flux density $\mathscr{P}$ known as the *Poynting vector*:

$$\mathscr{P} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi\gamma}$$

A similar derivation can be carried out for momentum. If **P** denotes the total mechanical momentum, then

$$\begin{aligned}\frac{d\mathbf{P}}{dt} &= \sum \mathbf{F} = \iiint (\rho \mathbf{E} + \eta \mathbf{j} \times \mathbf{B})\, dV \\ &= \frac{1}{4\pi k} \iiint (\nabla \cdot \mathbf{E}) \mathbf{E}\, dV + \frac{\eta}{4\pi\gamma} \iiint (\nabla \times \mathbf{B}) \times \mathbf{B}\, dV \\ &\quad - \frac{\eta}{4\pi k} \iiint \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}\, dV \end{aligned} \tag{D.31}$$

using Maxwell's equations. Invoking the identity

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}$$

and using (D.29) and (D.23), we rewrite (D.31) as

$$\frac{d\mathbf{P}}{dt} + \frac{\eta}{4\pi k}\iiint \frac{\partial(\mathbf{E}\times\mathbf{B})}{\partial t}\,dV = \iiint \frac{1}{4\pi k}[(\nabla\times\mathbf{E})\times\mathbf{E} + (\nabla\cdot\mathbf{E})\mathbf{E}]\,dV$$
$$+ \iiint \frac{\eta}{4\pi\gamma}[(\nabla\times\mathbf{B})\times\mathbf{B} + (\nabla\cdot\mathbf{B})\mathbf{B}]\,dV \tag{D.32}$$

Employing tensor notation, we find

$$\begin{aligned}
[(\nabla\times\mathbf{E})\times\mathbf{E} + (\nabla\cdot\mathbf{E})\mathbf{E}]_i &= \varepsilon_{ijk}\varepsilon_{j\ell m}(\partial_\ell E_m)E_k + (\partial_\ell E_\ell)E_i \\
&= \varepsilon_{kij}\varepsilon_{\ell mj}(\partial_\ell E_m)E_k + (\partial_\ell E_\ell)E_i \\
&= (\delta_{k\ell}\,\delta_{im} - \delta_{km}\,\delta_{i\ell})(\partial_\ell E_m)E_k + (\partial_\ell E_\ell)E_i \\
&= (\partial_\ell E_i)E_\ell - (\partial_i E_m)E_m + (\partial_\ell E_\ell)E_i \\
&= \partial_\ell(E_\ell E_i) - \partial_i\left(\frac{E_m{}^2}{2}\right) \\
&= \partial_\ell\left(E_\ell E_i - \frac{\delta_{i\ell}|\mathbf{E}|^2}{2}\right)
\end{aligned}$$

Hence, the ith component of (D.32) can be expressed

$$\frac{dP_i}{dt} + \frac{\eta}{4\pi k}\iiint \frac{\partial(\mathbf{E}\times\mathbf{B})_i}{\partial t}\,dV = \iiint \partial_\ell T_{\ell i}\,dV \tag{D.33}$$

where $$T_{\ell i} = \frac{E_\ell E_i}{4\pi k} + \eta\frac{B_\ell B_i}{4\pi\gamma} - \delta_{\ell i}\left(\frac{|\mathbf{E}|^2}{8\pi k} + \frac{\eta|\mathbf{B}|^2}{8\pi\gamma}\right)$$

If we think of i as fixed, the right side of (D.33) looks like a divergence; therefore, the momentum equation becomes

$$\frac{dP_i}{dt} + \iiint \frac{\partial}{\partial t}\frac{\eta(\mathbf{E}\times\mathbf{B})_i}{4\pi k}\,dV = \iint_S T_{\ell i}n_\ell\,dS \tag{D.34}$$

where n_ℓ represents the components of the outward unit normal $\mathbf{n}$. The interpretation of Eq. (D.34) is to regard $\eta(\mathbf{E}\times\mathbf{B})/4\pi k$ as momentum stored in the field, and $T_{\ell i}$ as a "flux dyadic" or "stress tensor" giving, componentwise, the flow of momentum flux through the surface S. Elaboration of this *Maxwell stress tensor* will be found in the references.

Exercises

1. In free space with $\rho = 0$, $\mathbf{j} = \mathbf{0}$, show that both $\mathbf{E}$ and $\mathbf{B}$ satisfy the *wave equation*

$$\nabla^2\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} = \frac{\eta\gamma}{k}\frac{\partial^2}{\partial t^2}\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix}$$

2. In many electric conductors the currents and fields obey an experimental law known as *Ohm's law*: $\mathbf{j} = \sigma \mathbf{E}$ where σ is a constant depending on the conductor; it is called the conductivity. If Ohm's law holds and $\rho = 0$, show that both **E** and **B** satisfy the *telegrapher's equation*

$$\nabla^2 \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \frac{\eta\gamma}{k} \frac{\partial^2}{\partial t^2} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} + 4\pi\eta\gamma\sigma \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$$

3. If a wire loop is moved through a magnetic induction field $\mathbf{B}(\mathbf{r})$, the conduction electrons "feel" a force $\eta q \mathbf{v} \times \mathbf{B}$, where $\mathbf{v}$ is the velocity of the wire. However, an observer moving with the wire is unaware of any velocity and postulates that the source of this force is an electric field **E**. Use the flux transport theorem [Eq. (4.72)] to analyze this situation, and derive the relation (D.28) from Faraday's law.

References

1. JACKSON, J.D., *Classical Electrodynamics*, New York: John Wiley and Sons, Inc., 1962.
2. JEFFREYS, H., and JEFFREYS, B.S., *Mathematical Physics*, 3rd edition, New York: Cambridge University Press, 1956.
3. PANOFSKY, Wolfgang K.H., and PHILLIPS, M., *Classical Electricity and Magnetism*, 2nd edition, Reading, Mass.: Addison-Wesley, 1962.

Answers and Notes

Important: Not all the notes given here will be understood by a beginner. Some are intended for graduate students or teachers who may be teaching vector analysis for the first time.

In this book, vectors are represented by bold-faced letters such as **A**, **B**, **C**, . . . Since you cannot conveniently imitate this, the authors suggest that you either underline the letter, $\underline{\mathrm{A}}$, or put an arrow above it, $\vec{\mathrm{A}}$. Be sure to distinguish between the number 0 and the vector **0**.

SECTION 1.1

Note: If the reader has studied modern algebra or logic, he will recognize that a vector is an *equivalence class* of directed line segments. Note that parallel vectors having the same length in feet will also have the same length in meters or centimeters. That is, vector equality is not a metric property; it does not depend on choice of unit of length.

SECTION 1.2

1. Arrow extending from the same initial point and forming the diagonal of the parallelogram determined by the two vectors, as shown in Fig. 1.2.
2. Notice that $\mathbf{C} - \mathbf{A} = \mathbf{C} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{C}$.
3. Yes, the statement is correct (the parallelogram may be "flat").
4. This is easy if you observe that a regular hexagon is composed of six equilateral triangles. (a) $\mathbf{B} - \mathbf{A}$, $-\mathbf{A}$, $-\mathbf{B}$, $\mathbf{A} - \mathbf{B}$, (b) the zero vector.

5. In problems of this kind, think of the vectors as displacements. The displacement **C** can be obtained by first moving backward along **F**, then moving along **E**, then upward in a direction opposite to **D**. Hence, $\mathbf{C} = -\mathbf{F} + \mathbf{E} - \mathbf{D}$.
6. $\mathbf{G} = -\mathbf{K} + \mathbf{C} + \mathbf{D} - \mathbf{E}$.
7. $\mathbf{x} = \mathbf{F} - \mathbf{B} = \mathbf{A}$.
8. $\mathbf{x} = \mathbf{D} - \mathbf{E} - \mathbf{H} = \mathbf{G}$.

Note: This kind of addition was called *geometrical addition* when it was first introduced by Möbius and others over a century ago. Observe that the length of $\mathbf{A} + \mathbf{B}$ does not equal the length of **A** plus the length of **B**. A student once announced happily that he had won a bet in a tavern by showing an instance in which three units added to four units produced five units (see Exercise 4, Sec. 1.4).

SECTION 1.3

1. No, length is never negative.
2. $|4\mathbf{A}| = 12$, $|-2\mathbf{A}| = 6$, $|s\mathbf{A}| \le 6$.
3. $|s\mathbf{A}| = 1$, $|-s\mathbf{A}| = 1$.

Note: If s is a nonzero number and **A** is a vector, the vector $s^{-1}\mathbf{A}$ is sometimes said to be "**A** divided by s". Thus, if we divide a nonzero vector by its own length, we obtain a vector of unit magnitude. This is the point of the first part of Exercise 3.

4. Equals the magnitude of A.
5. No, **A** might be the zero vector.
6. Yes.
7. Not necessarily true, since the vectors may not point in the same direction.
8. Two. Think of the plane as the top of your desk. One of the vectors points upward and the other downward. Many students say, "There are infinitely many." This is incorrect, since we do not distinguish between vectors that are equal.
9. Infinitely many. Think of the line as perpendicular to the xy plane. The unit vector might make any angle θ with the x axis.
10. Two, pointing in opposite directions.
11. $\mathbf{C} = \frac{1}{2}(\mathbf{A} + \mathbf{B})$.
12. $|\mathbf{A}| = |\mathbf{A} - \mathbf{B} + \mathbf{B}| \le |\mathbf{A} - \mathbf{B}| + |\mathbf{B}|$
    Hence $|\mathbf{A}| - |\mathbf{B}| \le |\mathbf{A} - \mathbf{B}|$. If you prefer a less tricky method, draw a diagram and use a well-known theorem in geometry.
13. $a = -2$, $b = c = 1$ is one possible answer. There are others.

SECTION 1.4

Note: I think the only reason some students have trouble with some of these exercises is that they think more is expected of them than simply writing down the answer. When I work one of these problems by drawing a diagram and looking at it, students sometimes say, "Oh, is that all you want?" It is not necessary to use any equations or formulas in giving the answer to a trivial exercise.

1. 1.
2. 0.
3. $\sqrt{2}$.
4. 5.
5. $-\mathbf{i}$, $-\mathbf{j}$, $\frac{1}{2}\sqrt{2}\mathbf{i} + \frac{1}{2}\sqrt{2}\mathbf{j}$.
6. $\mathbf{A} = \mathbf{i} - 3\mathbf{j}$.
7. $A_1 = |\mathbf{A}| \cos \theta$, $A_2 = |\mathbf{A}| \sin \theta$.
8. $A_1 = 3\sqrt{3}$, $A_2 = 3$.
9. (a) $\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}$.
   (b) $\frac{1}{2}\sqrt{3}\mathbf{i} - \frac{1}{2}\mathbf{j}$.
   (c) $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$.
   (d) $\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}$, $\frac{1}{2}\mathbf{i} - \frac{1}{2}\sqrt{3}\mathbf{j}$.
   (e) $\pm(\frac{1}{2}\sqrt{2}\mathbf{i} + \frac{1}{2}\sqrt{2}\mathbf{j})$.
10. 10, 3, $\sqrt{1 + s^2}$, 1.
11. $2\mathbf{i} + 6\mathbf{j}$.

SECTION 1.5

1. 5, 3, 5.
2. $5\mathbf{i} + 6\mathbf{j} - \mathbf{k}$, $4\mathbf{j} + 4\mathbf{k}$.
3. $4\sqrt{2}$.
4. $\pm\frac{1}{3}$.
5. $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$.
6. (a) $4\sqrt{2}$. (b) yz plane.
7. $\cos \alpha = \frac{3}{5}$.
8. $\pm\mathbf{j}$.
9. $\sqrt{3}$.
10. $\mathbf{i} - 5\mathbf{j} - \mathbf{k}$.
11. $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
12. $s = 2$, $t = 3$, $r = -1$.
13. $\frac{2}{3}$, $-\frac{2}{3}$, $\frac{1}{3}$.
14. Use the pythagorean theorem.
15. Cone concentric with the positive x axis.
16. Two.
17. $-\cos \alpha$, $\cos \beta$, $\cos \gamma$.
18. $\pm\frac{1}{3}\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

SECTION 1.6

1. $2\mathbf{i} - 5\mathbf{j} - 8\mathbf{k}$.
2. $\mathbf{i} + 2\mathbf{j} + 9\mathbf{k}$.
3. $32\mathbf{j} - 26\mathbf{k}$.
4. 10 miles.
5. 7 pounds.

SECTION 1.7

2. If $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 0$ and $\mathbf{A} = -\mathbf{C}$ then $\mathbf{B} = -\mathbf{D}$.
3. *Hint*: Let the sides be $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{B} - \mathbf{A}$. If parallel to $\mathbf{A}$, the line segment is $-\frac{1}{2}\mathbf{B} + \mathbf{A} + \frac{1}{2}(\mathbf{B} - \mathbf{A}) = \frac{1}{2}\mathbf{A}$.
4. Use the technique illustrated in Example 1.3 of the text.
6. $\cos^{-1}(-\frac{2}{15})$.
7. $\cos^{-1}(\frac{1}{3}\sqrt{3})$.

8. $\cos^{-1}\frac{1}{41}\sqrt{1435}$, $\cos^{-1}\frac{1}{41}\sqrt{246}$.

9. $90^\circ - \cos^{-1}\frac{1}{3}$.

10. $(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) = 0$ if and only if $x+y+z=0$. Hence, $\theta = 90^\circ$ if, and only if, $x+y+z=0$.

11. True.

12. True.

13. True.

14. False (radius is 3).

15. $(x-2)^2+(y-3)^2+(z-4)^2=9$.

16. $x^2+y^2=4$.

17. Line.

18. y axis.

19. The single point $(2,-3,4)$.

20. The three coordinate planes.

21. 5.

22. 8.

23. 1.

24. Cone of two sheets concentric with z axis.

25. Ellipsoid.

SECTION 1.8

1. $x=3t,\ y=-2t,\ z=7t$.

2. $x=1,\ y=2$.

3. $y=2,\ z=3$.

4. $\pm(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j})$.

5. $\pm(\frac{6}{7}\mathbf{i}+\frac{3}{7}\mathbf{j}+\frac{2}{7}\mathbf{k})$.

6. $\pm(\frac{3}{19}\sqrt{19}\mathbf{i}-\frac{3}{19}\sqrt{19}\mathbf{j}+\frac{1}{19}\sqrt{19}\mathbf{k})$.

7. $x=\frac{1}{4}y=-z$.

8. $x=3,\ y=4$.

9. $x-1=-\frac{1}{2}(y-4)=\frac{1}{8}(z+1)$. This may be written in other forms.

10. $\frac{1}{7}\sqrt{42}$

11. $\cos^{-1}\frac{3}{70}\sqrt{42}$, about 74°.

14. (a) $0<\lambda<\infty$; $-1<\lambda<0$; $-\infty<\lambda<-1$.

15. (a) $(2,2,3)$.
   (b) The lines coincide.
   (c) No intersection (parallel lines).
   (d) No intersection.

SECTION 1.9

1. 19.

2. $8+27-12=23$.

3. 20.

4. $\cos^{-1}\frac{2}{15}$.

5. $\cos^{-1}\frac{3}{5}$.

6. -2.

7. $\frac{10}{3}$.

8. $\sqrt{2}$.

9. $\frac{15}{13}\sqrt{26}$.

10. $\sqrt{5}\mathbf{i}+\sqrt{5}\mathbf{j}$.

11. Nothing. But $\mathbf{A}=\mathbf{0}$.

13. (a) Circle with diameter $|\mathbf{A}|$.
   (b) Sphere with diameter $|\mathbf{A}|$.

14. $|\sin\frac{1}{2}\theta|$.

17. Expand $(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})+(\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})$.

19. (a) $-(\mathbf{i}+\mathbf{j}+\mathbf{k})+(7\mathbf{i}-2\mathbf{j}-5\mathbf{k})$
   (b) $3(2\mathbf{i}-\mathbf{j}-2\mathbf{k})+\mathbf{0}$
   (c) $\mathbf{0}+(6\mathbf{i}-3\mathbf{j}-6\mathbf{k})$.

20. $-\frac{214}{49}\mathbf{i}+\frac{37}{49}\mathbf{j}-\frac{330}{49}\mathbf{k}$.

SECTION 1.10

Note: Quite often we speak of *the* equation of a plane where it would be better to speak of *an* equation, since distinct equations may represent the same plane. For example, $x + y + 2z = 3$ and $2x + 2y + 4z = 6$ both represent the same plane.

1. (a) $\pm(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k})$.
   (b) $\pm(\frac{1}{2}\sqrt{2}\mathbf{i} - \frac{1}{2}\sqrt{2}\mathbf{k})$.
   (c) $\pm(-\frac{1}{37}\sqrt{37}\mathbf{j} + \frac{6}{37}\sqrt{37}\mathbf{k})$.
   (d) $\pm\mathbf{i}$.
   (e) $\pm(\frac{1}{2}\sqrt{2}\mathbf{j} - \frac{1}{2}\sqrt{2}\mathbf{k})$.
   (f) $\pm(\frac{1}{2}\sqrt{2}\mathbf{i} - \frac{1}{2}\sqrt{2}\mathbf{j})$.

2. $x - 4y + z = 0$.

3. $2x - 2y + z + 3 = 0$.

4. $3x + y - z = 3$.

5. No.

6. $\frac{16}{3}$.

7. (a) $\sqrt{14}$. (b) $3\sqrt{2}$. (c) 2.

8. $\frac{1}{3}\sqrt{3}$.

9. $(\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \cdot (2\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}) = 0$.

10. $\sin^{-1} \frac{5}{9}\sqrt{3}$, about 74°.

11. $90° - \cos^{-1} \frac{5}{9}\sqrt{3}$.

12. $3x - y = C,\ z = 0$.

13. $\frac{5}{2}\sqrt{2}$.

14. $3x + 2y = 11,\ z = 0$.

15. $2x + 2y + z = 4$.

16. $\pm\frac{1}{3}$.

17. (a) The point $(-2,1,5)$.
   (b) No intersection.
   (c) The line $x = y + 3 = -\frac{1}{2}z$.
   (d) No intersection.

SECTION 1.11

Note to instructor: A k-dimensional vector space (or k-dimensional subspace) is oriented by selecting a linearly independent ordered set consisting of k vectors. Any other such linearly independent ordered set is said to have "positive" orientation if it can be obtained from the given set in the proper order by a linear transformation with positive determinant. If an n-dimensional space has been oriented, and if also an $(n - 1)$-dimensional subspace of the same space is oriented by an ordered set $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{n-1}$, then the same orientation of the subspace can be prescribed just as well by selecting a single vector $\mathbf{C}$ not in the subspace, using the following convention: the ordered set $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{n-1}, \mathbf{C}$ must have positive orientation.

1. Numerically they are equal to the areas of the projections of the area on the coordinate planes.

SECTION 1.12

1. (a) $2\mathbf{i} + 14\mathbf{j} + 4\mathbf{k}$.
   (b) $-8\mathbf{i} + 23\mathbf{j} - \mathbf{k}$.
   (c) $-11\mathbf{i} - 6\mathbf{j} + \mathbf{k}$.
   (d) $\mathbf{k}$.
   (e) $\mathbf{j} - \mathbf{i}$.

2. $\sqrt{26}$.

3. $\frac{1}{2}\sqrt{61}$.

4. **0**. **A** and **B** are parallel.

5. $\pm(\frac{1}{11}\sqrt{11}\mathbf{i} - \frac{3}{11}\sqrt{11}\mathbf{j} + \frac{1}{11}\sqrt{11}\mathbf{k})$.

6. $\frac{1}{8}(x - 2) = -\frac{1}{13}(y - 3) = -\frac{1}{3}(z - 7)$.

7. $x = -\frac{1}{4}y = \frac{1}{3}z$.

8. $-64\mathbf{j} + 16\mathbf{k}$, $16\mathbf{i} - 16\mathbf{j} + 16\mathbf{k}$. No.

9. $17x - y + 9z = 43$.

10. $\pm\frac{1}{25}\sqrt{5}(5\mathbf{i}+6\mathbf{j}-8\mathbf{k})$.

11. $\sin(\psi-\theta)=\sin\psi\cos\theta-\cos\psi\sin\theta$.

13. $\sqrt{65}/\sqrt{26}$.

14. $r=3,\ s=-\frac{27}{2}$.

15. One of them is zero.

16. $\pm 8\mathbf{i}$.

17. (a) No.

(b) $\frac{1}{2}x-\frac{52}{7}=-\frac{1}{4}y+\frac{52}{21}=z-\frac{208}{21}$. (This answer can be written in many other ways, so don't be discouraged if your answer differs from this in appearance.)

(c) $4/\sqrt{21}$.

18. $x^2+y^2+z^2-xy-yz-zx=2$; a cylinder of radius $\frac{2}{3}\sqrt{3}$ ft.

SECTION 1.13

1. (a) 30. (b) -13. (c) 5. (d) 1.

2. 5.

3. 0.

4. $\frac{2}{3}$.

5. 1.

6. $3x-17y-4z=0$.

7. $3x-7y+z=-20$.

8. (a) Their triple scalar product is zero. Alternatively, all three are perpendicular to $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

(b) $x+y+z=0$.

9. (a) $C_3=2$. (c) Draw a diagram.

10. $\frac{2}{19}\sqrt{38}$.

11. Yes.

12. They are coplanar.

13. (a) Compare $\mathbf{A}\cdot\mathbf{i}$ and $\mathbf{A}\cdot\mathbf{u}$.

(b) $\mathbf{A}$.

15. Only (a), (b), (c), (g), and (h) have meaning.

SECTION 1.14

5. $(\boldsymbol{\omega}\cdot\mathbf{R})\boldsymbol{\omega}-(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{R}$.

6. No.

7. 0.

SECTION 1.15

These exercises appeared already in Sec. 1.14.

SECTION 2.1

Note: Since different rules can be used to define the same function, the definition of vector function given in the text is now regarded as old-fashioned. The more modern way is to define a function as a set of pairs determined by some rule. For further details on the "new mathematics" listen to the appropriate Tom Lehrer record; most mathematicians find the old-fashioned definition quite adequate.

1. (a) $\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$.
  (b) True since $\mathbf{k} \cdot \mathbf{F}'(t) = 0$.
  (c) $t = n\pi$
    $(n = 0, \pm 1, \pm 2, \ldots)$.
  (d) Yes, $\sqrt{2}$.
  (e) Yes, 1.
  (f) $-\sin t\,\mathbf{i} - \cos t\,\mathbf{j}$.

2. (a) $3\mathbf{i} + 3t^2\mathbf{j}$.
  (b) $\cos t\mathbf{i} - e^{-t}\mathbf{j}$.
  (c) $-2t\mathbf{i} + (e^t + 5t^4)\mathbf{j} + (e^t - 3t^2)\mathbf{k}$.
  (d) $(\cos t + 3t^2)(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.
  (e) $\mathbf{0}$.

3. (a) $6t - 10t \sin t - 5t^2 \cos t$.
  (b) $(8t\sqrt{8t^2 + 1})/(8t^2 + 1)$.
  (c) $1 - 12t^3$.

4. Use Theorem 2.4, noting that one term vanishes in this case.

5. (a) 7.
  (b) 33.
  (c) $8\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$.
  (d) 0.
  (e) -2.
  (f) $\frac{3}{5}\mathbf{i} + \frac{2}{5}\mathbf{j} + \frac{6}{5}\mathbf{k}$.
  (g) $-42\mathbf{i} + 66\mathbf{j} - \mathbf{k}$.
  (h) $\mathbf{B}$.
  (i) $\mathbf{B} \times \mathbf{C}$.

SECTION 2.2

1. No, the tangent may be parallel to the y axis.

2. The arc $t^2\mathbf{i} + t^3\mathbf{j}$ has a cusp at $t = 0$. Physically, a particle can follow a curve with a sharp corner by decelerating to zero speed at the corner, then resuming with a different direction of velocity.

3. $x^2 - y^2 = 1$, $z = 0$.

4. At (0,0,0), corresponding to $t = 0$.

5. $(\mathbf{i} + 2\pi\mathbf{j})/\sqrt{1 + 4\pi^2}$.

6. Along a straight line, $\mathbf{T}$ is constant.

7. (a) $\int_0^1 \sqrt{14}\,dt = \sqrt{14}$. (b) Distance between points is $\sqrt{14}$, and the path is straight.

8. (a) $\sqrt{2}(e - 1)$. (b) $x = \dfrac{s + \sqrt{2}}{\sqrt{2}} \cos \log\left(\dfrac{s + \sqrt{2}}{\sqrt{2}}\right)$,

$$y = \frac{s + \sqrt{2}}{\sqrt{2}} \sin \log\left(\frac{s + \sqrt{2}}{\sqrt{2}}\right),\ z = 0.$$

9. (a) $2\sqrt{5}\pi^2$.
  (b) $\frac{1}{5}\sqrt{5}(\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 2\mathbf{k})$.
  (c) $\frac{1}{5}\sqrt{5}(-\mathbf{j} + 2\mathbf{k})$.

10. $\mathbf{i}$.

SECTION 2.3

1. (a) $\sqrt{2}e^t$.
  (b) $a_t = \sqrt{2}e^t$, $a_n = \sqrt{2}e^t$.
  (c) $\frac{1}{2}\sqrt{2}[(\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j}]$.
  (d) $\frac{1}{2}\sqrt{2}e^{-t}$.

2. (a) $\sqrt{9t^2+25}$.

(b) $9t/\sqrt{9t^2+25}$, $[9(t^2+4)-81t^2/(9t^2+25)]^{\frac{1}{2}}$.

(c) $\dfrac{3(\cos t - t\sin t)\mathbf{i} + 3(\sin t + t\cos t)\mathbf{j} + 4\mathbf{k}}{\sqrt{9t^2+25}}$.

(d) $3(9t^4+52t^2+100)^{\frac{1}{2}}/(9t^2+25)^{\frac{3}{2}}$.

3. (a) $\sqrt{3}e^t$.

(b) $a_t = \sqrt{3}e^t$, $a_n = \sqrt{2}e^t$.

(c) $\frac{1}{3}\sqrt{3}[(\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j} + \mathbf{k}]$.

(d) $\frac{1}{3}\sqrt{2}e^{-t}$.

4. (a) $10\sqrt{5}$.

(b) $a_t = 0$, $a_n = 80$.

(c) $\frac{1}{5}\sqrt{5}(2\cos 4t\,\mathbf{i} - 2\sin 4t\,\mathbf{j} + \mathbf{k})$.

(d) $\frac{4}{25}$.

5. (a) $v = \frac{3}{2}$.

(b) $\mathbf{a} = -\cos t(\mathbf{i}-\mathbf{j}) - \sin t(\mathbf{i}+\mathbf{j})$.

(c) $-\frac{2}{3}\sin t(\mathbf{i}-\mathbf{j}) + \frac{2}{3}\cos t(\mathbf{i}+\mathbf{j}) + \frac{1}{3}\mathbf{k}$.

(d) $k = 4\sqrt{2}/9$

6. $\dfrac{2(3t^4+2t^3-3t^2-2t+2)^{\frac{1}{2}}}{3(2t^4-4t^3+10t^2+1)^{\frac{3}{2}}}$

7. $\frac{1}{2}$; $-\frac{1}{2}$.

8. 6, $\frac{1}{2}$, 0, circle of radius 2 in the plane $x = y$.

9. $\mathbf{F} \times \dfrac{d\mathbf{F}}{dt} \cdot \dfrac{d^3\mathbf{F}}{dt^3}$.

10. (a) 1.

(b) 0.

(c) $a_t = d^2s/dt^2$.

(d) 0.

(e) ds/dt.

(f) τ.

(g) 1.

(h) k.

(i) $-\tau\mathbf{N}$.

13. (a) False. (b) False. (c) True.

SECTION 2.4

1. $$\left[\frac{d^3r}{dt^3} - 3\frac{dr}{dt}\left(\frac{d\theta}{dt}\right)^2 - 3r\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}\right]\mathbf{u}_r + \left[3\frac{d^2r}{dt^2}\frac{d\theta}{dt} + 3\frac{dr}{dt}\frac{d^2\theta}{dt^2} + r\frac{d^3\theta}{dt^3} - r\left(\frac{d\theta}{dt}\right)^3\right]\mathbf{u}_\theta.$$

3. $\mathbf{v} = 4b[(\sin\theta)\mathbf{u}_r + (1-\cos\theta)\mathbf{u}_\theta]$

$\mathbf{a} = 16b[(2\cos\theta - 1)\mathbf{u}_r + (2\sin\theta)\mathbf{u}_\theta]$.

4. $\mathbf{v} = b[(\cos t)\mathbf{u}_r - e^{-t}(1+\sin t)\mathbf{u}_\theta]$

$\mathbf{a} = b([-\sin t - e^{-2t}(1+\sin t)]\mathbf{u}_r + e^{-t}[1+\sin t - 2\cos t]\mathbf{u}_\theta)$.

5. If the particle is moving parallel to the field no force will be exerted. (In elementary books it is sometimes stated that the force is proportional to the rate at which the particle "cuts" the lines of flow.)

6. $v/r\,|\mathbf{B}|$. [$qv\,|\mathbf{B}|$ must equal the component a_n discussed in Sec. 2.3.]
   (a) The second term.
   (b) The second and third terms.
   (c) All are nonzero.
   (d) Many possibilities.

8. Yes, except when its velocity is zero.

9. (a) $\pi^2 r$ cm/sec^2 (if r is in cm) directed towards the center. Note that 30 rev/min = π rad/sec.
   (b) $4\pi\mathbf{u}_\theta$ cm/sec.

10. 24π, since $dr/dt = 3$ and $d\theta/dt = 4\pi$.

11. $E = \dfrac{m}{2}\left(\dfrac{dr}{dt}\right)^2 + \dfrac{m}{2}\left(r\dfrac{d\theta}{dt}\right)^2 - \displaystyle\int_0^t \left(F_r\dfrac{dr}{dt} + F_\theta r\dfrac{d\theta}{dt}\right)dt.$ Now differentiate and use $\dfrac{d}{dt}\displaystyle\int_0^t f(t)\,dt = f(t).$

SECTION 3.1

1. (a) $(\cos x + ye^{xy})\mathbf{i} + xe^{xy}\mathbf{j} + \mathbf{k}$. (b) $-\mathbf{R}/|\mathbf{R}|^3$. (c) $\mathbf{k}$.

2. yz plane, where $x = 0$.

3. f depends only on y.

4. $f(x,y,z) = x^2 + yz + C$.

5. Unit vector directed away from the z axis, except at points on the z axis, where it is not defined.

6. (a) 0. (b) $-\frac{4}{3}$.

7. (a) $\frac{5}{3}$. (b) $-\frac{2}{3}$. (c) $-\frac{28}{3}$. (d) $\frac{1}{42}\sqrt{14}$.

8. (a) 10.
   (b) The maximum rate of increase of r^2 is in the direction of $\mathbf{R}$, where

$$\frac{d}{ds}(r^2) = \frac{d}{dr}(r^2) = 2r$$

which equals 10 at (3,0,4).

9. $150\sqrt{5}$. This function equals s^6, where s is the distance to the y axis. We have $(d/ds)(s^6) = 6s^5 = 150\sqrt{5}$ at this point.

10. Any scalar multiple of $4\mathbf{i} + \mathbf{j} + \mathbf{k}$.

11. $2x + 4y - z = 21$.

12. (a) From your diagram you see that any scalar multiple of $\mathbf{i} + \mathbf{k}$ will do.
   (b) $4\mathbf{i} + 4\mathbf{k}$.

13. $x + 2y - 8z = -28$.

14. $x = y$, $z = 0$.

15. $\pm\frac{1}{14}\sqrt{14}(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.

16. $4x + 6y - z = 13$.

17. $\pm\frac{1}{2}\sqrt{2}(\mathbf{i} - \mathbf{j})$. In (c), let $\mathbf{R} = 2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sqrt{5}\mathbf{k}$.

18. $\cos^{-1}\frac{31}{32}$.

19. $\sin^{-1}\frac{2}{3}\sqrt{2}$.

SECTION 3.2

1. See Fig. 3.3.
2. (a) $x(z + a) = -1$, $y(z + b) = -1$.
   (b) $x(z - 3) = -1$, $y(z - 3) = -1$.
3. Half lines extending from the origin.
4. The gradient is normal to these surfaces.

SECTION 3.3

1. $ye^{xy} + x \cos xy - 2x \cos zx \sin zx$.
2. 3.
3. $6y^3z + 18x^2yz$.
4. Zero except at the origin, where the field is not defined. The magnitude of this field at any point is $1/r^2$, so this field can be thought of as the electric field intensity due to a charge of suitably chosen magnitude at the origin. A physicist or electrical engineer might say that the divergence is "infinity" at the origin, since the divergence of an electrostatic field is proportional to the charge density, and the charge density at a point charge is "infinity".
6. Let $\mathbf{F} \cdot \mathbf{grad}\ \phi = 0$.
7. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ is one example.
8. There are infinitely many possible answers, for example $\mathbf{F} = -x\mathbf{i}$.
9. Again there are infinitely many acceptable answers. Two of them are $e^x\mathbf{i}$ and $e^x\mathbf{i} + ye^x\mathbf{j}$.
10. False (e.g., a constant field).
11. Divergence is zero everywhere, since $\partial F_1/\partial x = 0$, $F_2 = 0$, and (we assume) $F_3 = 0$. Some students observe that $\mathbf{F} = Cy\mathbf{i}$ for some constant C, and then compute the answer using the formula for the divergence. This is clever, but not the point of the exercise.
12. Divergence is zero everywhere. For example, consider point P. Along the x axis, $F_1 = 0$, so $\partial F_1/\partial x = 0$ at P. As we move through P along the flow line indicated, F_2 takes on its maximum value $|\mathbf{F}|$, therefore $\partial F_2/\partial s = 0$ at P, where s is measured along the flow line. But at point P we are moving parallel to the y axis, so $\partial F_2/\partial y = \partial F_2/\partial s$ at P, hence is zero at this point. Another method: Conjecture that $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ and use the formula.

SECTION 3.4

1. $x\mathbf{i} - y\mathbf{j} + y(1 - 2x)\mathbf{k}$.
2. $-z^2 \sin yz^2\, \mathbf{i} + (y \cos xy - xe^{xy})\mathbf{k}$.
3. $-(y^2 + z^2)\mathbf{i} + 2zx\mathbf{j}$.
4. (a) $1 + z^2 + x + y$.
   (b) $z\mathbf{i} + 2xz\mathbf{j} + y\mathbf{k}$.

5. The paddle wheel will not tend to rotate.

6. Think of the velocity field of a fluid swirling about the x axis. Assume constant angular velocity $\boldsymbol{\omega}$. The $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R}$, and since $\mathbf{curl\ F} = 2\boldsymbol{\omega}$ as stated in the text (to be proved later) we have $\boldsymbol{\omega} = \mathbf{i}$ and

$$\mathbf{v} = \mathbf{i} \times \mathbf{R} = \mathbf{i} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = y\mathbf{k} - z\mathbf{j}$$

This is one possible answer. Another is $2y\mathbf{k}$, which represents a shearing motion parallel to the xz plane.

7. No (Fig. 3.13).

8. No.

SECTION 3.5

1. 16.

2. $12\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

3. 64.

4. (a) $2xy + 1$.
   (b) $-2\mathbf{i} + \mathbf{j} - x^2\mathbf{k}$.
   (c) $2y\mathbf{i} + 2x\mathbf{j}$.

5. Scalar field.

6. Vector field.

7. 3, $\mathbf{0}$.

8. $(x^2 + z^2)e^{xz}$.

9. Always $\mathbf{0}$.

10. Always 0.

SECTION 3.6

1. $20x^3yz^3 + 6x^5yz$.

2. 0 except at the origin.

3. $-2yz^2(y^2z^2 + 3x^2z^2 + 6x^2y^2)\mathbf{k}$.

4. (a) and (b). Also (c) provided that $p^2 = q^2$.

5. (a) Vector field.
   (b) Scalar field.
   (c) Vector field.
   (d) Scalar field.
   (e) Zero vector field.
   (f) Meaningless.
   (g) Vector field.
   (h) Vector field.
   (i) Meaningless.
   (j) Vector field.

6. (b) $\dfrac{\sin x \sinh y}{\sinh 5} + \dfrac{\sin 2x \sinh 2y}{\sinh 10}$.

SECTION 3.7

5. As written, the right side is symmetrical in $\mathbf{F}$ and $\mathbf{G}$, but the left side is not, since $\mathbf{F} \times \mathbf{G} \neq \mathbf{G} \times \mathbf{F}$.

SECTION 4.1

1. (a) $\frac{1}{2}\sqrt{2}(\mathbf{i} + \mathbf{j})$.
   (b) $\mathbf{i}$.
   (c) $-\mathbf{j}$.

2. (a) $\sqrt{2}\,dx$ or $\sqrt{2}\,dy$.
   (b) dx.
   (c) $-dy$.

3. (a) $d\mathbf{R} = dx\,\mathbf{i} + dy\,\mathbf{j} = dx\,\mathbf{i} + dx\,\mathbf{j} = (\frac{1}{2}\sqrt{2}\mathbf{i} + \frac{1}{2}\sqrt{2}\mathbf{j})\sqrt{2}\,dx = \mathbf{T}\,ds.$
   (b) $d\mathbf{R} = dx\,\mathbf{i} + dy\,\mathbf{j} = \mathbf{i}\,dx = \mathbf{T}\,ds.$
   (c) $d\mathbf{R} = dx\,\mathbf{i} + dy\,\mathbf{j} = dy\,\mathbf{j} = \mathbf{T}\,ds.$
4. (a) Along this path, $\mathbf{F} = \sqrt{1-x^2}\;\mathbf{i} - x\mathbf{j}$ and

$$d\mathbf{R} = dx\,\mathbf{i} - \frac{x\,dx}{\sqrt{1-x^2}}\,\mathbf{j}$$

so

$$\mathbf{F}\cdot d\mathbf{R} = \frac{dx}{\sqrt{1-x^2}} \qquad \text{and} \qquad \int \mathbf{F}\cdot d\mathbf{R} = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi$$

   (b) π.
5. $\mathbf{F}\cdot d\mathbf{R} = -d(\tan^{-1} y/x) = -d\theta.$
6. (a) 8. (b) 8.
7. 36. (*Caution*: $\mathbf{R}\cdot d\mathbf{R} = s\,ds$ in this case because the points are collinear with the origin.)
8. $\pm 8\pi$, depending on direction.
9. 40. (This can also be done by observing that $\mathbf{F}\cdot d\mathbf{R} = d\phi$ where $\phi = x^2y + zy$, so that the integral is $\phi(3,4,1) - \phi(1,0,2)$. See Sec. 4.3 for further discussion of this "trick".)
10. Zero.
11. $\frac{41}{6}$.
12. (a) 0. (**F** is perpendicular to $d\mathbf{R}$.)
    (b) $\frac{4}{3}(\omega_1 - \omega_2)$.

SECTION 4.2

Note: In this book, any set of points is a *region* and a region is a *domain* if and only if it is open and connected. In some books other conventions are used; there is no standard agreement: for example, some books use *domain* to mean *domain of definition* and those domains that are open and connected are called regions.

1. Domain, not simply connected.
2. Simply connected domain.
3. Simply connected domain.
4. Not a domain. (Points on the plane $z = 0$ are not interior.)
5. Simply connected domain.
6. Domain, not simply connected.
7. Simply connected domain.
8. Not a domain (not connected).

SECTION 4.3

1. The integral over C equals that over C_1 minus that over C_2, so if the first of these is zero the other two are equal.

2. Many possibilities.
3. Many possibilities.
4. 2π or -2π, depending on which way the circle is oriented.
5. ϕ is a multiple-valued function, and hence not a scalar field as we have defined it.
6. $\phi = yx + \sin xz + C$.
7. $\mathbf{F} = \mathbf{grad}\,\phi$ where $\phi = x^2y + yz$.

SECTION 4.4

1. (a) Conservative, $\phi = 6x^2y + xyz + C$.
   (b) Conservative, $\phi = e^{xz} + C$.
   (c) Conservative, $\phi = -\cos x + \frac{1}{3}y^3 + e^z + C$.
   (d) Not conservative.
   (e) Conservative, $\phi = \ln(x^2 + y^2) + z^2$.
2. (e), since the domain of definition is not simply connected. You must explicitly construct ϕ.
3. Yes. $\phi + \psi$.
6. $\phi(1,2,3) = -\frac{1}{14}\sqrt{14}$ and $\phi(2,3,5) = -\frac{1}{38}\sqrt{38}$; hence the work done is

$$\phi(2,3,5) - \phi(1,2,3) = \tfrac{1}{14}\sqrt{14} - \tfrac{1}{38}\sqrt{38}$$

7. No, provided the path avoids the origin.

Note: Conservative fields are sometimes called *potential fields.* The term *irrotational* is also used. It is not possible for a flow line of such a field to be a closed curve, for the integral of a field about a closed flow line is nonzero, and this would contradict (*ii*). Therefore the flow lines either have no endpoints (i.e., if they "extend to infinity" in both directions) or perhaps they start at a point (called the "source") and perhaps end at another point (called the "sink"). For this reason, such fields are also called *source fields.* A simple example is the electrostatic field due to a positive point charge at the origin. The origin is the "source" and the flow lines extend radially away from the origin.

SECTION 4.5

2. $-\frac{1}{2}x^2\mathbf{k}$.
3. If $\mathbf{F} = \nabla\phi$, a vector potential is given by $\phi\mathbf{G}$.

SECTION 4.6

This section makes no pretense to rigor.

3. (a) $\frac{1}{3}\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$.
   (b) $\mathbf{k} \cdot \mathbf{n} = \frac{1}{3}\sqrt{3}$.
   (c) $\displaystyle\int_0^1 \int_0^{1-y} \frac{dx\,dy}{|\cos\gamma|}$.
   (d) $\frac{1}{2}\sqrt{3}$.

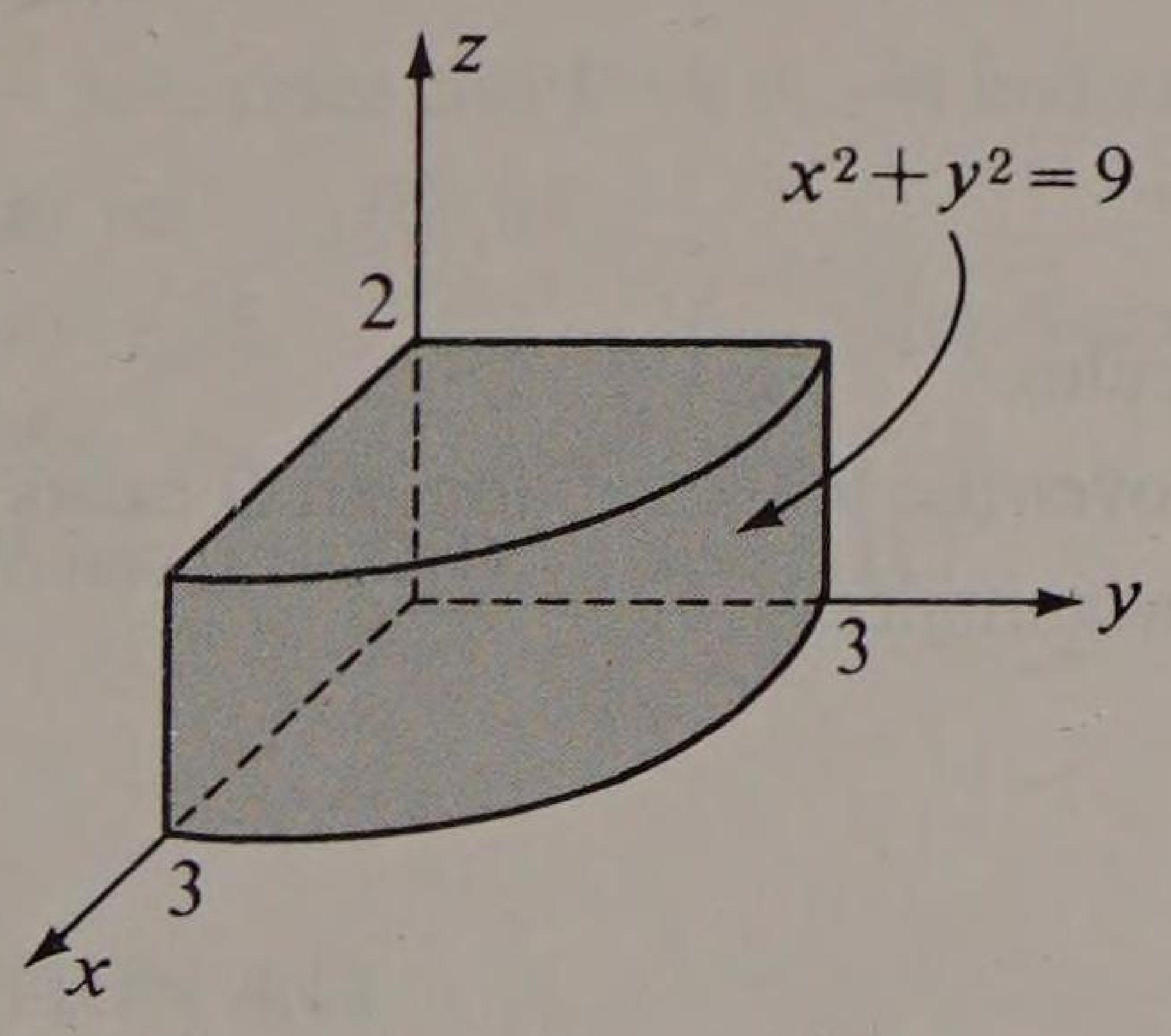
6. $\sqrt{4x^2 + 4y^2 + 1}\,dx\,dy$.
7. 11.

SECTION 4.7

1. 18π.
2. (a) 8.
   (b) 16.
   (c) 24.
   (d) 0.
   (e) 0.
   (f) 0.
   (g) 0.
3. $\int_0^1 \int_0^{2-2x} \frac{7}{2} \cdot \frac{6}{7} x\,dy\,dx = 1$.
4. $|\mathbf{E}| = \lambda/2\pi\varepsilon_0 r$.
5. (a) $T_r = T_a + \dfrac{1/r - 1/a}{1/b - 1/a}(T_b - T_a)$.
   (b) No.
6. Zero.
7. $3\pi a^2$.
8. -1.
9. (a) No.
   (b) Yes.
   (c) 4.
10. $13\frac{7}{8}$.
11. The term *source* is used rather than *sink* in Exercise 11 because in electrostatics it is conventional to take the electric field to be the *negative* of the potential.
    (a) By Gauss's law and symmetry, $\mathbf{F} = \mathbf{0}$ within the sphere so ϕ is constant within the sphere, and at the center $r = a$ is a constant so $\phi = q/a = 4\pi a^2\sigma/a$.
    (b) By Gauss's law and symmetry, the electric field outside the sphere is the same as that due to a point charge of magnitude $4\pi a^2\sigma$ located at the center.
12. (a) 8π. (Point is within sphere.)
    (b) $\frac{16}{3}\pi$. (Point is outside sphere.)
13. 20π.

SECTION 4.8

3.



4. (a) 3.
   (b) 3.
   (e) This will be discussed later.

5. $3v$.

6. $\pi(1 - e^{-1})$.

7. $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$ at each point in space. Here we assume charge to be distributed continuously, i.e., no point charges in the domain.

8. (a) $\frac{22}{3}\pi$. (b) $\frac{32}{9}\pi$.

9. $\frac{32}{255}\sqrt{85}\pi$.

SECTION 4.9

6. (a) 0. (d) 0.
   (b) -2. (e) -1.
   (c) 4.

7. The divergence is identically zero, so the desired integral equals the negative of the integral over the missing top, which, in this case, is trivial to compute.

8. The field is $\frac{1}{3}x^3\mathbf{i} + \frac{1}{3}y^3\mathbf{j}$.

10. $\pm 2\pi$, depending on direction of integration.

11. (a) 16.8 if volume is proportional to the cube of the minimal diameter.
    (b) Yes.

12. $5\varepsilon_0$.

13. (a) $4\pi b^4$.
    (b) To avoid a triple integral, take $dV = 4\pi r^2\,dr$, so that the integral is

$$\int_0^b 16\pi r^3\,dr$$

14. 8π.

16. 0.

17. (a) Use (3.20). 18. (a) 108π.
    (b) $\frac{928}{3}\pi$. (b) 1944π.

[Exercise 17(a) does not apply in part (b) since this function is not harmonic.]

SECTION 4.10

1. In applying the fundamental theorem of calculus.

2. To ensure that the volume integral of div **F** over the bounded domain D exists.

5. $\cos\gamma = 0$, so the expression $dx\,dy/|\cos\gamma|$ is meaningless.

6. 30.

7. $5v$.

8. Yes.

9. (a) An outer sphere with **n** pointing away from the origin and an inner sphere with **n** pointing towards the origin.
   (b) Sum of two integrals.

(c) They are equal.
(d) No.
(e) 4π.

11. (b) The lumpiness equals the negative of the laplacian.
(c) The lumpiness is zero.

12. *Second hint*: In steady state, rate of heat flow out of a domain equals the rate of heat flow into the domain; otherwise the temperature would be changing. Hence,

$$\iint \frac{\partial \phi}{\partial n}\, dS = 0$$

over arbitrary closed surfaces. Also note that the limit, as $V \to 0$, of

$$\frac{1}{V} \iiint_D \nabla^2 \phi \, dV$$

as the domain D shrinks down to a point, is the value of $\nabla^2\phi$ at that point, if $\nabla^2\phi$ is continuous.

13. $c\rho(\partial\phi/\partial t) = k\nabla^2\phi$.
Note: These derivations can be placed on a more rigorous level by making use of the theorem that if f is continuous and

$$\iiint f(x, y, z)\, dV = 0$$

for every domain D, then f is identically zero. For instance, in Exercise 13 this theorem is used, taking

$$f = c\rho \frac{\partial \phi}{\partial t} - k\nabla^2\phi$$

14. $-4\pi\phi(P)$.

15. They are equal.

17. $4\pi\phi(0,0,0) = 20\pi$.

18. (a) $-4\pi\phi(0,1,0) = -20\pi$.
(b) $-4\pi\phi(2,1,3) = 0$.

19. Zero.

SECTION 4.11

6. Zero.
7. -36π, since $\mathbf{curl\ F} \cdot \mathbf{k} = -4$ and area enclosed by C is 9π.
8. (a) 6π. (b) 6π.
9. (a) -16. (b) -16. (c) Second term.
10. 28π.
11. What is the title of Sec. 1.11?

SECTION 4.12

1. The curl of a vector field **F** points in the direction of maximum swirl and its magnitude equals this maximum swirl.

2. (a) Zero.
(b) Zero.
(c) Zero. More rigorously, use the theorem mentioned in the answer to Exercise 13, Sec. 4.10, to avoid having to speak of "very small" laundry bags.
(d) div **curl F** = 0.

Note: The divergence of a vector field at a point is sometimes called the *source density* of the field at that point. This is because the divergence of the electric intensity of an electrostatic field is equal (within a factor) to the charge density, and electric charge is the "source" or "cause" of the field. The statement "a field has zero divergence in any region that is free of sources" has an intuitive appeal to many students. The above exercise can be worded: the curl of a vector field is another vector field that is free of sources.

3. (a) Zero. (b) Zero. (c) Zero.

4. (a) Zero. (b) Zero. (c) Zero vector. (d) **curl grad** $\phi = \mathbf{0}$.

Note: The curl of a vector field at a point is sometimes called the *vortex density* of the field at that point. This is because, in some sense, the curl describes the "eddy" or "whirlpool" nature of the field. Note that vortex density is a vector quantity. Just as engineers sometimes think of a point source as a point where the divergence is "infinite", so also do they think of a *vortex filament* as a curve in space along which the magnitude of the curl is "infinite". The central part of a tornado provides an approximate idea. We leave to the reader the precise formulation of the definition. The intuitive content of Exercise 4 is that any field that can be derived from a scalar potential must be vortex-free. It should be noted, however, that if we allow the scalar potential to be a multiple-valued function, it is sometimes possible to find a scalar potential for the velocity field of fluid swirling about a vortex filament. We heartily recommend the chapter on vector analysis in *Mathematics of Circuit Analysis*, by E. A. Guillemin (Wiley, 1949), in which these matters are taken up in greater detail.

Let us now briefly review and extend some of the earlier ideas. We consider only continuously differentiable vector fields.

If a vector field defined in a domain D has any one of the following properties, it has all of them:

(*i*) Its curl is zero at every point.
(*ii*) Its integral around any closed contour is zero, provided that there is a surface enclosed by the contour entirely within D.
(*iii*) It is the gradient of a scalar function, but this function may possibly be multiple-valued.

If the domain D is simply connected, we can omit the clauses starting "provided that . . ." and "but this . . ." from these properties. When D is simply connected, the following terms are used for these fields: conservative field, irrotational field, potential field, source field.

Similarly, any one of the following properties of a continuously differentiable vector field implies the others:

(*i*) Its divergence is zero at every point in D.
(*ii*) Its integral over every surface is zero, provided that we consider only closed surfaces enclosing points all of which are in D.
(*iii*) It is the curl of another (possibly multiple-valued) vector field.

These statements are not precise and should not be taken very seriously. Terms sometimes used for such fields are: solenoidal field, rotational field, turbulent field, source-free field, vortex field. The terminology is not standardized; in modern usage, the term "turbulent" has an altogether different meaning. In applications, vector fields that are discontinuous along a surface are of considerable importance. We have not discussed such fields because they arise more naturally in courses dealing with applications, where the motivation for studying them is more apparent. The above statements are utterly false for such fields.

6. (a) $2z$.
(b) $-5\mathbf{k}$.
(c) -20π.

7. (a) 27π.
(b) 0.

SECTION 5.1

6. See the end of Sec. 5.2.

8. All zero. (Except, of course, at the origin.)

9. π.

10. $\sqrt{2} + \log(1 + \sqrt{2})$.

11. Zero.

12. 2π.

13. $nr^{n-1}\mathbf{e}_r$.

14. $\frac{16}{3}\pi$.

15. (d) Zero.

16. $(-2\cos\phi\,\mathbf{e}_r - \sin\phi\,\mathbf{e}_\phi)/r^3$.

17. $\nabla \cdot \mathbf{F} = \dfrac{2}{r} + \cot\phi - \dfrac{\sin\theta}{\sin\phi}$

$\nabla \times \mathbf{F} = \cot\phi\cos\theta\,\mathbf{e}_r - 2\cos\theta\,\mathbf{e}_\phi + 2\mathbf{e}_\theta$.

18. 2π.

20. $n = -2$.

21. All n.

22. (a) $F_r = r^{m+1}/(m+3)$.

SECTION 5.2

4. $(1/u_1)\,du_1\,du_2\,du_3$.

5. $(12u_1u_3 + 12u_2u_3 + 3)/4u_3$.

6. (a) Yes.
(b) $x = (u_1 + u_2)/2$, $y = (u_1 - u_2)/2$, $z = u_3/2$.
(c) $h_1 = \frac{1}{2}\sqrt{2}$, $h_2 = \frac{1}{2}\sqrt{2}$, $h_3 = \frac{1}{2}$.
(d) $\nabla^2 f = 2\dfrac{\partial^2 f}{\partial u_1{}^2} + 2\dfrac{\partial^2 f}{\partial u_2{}^2} + 4\dfrac{\partial^2 f}{\partial u_3{}^2}$.
(e) $\sqrt{2}\mathbf{u}_1 + \sqrt{2}\mathbf{u}_2 + 4\mathbf{u}_3$.

7. (a) $x = \frac{1}{3}(2u_1 + u_2)$, $y = \frac{1}{3}(u_1 - u_2)$, $z = \frac{1}{2}u_3$.
(c) This coordinate system is not orthogonal.

8. (b) $h_1 = h_2 = 2(u_1{}^2 + u_2{}^2)^{\frac{1}{2}}$, $h_3 = 1$.

(c) $\left(\dfrac{\partial^2 g}{\partial u_1{}^2} + \dfrac{\partial^2 g}{\partial u_2{}^2}\right)\Big/ 4(u_1{}^2 + u_2{}^2) + \dfrac{\partial^2 g}{\partial u_3{}^2}$.

(d) $\nabla \cdot \mathbf{F} = \dfrac{u_1(u_2 + u_3)}{2(u_1{}^2 + u_2{}^2)^{\frac{3}{2}}}$

$$\nabla \times \mathbf{F} = \mathbf{e}_1/2(u_1{}^2 + u_2{}^2)^{\frac{1}{2}} + \mathbf{e}_2$$
$$+ (4u_1{}^2 + 2u_2{}^2 - 2u_2u_3)\mathbf{e}_3/(u_1{}^2 + u_2{}^2)^{\frac{1}{2}}.$$

10. (a) No. (b) The element of volume in spherical coordinates is different in shape and position from that in cartesian coordinates.

11. This coordinate system is not right-handed, hence the usual formula for curl does not apply.

12. (a) $1/uv$.
(b) $2w/uv$.

13. $\partial f/\partial n$ in the u_1 direction is $(\partial f/\partial u_1)/h_1$ and its surface integral over *abcd* is $h_2 h_3\, du_2\, du_3$ times this, and so the surface integral over this face and the opposite face is

$$\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) du_1\, du_2\, du_3$$

and similarly for the other pairs of faces. The laplacian is the overall sum divided by the volume $h_1 h_2 h_3\, du_1\, du_2\, du_3$.

SECTION 5.3

1. (a) 14. (b) $[1 \quad -1 \quad 7]$. (c) $\begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$. (d) $[1 \quad -1 \quad 7 \quad 10]$.

(e) $\begin{bmatrix} 1 & -1 & 7 \\ 2 & 0 & 1 \\ 1 & -1 & 7 \end{bmatrix}$. (f) $\begin{bmatrix} 10 & 0 & -10 \\ 0 & 10 & 10 \\ 0 & -10 & 20 \end{bmatrix}$.

10. Check your answer by substituting into the equations.

15. In general, no. (Yes, if they commute.)

SECTION 5.4

2. (b) $\begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}$

6. $f' = x'^2 + y'^2 \qquad \mathbf{V}' = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}'$
$\nabla f = 2x'\mathbf{i}' + 2y'\mathbf{j}' \qquad \nabla \cdot \mathbf{V}' = 3 \qquad \nabla \times \mathbf{V}' = \mathbf{0}$.

11. $ds = (dx'^2 + dy'^2 + dz'^2)^{\frac{1}{2}}$.

12. $dV' = dx'\,dy'\,dz'$.

14. $x = -y'$, $y = z'$, $z = -x'$.

18. The line is along $\mathbf{i} - \mathbf{j} - \mathbf{k}$. The angle around this axis is $-\pi/3$.

19. (d) Take the determinant.

20–22. Elaborate on the technique in Exercise 18; or read "Coordinate-Free Rotation Formalism," by J. Mathews, *Amer. Jnl. of Physics, 44*, 1210 (1976).

23. (c) The transformation might be left-handed.

REVIEW PROBLEMS

1. (a) $\frac{8}{29}\sqrt{29}$
  (b) $\frac{8}{77}\sqrt{77}$.
  (c) Any scalar multiple of $6\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.

3. $90° - \cos^{-1}\frac{1}{7}\sqrt{35}$.

4. (a) Any scalar multiple of $5\mathbf{i} - \mathbf{j} - \mathbf{k}$
  (b) $\frac{8}{27}\sqrt{27}$.

5. $\sqrt{34}$.

6. $\frac{1}{5}(x - 4) = -\frac{1}{8}(y - 2) = -\frac{1}{7}(z - 1)$.

7. (a) $\frac{1}{6}(101)$.
  (b) $x - 11y - 14z + 43 = 0$.
  (c) $\frac{1}{319}\sqrt{319}$.

8. $\pm\frac{5}{3}(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$.

9. (a) 13.
  (b) $\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$.
  (c) 3.
  (d) -12.

10. (a) $\frac{1}{2}\sqrt{26}$.
  (b) 2.
  (c) $\cos^{-1}\frac{1}{9}$.

11. Let C be the center, A, B the ends of diameter, and let P be another point on the circumference. Write PA and PB in terms of PC and CA or CB; then show $PA \cdot PB = 0$. You will need to use the fact that $|CA| = |BC| = |PC|$.

12. Show that the vector sum of the medians is the zero vector.

13. (a) Infinitely many. (b) $\mathbf{u} = (19\mathbf{i} + 10\mathbf{j} + 17\mathbf{k})/\sqrt{750}$.
Note that $\mathbf{u} \cdot (\mathbf{a}/|\mathbf{a}|) = \mathbf{u} \cdot (\mathbf{b}/|\mathbf{b}|)$, $\mathbf{u} \cdot \mathbf{a} \times \mathbf{b} = 0$, and $|\mathbf{u}| = 1$ determine $\pm\mathbf{u}$. How do we know the answer just given is $\mathbf{u}$ and not $-\mathbf{u}$?

14. $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} \times (\mathbf{u} \times \mathbf{v})$. Expand the triple vector product. The answer is

$$\begin{vmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{u} \cdot \mathbf{v} & \mathbf{v} \cdot \mathbf{v} \end{vmatrix}$$

15. No.

16. Either $\mathbf{u} = \pm\mathbf{v}$ or both $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to $\mathbf{w}$.

17. 7. (Simply the distance $|AB|$.)

18. 6. (No calculations are required.)

19. $\frac{30}{7}$.

20. $3x + 2y + z = 18$.

21. The vector $\mathbf{i} + 8\mathbf{j} + 12\mathbf{k}$ is tangent to the curve and perpendicular to the plane at the point (2,8,8).

22. (2,4,8). (The vector extending from the center of the sphere to this point is perpendicular to the given plane.)

23. (1,1,1). [The gradient of $x^2 + 2y^2 + 3z^2$ is parallel to $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ at (1,1,1).]

24. (1,1,2). (At this point the tangent to the curve is perpendicular to **grad** ϕ. Of course, this can also be done without using vector methods, by observing that $\phi = t^2 - 6t + t^4$ has its minimum at $t = 1$.)

25. (1,1,2). This is the preceding exercise in a different format.

26. $90^\circ - \cos^{-1} 98/(\sqrt{157}\sqrt{65})$.

27. A region to the right of P, approximately cigar-shaped, with major axis parallel to the y axis.

28. $$\frac{d\mathbf{R}}{dt} = \frac{d}{dt}(R_1\mathbf{i} + R_2\mathbf{j} + R_3\mathbf{k}) = \left.\frac{d\mathbf{R}}{dt}\right|_m + R_1\frac{d\mathbf{i}}{dt} + R_2\frac{d\mathbf{j}}{dt} + R_3\frac{d\mathbf{k}}{dt}$$

By example 2.6, $d\mathbf{i}/dt$ is perpendicular to $\mathbf{i}$. Therefore,

$$\frac{d\mathbf{i}}{dt} = \alpha_1\mathbf{j} + \alpha_2\mathbf{k}$$

Similarly, $$\frac{d\mathbf{j}}{dt} = \alpha_3\mathbf{k} + \alpha_4\mathbf{i}$$

$$\frac{d\mathbf{k}}{dt} = \alpha_5\mathbf{i} + \alpha_6\mathbf{j}$$

Differentiating both sides of $\mathbf{i}\cdot\mathbf{j} = 0$, we obtain $\mathbf{i}\cdot(d\mathbf{j}/dt) + (d\mathbf{i}/dt)\cdot\mathbf{j} = 0$. Hence $\alpha_4 = -\alpha_1$. Similarly, $\alpha_5 = -\alpha_2$ and $\alpha_6 = -\alpha_3$. Let $\boldsymbol{\omega} = \alpha_3\mathbf{i} - \alpha_2\mathbf{j} + \alpha_1\mathbf{k}$ and verify that $\boldsymbol{\omega} \times \mathbf{R} = R_1\, d\mathbf{i}/dt + R_2\, d\mathbf{j}/dt + R_3\, d\mathbf{k}/dt$ (see also Appendix C).

29. If $\mathbf{T}$ is a unit tangent to the ellipse, $\nabla(|\mathbf{R}_1| + |\mathbf{R}_2|)\cdot\mathbf{T} = 0$ (why?). Also, $\nabla|\mathbf{R}_1|$ and $\nabla|\mathbf{R}_2|$ are unit vectors in the directions $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively, so the cosine of the angle between $\nabla|\mathbf{R}_2|$ and $\mathbf{T}$ equals the cosine of the angle between $\nabla|\mathbf{R}_1|$ and $-\mathbf{T}$.

30. $\cos^{-1}\frac{8}{63}\sqrt{21}$.

31. (a) $4x\mathbf{i} + \mathbf{j}$.
(b) 3.
(c) 4.

32. (d) $z\mathbf{i} - 4xz\mathbf{j} + (4xy - x)\mathbf{k}$.
(a) $2\mathbf{i}$.
(b) $2\mathbf{k}$.
(c) -2.

33. $40[(z - y)\mathbf{i} + (x - z)\mathbf{j} + (y - x)\mathbf{k}]$.

34. $6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

35. $2\mathbf{A}$.
(a) $2\mathbf{R}\cdot\mathbf{A}$.
(b) $2\mathbf{R}\times\mathbf{A}$.
(c) $2r^2\mathbf{A}$.
(d) $4(\mathbf{A}\cdot\mathbf{R})^3\mathbf{A}$.
(e) $(\mathbf{A}\cdot\mathbf{R})/r$.
(f) $(\mathbf{A}\cdot\mathbf{R})\mathbf{A}$.
(g) 0.
(h) $2\mathbf{A}$.
(i) 6.

37. (a) Any scalar multiple of $6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.
(b) 7.

38. (a) $x + y - 4z + 6 = 0$.
(b) $x + 2 = y - 4 = -\frac{1}{4}(z - 2)$.
(c) $\frac{23}{7}$.

39. $90^\circ - (\cos^{-1}\frac{3}{305}\sqrt{122})$.

40. Zero vector field except where $r = 0$; not defined where $r = 0$.

41. div (**curl F**) = 0. Hence $2 + C = 0$ and $C = -2$.

42. 0. By symmetry, it is fairly clear that a paddle wheel in such a force field will not tend to rotate, no matter how the wheel is oriented.

43. $\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (-\partial\mathbf{H}/\partial t) = -\partial/\partial t(\nabla \times \mathbf{H}) = -\partial^2\mathbf{E}/\partial t^2$
Also, $\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2\mathbf{E} + \nabla(\nabla\cdot\mathbf{E}) = -\nabla^2\mathbf{E}$
Hence $\nabla^2\mathbf{E} = \partial^2\mathbf{E}/\partial t^2$. The derivation for $\mathbf{H}$ is similar.

45. Since $\psi = \tan\phi$, the two functions have the same isotimic surfaces, so $\nabla\psi$ and $\nabla\phi$ are parallel.

46. Since $\nabla\phi$ and $\nabla\psi$ are parallel, every isotimic surface of ϕ is also an isotimic surface of ψ and vice versa.

47. (a) $$\begin{aligned}\nabla w \cdot \nabla u \times \nabla v &= (u\nabla v + v\nabla u) \cdot \nabla u \times \nabla v \\ &= u(\nabla v \cdot \nabla u \times \nabla v) + v(\nabla u \cdot \nabla u \times \nabla v) = 0\end{aligned}$$

(b) At any point in space, the isotimic surfaces $u =$ constant and $v =$ constant intersect in a curve along which both u and v and, hence, w are constant. $\nabla u \times \nabla v$ is tangent to this curve and, hence, perpendicular to ∇w.

48. If u, v, and w are functionally related, $\nabla w \cdot \nabla u \times \nabla v = 0$.

49. No. div $(\mathbf{F} \times \mathbf{G}) = 0$.

50. 60. The z component of $\mathbf{F}$ can be ignored, so replace $\mathbf{F}$ by $\mathbf{G} = y\mathbf{i} + (x+2)\mathbf{j}$. Div $\mathbf{G} = 0$ so (by the divergence theorem) the desired integral is the negative of the integral of $\mathbf{G}$ over the four *flat* faces of a certain five sided closed surface. Only two faces contribute nonzero values to the integral. The average value of $x + 2$ over the face in the xz plane is $\frac{7}{2}$ and the average value of y over the face in the yz plane is $\frac{3}{2}$ so the arithmetic is $\frac{7}{2}(3)(4) + \frac{3}{2}(3)(4) = 60$.

51. Interchanging the role of x and y amounts to reflecting the plane in the line $y = x$; following this by an inversion (replacing x by $-x$ and y by $-y$) we obtain an integral that "means" the same as before. But the effect of these replacements is to change the sign of the integrand. Therefore, the integral equals its own negative and must be zero. If this seems too abstruse, let $x = a\cos\theta$, $y = a\sin\theta$, $ds = a\,d\theta$, and the integral becomes $\int_0^{2\pi} a^3 \cos 2\theta\, d\theta = 0$.

52. 0. Use Stokes' theorem and the preceding problem.

53. $\int \mathbf{F} \cdot d\mathbf{R} = \frac{1}{2}\int x\,dy - y\,dx$. Let $x = a\cos\theta$, $y = b\sin\theta$. The integral becomes $\frac{1}{2}\int_0^{2\pi} ab\,d\theta = \pi ab$.

54. -1. The integral is independent of the path.

55. $\iiint \nabla \cdot \mathbf{F}\,dV = \iint \mathbf{F} \cdot \mathbf{n}\,dS$ leads to $\mathbf{C} \cdot \iiint \nabla\phi\,dV = C \cdot \iint \phi\mathbf{n}\,dS$ and, since this is valid for every constant vector $\mathbf{C}$, the identity follows.

56. In the divergence theorem, let $\mathbf{F} = \mathbf{A} \times \mathbf{C}$ where $\mathbf{C}$ is a constant vector field, and proceed as in the preceding problem.

57. (a) ∇f.

(b) $\nabla \times \mathbf{F}$. (Make use of the two preceding problems.)

58. $h_u = h_v = \sqrt{u^2 + v^2}$, $h_z = 1$.

59. $(u^2 + v^2)\,du\,dv\,dz$.

60. (a) $$\frac{1}{u^2 + v^2}\left(\frac{\partial}{\partial u}(\sqrt{u^2 + v^2}A_u) + \frac{\partial}{\partial v}(\sqrt{u^2 + v^2}A_v)\right) + \frac{\partial A_z}{\partial z}$$

(b) $$\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2} + (u^2 + v^2)\frac{\partial^2\phi}{\partial z^2} = 0.$$

APPENDIX A

1. Simply multiply $(u_1 i + u_2 j + u_3 k)(v_1 i + v_2 j + v_3 k)$.

2. (a) $uv + vu = (-u \cdot v + u \times v) + (-v \cdot u + v \times u)$. The cross products disappear.

3. (a) This is simple vector algebra. (See Fig. 1.20).
(b) $v' = v - 2(v \cdot n)n = v + (vn + nv)n = v + vnn + nvn$. But $n \cdot n = -nn$ and since n is a unit vector $nn = -1$. So $v' = v - v + nvn = nvn$.

4. $n'n = -(n' \cdot n) + n' \times n = -\cos \frac{1}{2}\theta - u \sin \frac{1}{2}\theta$.

5. Since u is a unit vector, $u^2 = -1$, so $u^3 = -u$, $u^4 = 1$, $u^5 = u, \ldots$ and

$$\begin{aligned} &1 + \phi u + \phi^2 u^2/2! + \phi^3 u^3/3! + \cdots \\ &\quad = (1 - \phi^2/2! + \phi^4/4! - \cdots) + (\phi - \phi^3/3! + \cdots)u \end{aligned}$$

6. $v^u = e^{(\theta/2)u} v e^{-(\theta/2)u}$.

Index

Acceleration, 72–77, 286
  centripetal, 72
  Coriolis, 85
  normal to curve, 74, 77
  tangential to curve, 74, 77
Addition (*see* Sum)
Ampere's law, 306
Angle:
  direction, 19
  positive sense, 34–36
Angle between planes, 32
Angle between vectors, 17
Angular momentum, 85, 286, 295–296
  velocity, 39, 108, 294
Arc (*see* Curve)
Arc length, 67–69
  in orthogonal coordinates, 240
Area:
  cosine principle, 161–164
  of a surface, 160
Antisymmetric matrix, 258

Binormal, 79
Boundary, oriented, 155–156
Boundary conditions, 308–310

Cartesian coordinates, 7–8, 259
Center of mass, 14, 288
Components, 9
Conservative, 138
Continuity equation, 104–105
Coordinates, 18
  Cartesian, 7–8, 259
  cylindrical, 221–229
  orthogonal, 237–244
  polar, 82–84
  right-handed, 187
  spherical, 229–235
Coriolis:
  acceleration, 85
  law, 294
Cosine law, 18
Cosine principle, 161–164
Cosines, direction, 19
Cramer's rule, 257
Cross product (*see* Vector product)
Curl, 106–109
  formula, 109
    in orthogonal coordinates, 243
  physical significance, 106–109, 203
Current, electric, 304
Curvature, 73, 76, 79
Curve, 62–67
  arc length, 67–69
  characteristic, 98
  closed, 66
  curvature, 73, 76, 79
  oriented, 66
  parametric equations, 62, 70
  regular, 66
  smooth, 66
  space, 62
  tangent to, 64
Cylindrical coordinates, 221–229

∇, 114
α_i, 123
δ_{ij}, 53
Darboux vector, 82
Degenerate line segment, 1
Del, 112–114
Delta, Kronecker, 53
Derivative:
  directional, 91–92, 282
  as an operator, 114
Desargues's theorem, 56
Determinant, 38, 45, 52
  Cramer's rule, 257
  Jacobian, 267, 270
  symbolic, 38
Direction:
  angles, 10
  cosines, 11
Directional derivative, 91–92, 282
Displacement, 12–13, 63
  current, 307
Divergence, 100–102
  formula, 102
    in orthogonal coordinates, 242
  physical significance, 100–102, 105, 192
Divergence theorem, 183, 191
  and Gauss's law, 184
  heuristic proof, 183–184
  rigorous proof, 189–192
Division, points of, 25
Domain, 133–137
  simply connected, 135–136
  star-shaped, 137, 148
Dot product (*see* Scalar product)
Dual vectors, 254
Dyadic, 117–118

ϵ_{ijk}, 52
Electric field, 170, 298
Electrostatics, 170, 298–303
  potential, 299
Energy:
  electromagnetic, 311
  kinetic, 287, 296, 310
  potential, 287
Equipotential, 90
Equivalence of line segments, 1
Euler:
  angles, 296
  equations, 295
  expansion formula, 212
  theorem, 269
Evolute, 89

Factor, scale, 239
Faraday, 307
Field:
  conservative, 138
  irrotational, 145
  scalar, 90
  solenoidal, 152
  vector, 97
  velocity, 100
Flow:
  fluid, 170
  heat, 169–170
Flow line, 98
Fluid flow, 170
Flux, 101, 166, 208, 307
Force, 3, 13, 286
  central, 85
  centrifugal, 72
  centripetal, 72
  lines of, 98
  Lorentz, 308
Frenet formulas, 79–81
Functions: (*see also* Field)
  harmonic, 116
  linear, 259–268
  and operators, 114
  vector-valued, 58

Gauss's law, 170–172, 300, 302
Gradient, 92–93
  in orthogonal coordinates, 241–242
  physical significance, 92–93
Green's formulas, 194, 195
  theorem, 195–198

Harmonic, 116
Heat flow, 169–170, 172, 194

Identities:
  algebraic, 49
  analytic, 120
Identity matrix, 253
Inertia:
  moment of, 188, 296–297
  tensor, 295
Inner product (*see* Scalar product)
Integrals (*see* Line integrals, Surface integrals, etc.)
Inverse of a matrix, 253
Irrotational, 145
Isobar, 91
Isotherm, 91
Isotimic surface, 87

Jacobian, 267, 270

Kepler's second law, 85
Kronecker delta, 53

Laplace's equation, 116
Laplacian, 115-117
  formula, 115
    in orthogonal coordinates, 243
  physical significance, 116, 194
Law of cosines, 18
Line,
  equations, 20-23, 25
  flow, 98
  stream, 98
Line integrals, 127-129
  dependence on path, 139, 145
Lines of force, 98
Lorentz force, 308

Magnetic field, 303-307
Magnitude of a vector, 2
Mass, 286
Matrix, 248-257
  antisymmetric, 258
  column, 248
  identity, 253
  inverse, 254-255
  orthogonal, 258, 261
  row, 248
  symmetric, 258, 295
  transpose, 258, 261
Maximum principle, 27
Maxwell equations, 308
  stress tensor, 312
Mechanics (*see* Force, Velocity, etc.)
Mirror reflection, 12, 28, 281
Möbius strip, 156
Momentum, 286
  angular, 85, 286, 295-296
Multiplication (*see* Product, Scalar product, Vector product, etc.)

Nabla, 114
Neighborhood, 133-134
Newton's second law, 286
  third law, 290, 303
Nonorientable surface, 156
Normal:
  to a curve, 74, 77
  to a plane, 31
  to a surface, 154-155, 158

Operators, 114
Orientation, 33-36
  of arcs, 66
  of boundaries, 155-156
  of planes, 34
  of surfaces, 154-156
Orthogonal curvilinear coordinates, 237-244
Orthogonal transformation, 259-268, 293
Orthogonal matrix, 258, 261
Orthogonal projection, 9, 118
Osculating circle, 74

Parabolic cylindrical coordinates, 276
Parallel–perpendicular decomposition, 28, 43-44
Parallelepiped, volume of, 46
Parallelogram:
  area, 37
  equality, 29
  geometry of, 3
Parametrization, 21, 30, 157
Planes:
  angle between, 32
  distance between, 32
  equations, 30-31, 47
  normals to, 31
Point:
  boundary, 134
  interior, 134
Point of division, 25
Polar coordinates, 82-84
Position vector, 12, 20
Potential, 138, 143
  vector, 152, 305-306
Power, 287
Poynting vector, 311
Principal normal, 77
Product, 42 (*see also* Scalar product, Vector product)
  matrix, 250-251
  of a vector by a scalar, 6
Pythagorean theorem, 8, 10

Quaternions, 277-280

Radius of curvature, 73, 76, 79
Reflection, 12, 28, 281
Reynold's transport theorem, 211
Right-hand system, 35, 187
Rigid body, 292
Rotations, 13-14, 296

Σ, 52
Scalar, 3
  field, 90

Scalar product, 26–27
  triple, 45–47
Scale factor, 239
Screw sense, 34
Simply connected, 135–136
Solenoidal, 152
Solid angle, 176, 218
Speed, 63
Spherical coordinates, 229–235
Stokes's theorem, 184, 202
  heuristic proof, 185–186
  rigorous proof, 201–203
Substitution tensor, 53
Sum:
  forces, 13
  matrix, 250
  vector, 3–5, 10
Summation convention, 53
Surface:
  area, 160
  closed, 156
  equipotential, 90
  isobaric, 91
  isotimic, 87
  moving, 206
  normal to, 154, 155, 158
  oriented, 154–156
  parametric equations, 157
  piecewise smooth, 155
  second fundamental form, 165
  smooth, 155
Surface integrals, 165–166
Swirl, 203
Symmetric matrix, 258, 295
  tensor, 295

Tangent:
  to a curve, 64
  to a surface, 97
Telegrapher's equation, 313
Tensor:
  inertia, 295
  notation, 51–55, 87, 123–125
Tetrahedron, volume of, 48
Torque, 36, 286
Torsion, 80
Torus, 219
Transport theorems, 205–212
Transpose of a matrix, 258, 261
Triple scalar product, 45–47
  significance of sign, 46
Triple vector product, 49–50

Unit vector, 6
  maximum principle, 27
  normal to a surface, 154, 155, 158
  tangent to a curve, 64, 71

Vector, 1 (*see also* Field, Position vector, etc.)
Vector potential, 152, 305–306
Vector product, 36–38
  triple, 49–50
Velocity, 63
  angular, 39, 108, 294
Velocity field, 100
Volume:
  of a parallelepiped, 46
  of a tetrahedron, 48
Volume integral, 176–177, 241

Wave equation, 312
Work, 27
  as a line integral, 132

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